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Dual mixed volumes of polytopes

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Abstract. We define and study the dual mixed volume rational function of a sequence of polytopes, a dual version of the mixed volume polynomial. This concept has direct relations to the adjoint polynomials and the canonical forms of polytopes. We show that dual mixed volume is additive under mixed subdivisions, and is related by a change of variables to the dual volume of the Cayley polytope. We study dual mixed volume of zonotopes, generalized permutohedra, and associahedra. The latter reproduces the planar ϕ^3-scalar amplitude at tree level.

Keywords: Polytopes, mixed volume, canonical form, mixed subdivisions, scattering amplitude

1 Introduction

Let $\mathrm{Vol}(\cdot)$ be the normalized volume function in $\mathbb{R}^d$ where the coordinate simplex has volume 1. For a sequence of convex bodies $\mathbf{S} = (S_1, S_2, \ldots, S_r)$ in $\mathbb{R}^d$ and $x_1, \ldots, x_r > 0$,

$$\mathrm{Vol}_{\mathbf{S}}(\mathbf{x}) := \mathrm{Vol}(x_1 S_1 + \cdots + x_r S_r) = \sum_{i_1, i_2, \ldots, i_d = 1}^{r} V(S_{i_1}, S_{i_2}, \ldots, S_{i_d}) x_{i_1} \cdots x_{i_d}, \tag{1.1}$$

where $V(S_{i_1}, S_{i_2}, \ldots, S_{i_d})$ are the *mixed volumes* of $S_{i_1}, \ldots, S_{i_d}$, and we call $\mathrm{Vol}_{\mathbf{S}}(\mathbf{x})$ the *mixed volume polynomial*. Mixed volumes are nonnegative and satisfy various inequalities, among them the *Alexandrov–Fenchel inequality*:

$$V(S_1, S_2, S_3, \ldots, S_d)^2 \geq V(S_1, S_1, S_3, \ldots, S_d) \cdot V(S_2, S_2, S_3, \ldots, S_d), \tag{1.2}$$

a deep result in the geometry of convex bodies, with exciting applications in combinatorics (see [15, 16]).

In this work, we study the *dual mixed volume function*, defined by

$$m_{\mathbf{S}}(\mathbf{x}) = m_{\mathbf{S}}(x_1, \ldots, x_r) := \mathrm{Vol}((x_1 S_1 + \cdots + x_r S_r)^{\vee})$$

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where $S^\vee$ denotes the polar of a convex set S. For general convex bodies, $m_{\mathbf{S}}(\mathbf{x})$ is a complicated analytic object, but for a sequence of polytopes $\mathbf{P} = (P_1, \ldots, P_r)$, the dual volume function $m_{\mathbf{P}}(\mathbf{x})$ is a rational function. We focus on polytopes in this work, leaving the general dual mixed volume function for future investigation.

Besides the natural parallel with the mixed volume polynomial, the dual mixed volume function is motivated by recent developments in *positive geometry* [3, 12], occurring at the interface of combinatorial algebraic geometry and the physics of scattering amplitudes. The dual mixed volume function specializes to the *canonical form* of a polytope and produces in a special case the scalar ϕ^3-amplitude (see Section 8), the field theory limit of the open string amplitude.

This extended abstract is organized as follows. In Section 2, we define the *dual volume* $\mathrm{Vol}^\vee(P)$ and the *dual volume function* $\mathrm{Vol}^\vee_{\mathbf{z}}(P)$ of a polyhedron P in a rigorous way, including those polyhedra that do not contain the origin in its interior, whose polar duals are unbounded. In Section 3, we show that $\mathrm{Vol}^\vee(P)$ and $\mathrm{Vol}^\vee_{\mathbf{z}}(P)$ are valuative, generalizing results of Filliman [7] and Kuperberg [11]. We also illustrate that the numerator of $\mathrm{Vol}^\vee_{\mathbf{z}}(P)$, suitably normalized, coincides with the *adjoint polynomial* of (the cone of) the dual polytope, originally defined by [17], and that the *canonical form*, in the sense of positive geometry, is given by $\Omega(P) = \mathrm{Vol}^\vee_{\mathbf{z}}(P) dz_1 \cdots dz_d$. In Section 4, we define our main object of study, the *dual mixed volume function* $m_{\mathbf{P}}(\mathbf{x}, \mathbf{z})$ for a sequence of polytopes $\mathbf{P} = (P_1, \ldots, P_r)$ in $\mathbb{R}^d$. We provide formulae for the dual mixed volumes via integrals, mixed subdivisions and Cayley polytopes (Section 5). The most important use cases of our study are *generalized permutohedra* [14]. We first setup necessary tools to discuss polytopes that live in an affine hyperplane (Section 6). Our results lead to curious identities on generalized permutohedra that are omitted here[1]. Specific cases of generalized permutohedra include zonotopes (Section 7) and associahedra (Section 8). In particular, we relate the dual mixed volume of an associahedron to the planar ϕ^3-amplitude.

2 Definitions

The main goal of this section is to define the *dual volume* $\mathrm{Vol}^\vee(P)$ of a polyhedron $P \subseteq \mathbb{R}^d$, but we will start with some basic preliminaries.

A *polyhedral cone* $C \subseteq \mathbb{R}^d$ is a non-empty intersection of finitely many closed half spaces, each passing through the origin $\mathbf{0}$. A polyhedral cone C is *pointed* if it does not contain any line. A *face* of C is an intersection of C with a supporting hyperplane. A *(polyhedral) fan* $\mathcal{F}$ is a finite set of polyhedral cones such that if $C \in \mathcal{F}$ and $F \subseteq C$ is a face, then $F \in \mathcal{F}$, and that if $C_1, C_2 \in \mathcal{F}$, then $C_1 \cap C_2$ is a face of both C_1 and C_2. A fan $\mathcal{F}$ in $\mathbb{R}^d$ is called *complete* if the union of cones in it is the whole space $\mathbb{R}^d$.

A *polyhedron* $P \subseteq \mathbb{R}^d$ is a non-empty intersection of finitely many closed half spaces

[1] A full version of the work summarized in this extended abstract appears in [8].

(not necessarily passing through the origin). A polyhedron P is a *polytope* if it is a bounded subset of $\mathbb{R}^d$.

For any non-empty closed convex set $S \subseteq \mathbb{R}^d$, the *support function* of S describes the (signed) distances from its supporting hyperplanes to the origin. It is given by

$$\begin{aligned} h_S : \mathbb{R}^d &\to \mathbb{R} \cup \{\infty\} \\ \mathbf{v} &\mapsto -\min_{\mathbf{y}\in S} \langle \mathbf{v}, \mathbf{y} \rangle. \end{aligned} \tag{2.1}$$

For a polyhedron P, the *normal fan* $\mathcal{N}(P)$ consists of the cones

$$C_F := \{\mathbf{v} \in \mathbb{R}^d \mid h_P(\mathbf{v}) = -\langle \mathbf{v}, \mathbf{y} \rangle \neq \infty \text{ for every } \mathbf{y} \in F\}$$

for each face F of P. In particular, h_P is a linear function on each C_F. When P is a polytope, its normal fan $\mathcal{N}(P)$ is a complete fan.

Definition 2.1. Let $\mathcal{F}$ be a polyhedral fan generated by g rays $\mathbf{v}_1, \ldots, \mathbf{v}_g$, and let $\mathcal{T}$ be any triangulation of $\mathcal{F}$ into full-dimensional *simplicial cones* without adding generating rays, i.e., cones whose generators are linearly independent. Define the following rational function

$$f_{\mathcal{F},\mathbf{v},\mathcal{T}}(u_1, \ldots, u_g) := \sum_{C=\operatorname{span}_{\mathbb{R}_{\geq 0}}(\mathbf{v}_{j_1},\ldots,\mathbf{v}_{j_d})\in\mathcal{T}} \frac{|\det(\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_d})|}{u_{j_1} \cdots u_{j_d}}.$$

If $\mathcal{F}$ is a polyhedral fan in $\mathbb{R}^d$ pure of dimension $r < d$, then we define $f_{\mathcal{F},\mathbf{v},\mathcal{T}} := 0$.

When $u_1, u_2, \ldots, u_g$ are all positive, $|\det(\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_d})|/(u_{j_1} \cdots u_{j_d})$ equals the normalized volume of the simplex formed by $\mathbf{0}$, $\mathbf{v}_{j_1}/u_{j_1}$, $\ldots$, $\mathbf{v}_{j_d}/u_{j_d}$. While the formula for $f_{\mathcal{F},\mathbf{v},\mathcal{T}}$ may initially look daunting, it is simply the volume of some regions when the u_i-s are positive positive. We show that $f_{\mathcal{F},\mathbf{v},\mathcal{T}}$ does not depend on the triangulation $\mathcal{T}$. Therefore we write $f_{\mathcal{F},\mathbf{v}}(u_1, \ldots, u_g) := f_{\mathcal{F},\mathbf{v},\mathcal{T}}(u_1, \ldots, u_g)$ for any triangulation $\mathcal{T}$ of $\mathcal{F}$.

We say that P is *non-codegenerate* if the origin $\mathbf{0}$ is not contained in the affine span of any of the facets of P and show that.

Lemma 2.2. *A polyhedron P is non-codegenerate if and only if for any ray $\mathbb{R}_{\geq 0} \cdot \mathbf{v}$ of $\mathcal{N}(P)$, we have $h_P(\mathbf{v}) \neq 0$.*

The following is the key definition of this section.

Definition 2.3. Let $P \subseteq \mathbb{R}^d$ be a full-dimensional non-codegenerate polyhedron. For each ray in $\mathcal{N}(P)$, pick a vector that spans it, and collect them as $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_g)$. Define the *dual volume*

$$\operatorname{Vol}^\vee(P) := f_{\mathcal{N}(P)}(h_P) = f_{\mathcal{N}(P),\mathbf{v}}(h_P(\mathbf{v_1}), \ldots, h_P(\mathbf{v_g})).$$

When P is not full dimensional, set $\operatorname{Vol}^\vee(P) = 0$. Also define the *dual volume function*

$$\operatorname{Vol}^\vee_{\mathbf{z}}(P) := \operatorname{Vol}^\vee(P - \mathbf{z})$$

viewed as a rational function in the coordinates $z_1, \ldots, z_d$ of $\mathbf{z}$.

Lemma 2.4. *The dual volume* $\mathrm{Vol}^\vee(P)$ *and dual volume function* $\mathrm{Vol}^\vee_{\mathbf{z}}(P)$ *do not depend on the choice of* $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_g)$.

The name "dual volume" comes from its connection to the polar dual. Recall

Definition 2.5. For a polyhedron $P \subseteq \mathbb{R}^d$, its *polar dual* is

$$P^\vee := \{\mathbf{v} \in \mathbb{R}^d \mid h_P(\mathbf{v}) \leq 1\} = \{\mathbf{v} \in \mathbb{R}^d \mid \langle \mathbf{v}, \mathbf{y}\rangle \geq -1 \text{ for all } \mathbf{y} \in P\}.$$

Proposition 2.6 ([6, Chapter 4 (1.2)])**.** *If P is a polytope with $\mathbf{0}$ in its interior, then $P^\vee$ is also a polytope with $\mathbf{0}$ in its interior. In this case $(P^\vee)^\vee = P$.*

Lemma 2.7. *If $P \subseteq \mathbb{R}^d$ is a full dimensional polytope and $\mathbf{0}$ is in its interior, then we have* $\mathrm{Vol}^\vee(P) = \mathrm{Vol}(P^\vee)$.

Definition 2.3 is motivated by Lemma 2.7, and intuitively, one views $\mathrm{Vol}^\vee$ as a volume function. To be precise, the notion of the dual volume $\mathrm{Vol}^\vee$ is much more powerful, in the sense that $\mathrm{Vol}^\vee(P)$ is always defined whenever $\mathbf{0}$ is not contained in the affine span of any facets of P, and the rational function $\mathrm{Vol}^\vee_{\mathbf{z}}(P)$ is always well-defined.

Example 2.8. Consider a polytope $P \subset \mathbb{R}^2$ as the convex hull of $(1,1)$, $(2,1)$, $(3,-1)$, $(1,-1)$, with its normal fan $\mathcal{N}(P)$ shown in Figure 1. We can pick $\mathbf{v_1} = (0,1)$, $\mathbf{v_2} = (1,0)$, $\mathbf{v_3} = (0,-1)$ and $\mathbf{v_4} = (-2,-1)$. Summing cyclically with $\mathbf{v_5} = \mathbf{v_1}$,

$$f_{\mathcal{N}(P),\mathbf{v}}(u_1,u_2,u_3,u_4) = \sum_{i=1}^{4} \frac{|\det(\mathbf{v_i}, \mathbf{v_{i+1}})|}{u_i u_{i+1}} = \frac{1}{u_1u_2} + \frac{1}{u_2u_3} + \frac{2}{u_3u_4} + \frac{2}{u_4u_1}.$$

We have $h_P(\mathbf{v_1}) = 1$, $h_P(\mathbf{v_2}) = -1$, $h_P(\mathbf{v_3}) = 1$ and $h_P(\mathbf{v_4}) = 5$. Assigning $u_i = h_P(\mathbf{v_i})$ in $f_{\mathcal{N}(P),\mathbf{v}}$ for all i, we get $\mathrm{Vol}^\vee(P) = (-1) + (-1) + \frac{2}{5} + \frac{2}{5} = -\frac{6}{5}$. Assigning $u_i = h_{P-\mathbf{z}}(\mathbf{v_i}) = h_P(\mathbf{v_i}) + \langle \mathbf{z}, \mathbf{v_i}\rangle$ for all i, we have

$$\begin{aligned}&\mathrm{Vol}^\vee_{\mathbf{z}}(P)\\ &= \frac{1}{(1+z_2)(-1+z_1)} + \frac{1}{(-1+z_1)(1-z_2)} + \frac{2}{(1-z_2)(5-2z_1-z_2)} + \frac{2}{(5-2z_1-z_2)(1+z_2)}.\end{aligned}$$

Example 2.9. Consider the unbounded polyhedron $P \subset \mathbb{R}^2$ defined by the inequalities $3x_1 + x_2 + 3 \geq 0$, $x_1 + x_2 + 1 \geq 0$, $-2x_1 + x_2 + 4 \geq 0$. Its normal fan, $\mathcal{N}(P)$, is not a complete fan, shown in Figure 2. We can pick $\mathbf{v_1} = (-2,1)$, $\mathbf{v_2} = (1,1)$ and $\mathbf{v_3} = (3,1)$. Compute that $h_P(\mathbf{v_1}) = 4$, $h_P(\mathbf{v_2}) = 1$, $h_P(\mathbf{v_3}) = 3$ and

$$f_{\mathcal{N}(P),\mathbf{v}}(u_1,u_2,u_3) = \frac{3}{u_1u_2} + \frac{2}{u_2u_3}.$$

Substituting $u_i = h_P(\mathbf{v_i})$ for all i gives $\mathrm{Vol}^\vee(P)$ and substituting $u_i = h_P(\mathbf{v_i}) + \langle \mathbf{z}, \mathbf{v_i}\rangle$ gives $\mathrm{Vol}^\vee_{\mathbf{z}}(P)$. We therefore obtain $\mathrm{Vol}^\vee(P) = 17/12$ and

$$\mathrm{Vol}^\vee_{\mathbf{z}}(P) = \frac{3}{(4-2z_1+z_2)(1+z_1+z_2)} + \frac{2}{(1+z_1+z_2)(3+3z_1+z_2)}.$$

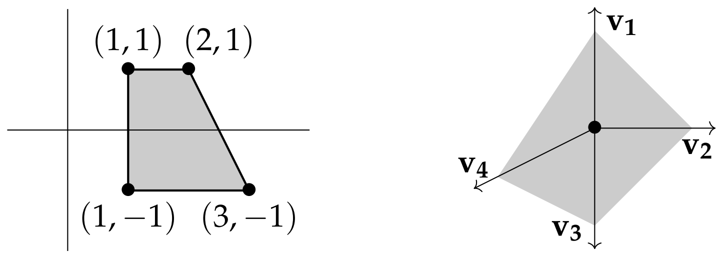

Figure 1: A polytope P and its normal fan $\mathcal{N}(P)$

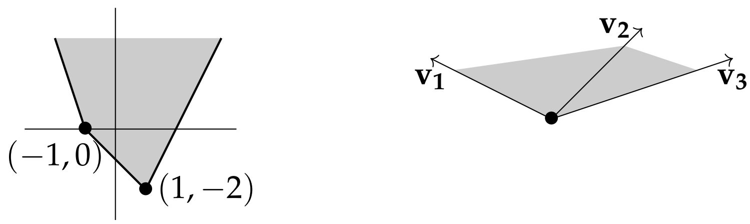

Figure 2: An unbounded polyhedron P and its normal fan $\mathcal{N}(P)$

3 Properties of dual volume

3.1 An integral formula

Theorem 3.1. *Let $P \subset \mathbb{R}^d$ be a polyhedron and $\mathbf{z} \in \mathbb{R}^d$ be any point such that $P - \mathbf{z}$ is non-codegenerate. Then*

$$\mathrm{Vol}^{\vee}_{\mathbf{z}}(P) = \int_{\mathbb{R}^d} \exp(-h_P(\mathbf{v}) - \langle \mathbf{v}, \mathbf{z} \rangle) d\mathbf{v}. \tag{3.1}$$

For a cone $C \subseteq \mathbb{R}^d$, define the *dual cone* C^* by

$$C^* := \{\mathbf{v} \in \mathbb{R}^d \mid \langle \mathbf{v}, \mathbf{y} \rangle \geq 0 \text{ for all } \mathbf{y} \in C\}. \tag{3.2}$$

Note that we use * for polarity/duality on cones and $^\vee$ for polarity on polytopes.

Corollary 3.2. *Let $P \subset \mathbb{R}^d$ be a non-codegenerate polyhedron. Then*

$$\mathrm{Vol}^{\vee}(P) = \int_{\mathbb{R}^d} e^{-h_P(\mathbf{v})} d\mathbf{v}. \tag{3.3}$$

3.2 Valuative property

Let $[P]$ be the indicator function of a polyhedron P. Let $\mathcal{P}$ denote the space spanned by indicator functions of polyhedra. We show that $\mathrm{Vol}^{\vee}(P)$ and $\mathrm{Vol}^{\vee}_{\mathbf{z}}(P)$ are valuative.

Theorem 3.3. *Suppose $\sum_{i=1}^r \alpha_i [P_i] = 0$ in $\mathcal{P}$, for polyhedra $P_1, \ldots, P_r \in \mathbb{R}^d$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$. Then*

$$\sum_{i=1}^r \alpha_i \operatorname{Vol}^\vee_{\mathbf{z}}(P_i) = 0. \tag{3.4}$$

If each P_i is non-codegenerate, then

$$\sum_{i=1}^r \alpha_i \operatorname{Vol}^\vee(P_i) = 0. \tag{3.5}$$

We deduce Theorem 3.3 from the classical result of Lawrence stating that the algebra of indicator functions of cones has a polarity involution. The result is also known to Alexander Barvinok [5]. It generalizes duality results of Filliman [7] and Kuperberg [11].

3.3 Relation to adjoint polynomials

The adjoint polynomial of a polytope was first introduced by Warren in [17]. We use the version given in Aluffi [1]; see also [10].

Definition 3.4 ([1, Definition/Theorem 4.1])**.** Let C be a polyhedral cone in $\mathbb{R}^{d+1}$ generated by the extreme rays $V(C)$, and let T be a triangulation of C. The *adjoint polynomial* of C is given by

$$\operatorname{adj}_C(\mathbf{z}) = \sum_{F \in T} |\det(F)| \prod_{\mathbf{v} \in V(C) \setminus V(F)} \langle \mathbf{v}, \mathbf{z} \rangle \tag{3.6}$$

where the sum is over all simplicial cones $F = \operatorname{span}_{\mathbb{R}_{\geq 0}}(\mathbf{v}_1, \ldots, \mathbf{v}_{d+1})$ in T, and $\det(F) = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{d+1})$.

This definition is independent of the choice of the triangulation T.

Combine the summands in the definition of $\operatorname{Vol}^\vee_{\mathbf{z}}(P)$ into a single fraction to get:

$$\operatorname{Vol}^\vee_{\mathbf{z}}(P) = \frac{A_{\mathbf{z}}(P)}{B_{\mathbf{z}}(P)}, \tag{3.7}$$

where

$$\begin{aligned} A_{\mathbf{z}}(P) &:= \sum_{C \in \mathcal{T}} |\det(C)| \prod_{\mathbf{v} \in V(\mathcal{N}(P)) \setminus V(C)} h_{P-\mathbf{z}}(\mathbf{v}), \\ B_{\mathbf{z}}(P) &:= \prod_{\mathbf{v} \in V(\mathcal{N}(P))} h_{P-\mathbf{z}}(\mathbf{v}) \end{aligned} \tag{3.8}$$

and $\mathcal{T}$ is any triangulation of $\mathcal{N}(P)$, C is a simplicial cone in $\mathcal{T}$ generated by its extreme rays $V(C) = \{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$, and $\det(C) = \det(\mathbf{v}_1, \ldots, \mathbf{v}_d)$.

We show that the numerator of the dual volume function $\operatorname{Vol}_{\mathbf{z}}(P)$ coincides with the adjoint polynomial of (the cone of) the dual polytope $C(P)^*$.

Theorem 3.5. *For any vector* $\mathbf{z} \in \mathbb{R}^d$, *let* $\bar{\mathbf{z}} = (1, \mathbf{z}) \in \mathbb{R}^{d+1}$. *Then*

$$\mathrm{Vol}^{\vee}_{\mathbf{z}}(P) = \frac{\mathrm{adj}_{C(P)^*}(\bar{\mathbf{z}})}{B_{\mathbf{z}}(P)}, \qquad \textit{or equivalently,} \qquad \mathrm{adj}_{C(P)^*}(\bar{\mathbf{z}}) = A_{\mathbf{z}}(P).$$

3.4 Canonical forms

We connect the dual volume of a polyhedron to positive geometry. *Positive geometries* are semialgebraic sets endowed with a distinguished meromorphic form called the *canonical form*. *Projective polytopes* $\bar{P} \subseteq \mathbb{P}^d(\mathbb{R})$ are examples of positive geometries (see [3, 12]).

Theorem 3.6. *Let* $P \subseteq \mathbb{R}^d$ *be a full-dimensional polyhedron that does not contain lines. Then the canonical form* $\Omega(\bar{P})$ *of the projective polytope* $\bar{P}$ *is given by*

$$\Omega(\bar{P}) = \mathrm{Vol}^{\vee}_{\mathbf{z}}(P) dz_1 dz_2 \cdots dz_d.$$

4 Dual mixed volumes

We move on to consider a sequence $\mathbf{P} = (P_1, \ldots, P_r)$ of polyhedra in $\mathbb{R}^d$. Each h_{P_i} is piecewise-linear, and their common domains of linearity give a fan $\mathcal{F}$ in $\mathbb{R}^d$, which coincides with the normal fan $\mathcal{N}(P)$ of the Minkowski sum $P = P_1 + \cdots + P_r$. The fan is complete exactly when P (and thus each P_i) is a polytope. In the following assume that the Minkowski sum P is full-dimensional in $\mathbb{R}^d$, so the maximal cones of $\mathcal{F}$ are pointed.

Definition 4.1. Let $\mathbf{P} = (P_1, \ldots, P_r)$ be a regular sequence of polyhedra with normal fan $\mathcal{N}(P)$ and let $\mathbf{x} = (x_1, \ldots, x_r)$. The *dual mixed volume rational function* $m_{\mathbf{P}}(\mathbf{x})$ is

$$m_{\mathbf{P}}(\mathbf{x}) := \mathrm{Vol}^{\vee}(h_{\mathbf{xP}}) = f_{\mathcal{N}(P),\mathbf{v}}(h_{\mathbf{xP}}(\mathbf{v_1}), \ldots, h_{\mathbf{xP}}(\mathbf{v_g}))$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_g$ are the generating rays of $\mathcal{N}(P)$, with notation as in Definition 2.3. For any sequence $\mathbf{P}$ that is full-dimensional, also define

$$m_{\mathbf{P}}(\mathbf{x}, \mathbf{z}) := m_{(P_1, P_2, \ldots, P_r, -\mathbf{e}_1, \ldots, -\mathbf{e}_r)}(x_1, \ldots, x_r, z_1, \ldots, z_d) = \mathrm{Vol}^{\vee}(h_{\mathbf{xP}-\mathbf{z}}).$$

If $\mathbf{P}$ is not full-dimensional, we set $m_{\mathbf{P}}(\mathbf{x}) := 0$ and $m_{\mathbf{P}}(\mathbf{x}, \mathbf{z}) := 0$. By Definition 2.1 we can write $m_{\mathbf{P}}$ as a rational function in $x_1, x_2, \ldots, x_r$, with degree $-d$. The denominator is a product of linear factors, each corresponding to a ray $\mathbf{v}_i$ of $\mathcal{N}(P)$. Note that $m_{\mathbf{P}}(\mathbf{x})$ generalizes both $\mathrm{Vol}^{\vee}$ and $\mathrm{Vol}^{\vee}_{\mathbf{z}}$. For a full-dimensional polytope P, we have the specializations

$$m_P(1) = \mathrm{Vol}^{\vee}(P), \qquad m_{(P, -\mathbf{e}_1, -\mathbf{e}_2, \ldots, -\mathbf{e}_d)}(1, z_1, z_2, \ldots, z_d) = \mathrm{Vol}^{\vee}_{\mathbf{z}}(P).$$

Theorem 4.2. *Let $\mathbf{P} = (P_1, \ldots, P_r)$ be a sequence of polyhedra in $\mathbb{R}^d$, $\mathbf{x} = (x_1, \ldots, x_r)$ in $\mathbb{R}^r$, and $\mathbf{z} \in \mathbb{R}^d$ be any point such that $\mathbf{xP} - \mathbf{z}$ is non-codegenerate. Then*

$$m_{\mathbf{P}}(\mathbf{x}, \mathbf{z}) = \int_{\mathbb{R}^d} \exp(-h_{\mathbf{xP}-\mathbf{z}}(\mathbf{v})) d\mathbf{v}.$$

If $\mathbf{xP}$ is non-codegenerate, we can let $\mathbf{z} \to \mathbf{0}$ and take limits on both sides to get:

$$m_{\mathbf{P}}(\mathbf{x}) = \int_{\mathbb{R}^d} \exp(-h_{\mathbf{xP}}(\mathbf{v})) d\mathbf{v}.$$

We study the behavior of the dual mixed volume function $m_{\mathbf{P}}(\mathbf{x})$ under *mixed subdivisions*: those subdivision of $P_1 + \cdots + P_r$ that are compatible with the Minkowski sum. Our main result (Theorem 4.3) states that dual mixed volume is additive under mixed subdivisions, which generalizes Theorem 3.3 to the mixed setting.

Theorem 4.3. *Let $\mathcal{S} = \{\mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}, \ldots, \mathbf{Q}^{(n)}\}$ be a mixed subdivision of $\mathbf{P} = (P_1, \ldots, P_r)$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_r)$. Then the dual mixed volume of $\mathbf{P}$ can be written as a sums of the dual mixed volumes of the cells in $\mathcal{S}$:*

$$m_{\mathbf{P}}(\mathbf{x}, \mathbf{z}) = \sum_{\mathbf{Q} \in \mathcal{S}} m_{\mathbf{Q}}(\mathbf{x}, \mathbf{z}). \tag{4.1}$$

5 Formulae from the Cayley polytope

Definition 5.1. For a sequence of polytopes $\mathbf{P} = (P_1, P_2, \ldots, P_r)$ in $\mathbb{R}^d$, let $\mu_i(\mathbf{z}) := (\mathbf{z}, \mathbf{e}_i)$ for $i \in [r]$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{R}^r$. The *Cayley embedding* of $\mathbf{P}$ is the map $\mathcal{C}$ sending $\mathbf{P}$ to following polytope in $\mathbb{R}^{d+r}$.

$$\mathcal{C}(\mathbf{P}) := \mathcal{C}(P_1, P_2, \ldots, P_r) := \operatorname{conv}\{\mu_1(P_1) \cup \mu_2(P_2) \cup \cdots \cup \mu_r(P_r)\}.$$

We call $\mathcal{C}(P_1, P_2, \ldots, P_r)$ the *Cayley polytope* of $(P_1, P_2, \ldots, P_r)$.

The following proposition is known as the Cayley trick.

Proposition 5.2 ([9, Theorem 3.1])**.** *There is a bijection between the mixed subdivisions of $P_1 + P_2 + \cdots + P_r$ and the subdivisions of the Cayley embedding $\mathcal{C}(P_1, P_2, \ldots, P_r)$. Furthermore, it restricts to a bijection between the fine mixed subdivisions of $P_1 + P_2 + \cdots + P_r$ and the triangulations of $\mathcal{C}(P_1, P_2, \ldots, P_r)$.*

The Cayley trick allows us to write the dual mixed volume function as a single dual volume function calculated in an affine hyperplane, where volumes are normalized by the normal vector $\mathbf{u} = (0, \ldots, 0, 1, \ldots, 1)$.

Theorem 5.3. *Let $\mathbf{P} = (P_1, \ldots, P_r)$ be full-dimensional. For $t = \langle \mathbf{1}, \mathbf{x} \rangle$,*

$$m_{\mathbf{P}}(\mathbf{x}, \mathbf{z}) = \frac{\prod_{i=1}^n x_i}{\sum_{i=1}^n x_i} \cdot \operatorname{Vol}^{\vee}_{(\mathbf{z},\mathbf{x})}(t\mathcal{C}(\mathbf{P}))$$

where the right hand side is calculated in the affine hyperplane $\{(\mathbf{z}, \mathbf{x}) \mid \langle \mathbf{1}, \mathbf{x} \rangle = t\} \subset \mathbb{R}^{d+r}$.

6 Polytopes in an affine hyperplane

Many polytopes of interest in combinatorics, such as matroid polytopes, alcoved polytopes, and generalized permutohedra, are naturally defined to live in an affine hyperplane. We develop the following formalism to study this case.

Definition 6.1. let P be a polytope whose affine span is the hyperplane $H \subseteq \mathbb{R}^d$. Fix a triangulation $\mathcal{T}$ of the *boundary* of $\mathrm{cone}(P)^*$, and let $\{\mathbf{v}_i\}_{i=1}^{g}$ be generators for the extremal rays in $\mathcal{T}$. Let $F \in \mathcal{T}$ denote a top-dimensional ($(d-1)$-dimensional) simplicial cone F in $\mathcal{T}$. The *hyperplane dual volume function* of the "affine polytope" P is

$$\mathrm{EVol}^{\vee}_{\mathbf{z}}(P) := \sum_{F=\mathrm{span}(\mathbf{v}_{i_1},\ldots,\mathbf{v}_{i_{d-1}})\in\mathcal{T}} \frac{|\det(\mathbf{v}_{i_1},\mathbf{v}_{i_2},\ldots,\mathbf{v}_{i_{d-1}},\mathbf{1})|}{\langle \mathbf{v}_{i_1},\mathbf{z}\rangle\langle \mathbf{v}_{i_2},\mathbf{z}\rangle\cdots\langle \mathbf{v}_{i_{d-1}},\mathbf{z}\rangle}. \tag{6.1}$$

If P does not have full dimension in H, i.e. $\dim(P) < d-1$, we define $\mathrm{EVol}^{\vee}_{\mathbf{z}}(P) := 0$.

Similar to the case of the dual volume (Definition 2.3), $\mathrm{EVol}^{\vee}_{\mathbf{z}}(P)$ does not depend on the triangulation $\mathcal{T}$, and the following valuative property holds (similar to Theorem 3.3).

Theorem 6.2. *Suppose $\sum_{i=1}^{r}\alpha_i[P_i] = 0$ for polytopes $P_1,\ldots,P_r \in H$ and $\alpha_1,\ldots,\alpha_r \in \mathbb{R}$, then*

$$\sum_{i=1}^{r}\alpha_i \mathrm{EVol}^{\vee}_{\mathbf{z}}(P_i) = 0.$$

Definition 6.3. The *hyperplane dual mixed volume function* of $\mathbf{P} = (P_1,\ldots,P_r)$ is

$$\tilde{m}_{\mathbf{P}}(\mathbf{x},\mathbf{z}) := \sum_{F=\mathrm{span}(\mathbf{v}_{i_1},\ldots,\mathbf{v}_{i_{d-1}})\in\mathcal{T}} \frac{|\det(\mathbf{v}_{i_1},\mathbf{v}_{i_2},\ldots,\mathbf{v}_{i_{d-1}},\mathbf{1})|}{h_{\mathbf{xP}-\mathbf{z}}(\mathbf{v}_{i_1})h_{\mathbf{xP}-\mathbf{z}}(\mathbf{v}_{i_2})\cdots h_{\mathbf{xP}-\mathbf{z}}(\mathbf{v}_{i_{d-1}})}, \tag{6.2}$$

a rational function in $\mathbf{x} = (x_1,\ldots,x_r)$ and $\mathbf{z} = (z_1,\ldots,z_d)$, with $h_{\mathbf{xP}}(\mathbf{v})$ defined as before.

From Theorem 4.3 we obtain the following proposition.

Proposition 6.4. *Let $\mathbf{P} = (P_1,\ldots,P_r)$ be a sequence of polytopes in H_1 such that $P = P_1 + \cdots + P_r$ is full-dimensional in H_r. Let $\mathcal{S} = \{\mathbf{Q}^{(1)},\ldots,\mathbf{Q}^{(N)}\}$ be a fine mixed subdivision of $\mathbf{P}$. Then*

$$\tilde{m}_{\mathbf{P}}(\mathbf{x},\mathbf{z})|_{\langle \mathbf{x},\mathbf{1}\rangle=\langle \mathbf{z},\mathbf{1}\rangle} = \sum_{\mathbf{Q}\in\mathcal{S}} \tilde{m}_{\mathbf{Q}}(\mathbf{x},\mathbf{z})|_{\langle \mathbf{x},\mathbf{1}\rangle=\langle \mathbf{z},\mathbf{1}\rangle}.$$

7 Zonotopes

We study *zonotopes*, polytopes that are Minkowski sums of intervals and show that their dual mixed volumes uniquely determined by dual mixed volumes of the deletion and contraction.

Let $P \subseteq \mathbb{R}^d$ be a full-dimensional polytope and $\mathbf{p} \in \mathbb{R}^d$ be a vector. Let H be the hyperplane normal to $\mathbf{p}$ and let $H_{>0}$ (resp. $H_{\geq 0}$) and $H_{<0}$ (resp. $H_{\leq 0}$) denote the corresponding closed (resp. open) halfspaces. Let

$$P(x) := P + x[-\mathbf{p}, \mathbf{p}]$$

The aim is to give a recursive description of the rational function

$$V(x, \mathbf{z}) := \mathrm{Vol}^{\vee}_{\mathbf{z}}(P(x)) = \mathrm{Vol}^{\vee}_{\mathbf{z}}(P + x[-\mathbf{p}, \mathbf{p}]).$$

Let $V(\mathbf{z}) = V(0, \mathbf{z}) = \mathrm{Vol}^{\vee}_{\mathbf{z}}(P)$. Let $P_{/\mathbf{p}} := \mathrm{proj}_{\mathbf{p}} P$ denote the orthogonal projection of P into H, in the direction of $\mathbf{p}$.

Let $B(\mathbf{z})$ be the denominator of $V(\mathbf{z})$ as defined in (3.8). The cones of the normal fan $\mathcal{N}'$ of $P + [-\mathbf{p}, \mathbf{p}]$ are obtained from $\mathcal{N}(P)$ by intersecting with the cones $H_{\geq 0}$, H, and $H_{\leq 0}$. Let $D(\mathbf{z})$ be the product of linear factors $h_{P-\mathbf{z}}(\mathbf{v}_i)$'s where $\mathbf{v}_i$'s are rays of $\mathcal{N}'$ that are not rays of $\mathcal{N}$.

Theorem 7.1. *There exists a unique pair of rational functions $W_+ = W_+(\mathbf{z})$ and $W_- = W_-(\mathbf{z})$ satisfying the following properties:*

(1). $W_+(\mathbf{z}) + W_-(\mathbf{z}) = V(\mathbf{z})$;

(2). $W_\pm(\mathbf{z}) = A_\pm(\mathbf{z}) / \big(B(\mathbf{z}) D(\mathbf{z})\big)$ *for some polynomial* $A_\pm(\mathbf{z})$;

(3). writing $\mathbf{z} = \mathbf{z}_0 + t\mathbf{p}$ *with* $\mathbf{z}_0 \in H$*, then* $\lim_{t\to\infty} tW_\pm(\mathbf{z}_0, t) = \pm \mathrm{Vol}^{\vee}_{\mathbf{z}_0}(P_{/\mathbf{p}}) / ||\mathbf{p}||^2$.

Consequently, we have $V(x, \mathbf{z}) = W_+(\mathbf{z} + x\mathbf{p}) + W_-(\mathbf{z} - x\mathbf{p})$.

Recursively applying Theorem 7.1, one can compute the dual mixed volume of a zonotope, via contraction-deletion.

By the Bohne–Dress Theorem, mixed subdivisions of zonotopes, or zonotopal tilings, are in bijection with one-element liftings of the corresponding oriented matroid. Thus we obtain a formula for the dual mixed volume $m_{\mathbf{P}}(\mathbf{x})$ for each such lifting.

8 Associahedra

We give an explicit formula (Proposition 8.2) for the dual mixed volume of an *associahedron* and relate it to the planar ϕ^3-amplitude.

Definition 8.1. For $n \in \mathbb{Z}_{>0}$, let $\mathbf{P} = (\Delta_{[i,j]})_{1 \leq i \leq j \leq n}$. The *associahedron* is

$$\mathbf{xP} := \sum_{1 \leq i \leq j \leq n} x_{ij} \Delta_{[i,j]},$$

which is a special case of the generalized permutohedron.

Note that here we use x_{ij}, where $1 \le i \le j \le n$ to index a variable. Definition 8.1 is commonly referred to as the *Loday realization* [13] of the associahedra. Our presentation of the material largely follows [14].

We need some additional notations in order to describe the normal fan $\mathcal{N}(P)$ of the associahedron $\mathbf{xP}$. Let B be a plane binary tree and let $e \in B$ be an edge. Deleting e from this tree results in two connected components $[n] = L_{B,e} \sqcup U_{B,e}$ where $U_{B,e}$ contains the root of B.

Proposition 8.2. *When $\langle \mathbf{x}, \mathbf{1} \rangle = 0$, we have*

$$\tilde{m}_{\mathbf{P}}(\mathbf{x})|_{\langle \mathbf{x},\mathbf{1}\rangle=0} = (-1)^{n-1} \sum_{B \in \mathcal{PB}(n)} \prod_{e \in B} \frac{1}{\sum_{i,j \in L_{B,e}} x_{ij}}.$$

8.1 Relation to planar ϕ^3-amplitude at tree level

We relate Proposition 8.2 to a rational function $\mathcal{A}_n^{\phi_3}(\mathbf{s}_{ij})$ appearing in physics, called the ϕ^3-amplitude; see [4, 2]. This rational function was a main motivation for our study of dual mixed volumes.

Scattering amplitudes are functions that compute the outcome of scattering experiments in particle physics. Traditionally, they are computed as the sum over Feynman diagrams which depend on the choice of particles and their interactions. In "planar ϕ^3-theory", the Feynman diagrams are *planar cubic trees*. For n-particle scattering, the amplitude[2] $\mathcal{A}_n^{\phi_3}(\mathbf{s}_{ij})$ is a sum over all planar cubic trees with n nodes.

Comparing it with Proposition 8.2, we obtain the following.

Proposition 8.3. *Up to a sign, the dual mixed volume $\tilde{m}_{\mathbf{P}}(\mathbf{x})|_{\langle \mathbf{x},\mathbf{1}\rangle=0}$ of the associahedron is equal to the planar ϕ^3-amplitude $\mathcal{A}_n^{\phi_3}$ under the substitution $x_{ij} \mapsto s_{i,j+1}$.*

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