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Quantum bumpless pipe dreams

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Abstract. Schubert polynomials are polynomial representatives of Schubert classes in the cohomology of the complete flag variety and have a combinatorial formulation in terms of bumpless pipe dreams. Quantum double Schubert polynomials are polynomial representatives of Schubert classes in the torus-equivariant quantum cohomology of the complete flag variety, but no analogous combinatorial formulation had been discovered. We introduce a generalization of the bumpless pipe dreams called quantum bumpless pipe dreams, giving a novel combinatorial formula for quantum double Schubert polynomials as a sum of binomial weights of quantum bumpless pipe dreams. We give a bijective proof for this formula by showing that the sum of binomial weights satisfies a defining transition equation.

Keywords: quantum bumpless pipe dreams, quantum Schubert polynomials, quantum cohomology

1 Introduction

Schubert polynomials were introduced by Lascoux and Schützenberger [11] and they represent cohomology classes called Schubert classes of the complete flag variety. The original definition is algebraic and relies on divided difference operators; however, multiple combinatorial formulas for the monomial expansion of Schubert polynomials have been found [1, 2, 4, 9, 13]. Two such examples are Schubert polynomials as weight-generating functions of pipe dreams (originally called RC-graphs [2, 4]), or bumpless pipe dreams [9]. For example, for BPD(w), the set of (reduced) bumpless pipe dreams of a permutation w, one has

$$\mathfrak{S}_w(x,y) = \sum_{P \in \mathrm{BPD}(w)} \mathrm{bwt}(P)\,,$$

where bwt(P) is a product of binomials $(x_i - y_j)$ associated to $P \in \mathrm{BPD}(w)$ [9]. These pipe dream and bumpless pipe dream formulations generalize to some generalizations

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of Schubert polynomials, such as double Schubert polynomials and double Grothendieck polynomials [8, 15].

Motivated by ideas that stem from string theory, mathematicians defined quantum cohomology rings [14, 16]. See, e.g., [6] for more on the history of quantum cohomology. In the quantum cohomology of the complete flag variety, the Schubert classes correspond to quantum Schubert polynomials, another generalization of Schubert polynomials [3]. Quantum double Schubert polynomials, which generalize both quantum Schubert polynomials and double Schubert polynomials, were defined in [5, 7]. Like Schubert polynomials, there is a quantum double Schubert polynomial for each permutation w of $\{1, 2, \ldots, n\}$, denoted $\mathfrak{S}^q_w(x, y)$, lying in $\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n, q_1, \ldots, q_{n-1}]$. There is no known combinatorial formulation for the monomial expansion of quantum Schubert polynomials or quantum double Schubert polynomials. One major difficulty is the presence of unpredictable signs in the monomial expansion of $\mathfrak{S}^q_w(x, y)$. In this paper, we define combinatorial objects called quantum bumpless pipe dreams (QBPDs). They are a generalization of bumpless pipe dreams, and their weight-generating function gives the quantum double Schubert polynomials, i.e.,

$$\mathfrak{S}^q_w(x, y) = \sum_{P \in \mathrm{QBPD}(w)} \mathrm{bwt}(P)\,,$$

where $\mathrm{QBPD}(w)$ is the set of QBPDs of w and $\mathrm{bwt}(P)$ is a product of $(x_i - y_j)$'s and q_i's. This is stated precisely in Theorem 3.4. Unfortunately, this formula has internal cancellation, but the combinatorics seems quite natural.

We give the necessary background in Section 2. In Section 3, we define quantum bumpless pipe dreams, establish fundamental combinatorial properties, and state the main theorem. In Section 3.1, we provide a way to generate all QBPDs for a given permutation using droop moves as in [9], as well as new moves called *lift moves*. In Section 3.2, we establish that the quantity $\sum_{P \in \mathrm{QBPD}(w)} \mathrm{bwt}(P)$ satisfies the stability condition, i.e., it does not change under the natural inclusion map $i \colon S_n \to S_{n+1}$, which is needed for our proof of Theorem 3.4. We give an overview of the proof of Theorem 3.4 in Section 3.3. In Section 4, we provide some examples of (partial) cancellations of the binomial weight of QBPDs and provide some analysis of cancellations. This is an extended abstract for [12], and the full proof of Theorem 3.4 and other results stated here can be found there.

2 Background

We use the notation $[n] := \{1, 2, \ldots, n\}$. Let S_n be the symmetric group on $[n]$, i.e. the set of bijections from $[n]$ to $[n]$. To write a bijection $\sigma \colon [n] \to [n]$, we will use one-line notation, i.e., writing $\sigma(1)\sigma(2)\ldots\sigma(n)$. Multiplication of permutations are function composition (in that order), that is, $\sigma\tau = \sigma \circ \tau$ as functions from $[n]$ to $[n]$. We write

t_{ab} for the transposition that swaps a and b, and write s_i for the adjacent transposition $t_{i,i+1}$. The length of $w \in S_n$, denoted $\ell(w)$, is defined as the minimum number of adjacent transpositions s_i required to express w as their product. Any way to write w as a product of exactly $\ell(w)$ adjacent transpositions is called a *reduced word* for w. Let $w_0 := n\, n-1 \ldots 1$ be the longest permutation in S_n.

We write $\mathbb{Z}[x]$ for $\mathbb{Z}[x_1, \ldots, x_n]$, $\mathbb{Z}[x, y]$ for $\mathbb{Z}[x][y_1, \ldots, y_n]$, $\mathbb{Z}[x, q]$ for $\mathbb{Z}[x][q_1, \ldots, q_{n-1}]$ and $\mathbb{Z}[x, y, q]$ for $\mathbb{Z}[x, y][q_1, \ldots, q_{n-1}]$.

2.1 Rothe Diagrams

Throughout this paper, we use matrix coordinate notation: (i, j) means the box on row i column j. For a permutation $\sigma : [n] \to [n]$, the *Rothe diagram* of σ is defined as follows. The set of boxes $\{(i, \sigma(i)) : i \in [n]\}$ are first marked with a dot in the grid. Then, starting from each dot and ending on edges of the grid, vertical lines are drawn downward and horizontal lines are drawn rightward. The resulting figure is the Rothe diagram for σ.

Note that the Rothe diagram for a permutation σ can be turned into a bumpless pipe dream by "smoothing" the corners into tiles, as in Figure 1. (For the definition of bumpless pipe dream, see [9], or Definition 3.1 below.)

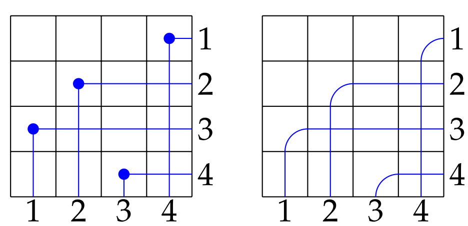


Figure 1: Rothe diagram for 4213

2.2 Double Schubert polynomials

Consider the action of S_n on $\mathbb{Z}[x, y]$ by permuting the y variables; in particular, s_i swaps y_i and y_{i+1}:

$$s_i f(x, y_1, \ldots, y_i, y_{i+1}, \ldots, y_n) = f(x, y_1, \ldots, y_{i+1}, y_i, \ldots, y_n).$$

We define divided difference operators ∂_i^y as follows

$$\partial_i^y(f) := \frac{f - s_i f}{y_i - y_{i+1}}.$$

The divided difference operators ∂_w^y for $w \in S_n$ are defined as follows. Let $s_{a_1} \cdots s_{a_k}$ be any reduced word for w. Then,

$$\partial_w^y = \partial_{a_1}^y \cdots \partial_{a_k}^y.$$

The double Schubert polynomials are then defined as follows:

$$\mathfrak{S}_w(x,y) = \begin{cases} \prod_{i+j \leq n} (x_i - y_j), & \text{if } w = w_0, \\ (-1)^{\ell(w_0)-\ell(w)} \partial_{ww_0}^y \mathfrak{S}_{w_0}(x,y), & \text{otherwise.} \end{cases} \tag{2.1}$$

Specializing the y variables to 0 recovers the Schubert polynomials.

2.3 Quantum double Schubert polynomials

As in [3], we define $E_i^k(x_1, \ldots, x_k)$ to be the coefficient of λ^i in the characteristic polynomial $\det(1 + \lambda G_k)$ where

$$G_k = \begin{bmatrix} x_1 & q_1 & 0 & \ldots & 0 \\ -1 & x_2 & q_2 & \ldots & 0 \\ 0 & -1 & x_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_k \end{bmatrix}.$$

These E_i^k are quantum analogues of the elementary symmetric polynomials e_i^k's which are defined by

$$e_i^k := \sum_{1 \leq a_1 < \cdots < a_i \leq k} x_{a_1} \cdots x_{a_i}.$$

Indeed, specializing all the q variables to 0 turns E_i^k into e_i^k. Fomin, Gelfand and Postnikov [3] originally defined quantum Schubert polynomials by passing the Schubert polynomials through a quantization map that sends e_i^k to E_i^k. There is, however, a way to define the quantum double Schubert polynomials via divided difference operators [5]. For the longest permutation w_0, we have

$$\mathfrak{S}_{w_0}^q(x,y) := \prod_{k=1}^{n-1} E_k^k(x_1 - y_{n-k}, \ldots, x_k - y_{n-k}), \tag{2.2}$$

and, for any permutation w, we have

$$\mathfrak{S}_w^q(x,y) = (-1)^{\ell(w_0)-\ell(w)} \partial_{ww_0}^y \mathfrak{S}_{w_0}^q(x,y). \tag{2.3}$$

Specializing the q variables to 0 recovers the double Schubert polynomials while specializing the y variables to 0 recovers the quantum Schubert polynomials. Setting both y and q variables to 0 recovers the Schubert polynomials.

Theorem 2.1 (Monk's rule for quantum double Schubert polynomials [10])**.** *For any k and any permutation w,*

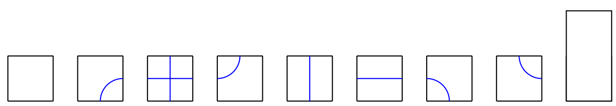
$$\mathfrak{S}^q_{s_k}(x,y)\mathfrak{S}^q_w(x,y) = \sum_{\substack{a\le k<b,\\ \ell(wt_{ab})=\ell(w)+1}} \mathfrak{S}^q_{wt_{ab}}(x,y) + \sum_{\substack{c\le k<d,\\ \ell(wt_{cd})=\ell(w)-\ell(t_{cd})}} q_{cd}\mathfrak{S}^q_{wt_{cd}}(x,y)$$
$$+ \sum_{i=1}^{k}(y_{w(i)} - y_i)\mathfrak{S}^q_w(x,y),$$

where $q_{cd} := q_c q_{c+1}\dots q_{d-1}$.

3 Quantum bumpless pipe dreams

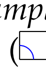
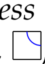
We will now define *quantum bumpless pipe dreams*, the central objects of this paper.

Definition 3.1. A *quantum bumpless pipe dream* (QBPD) is a tiling of an $n \times n$ grid filled with tiles



so that

- the tiling forms n pipes;
- each pipe starts horizontally at the right edge of the grid and ends vertically at the bottom edge of the grid;
- the pipes only move upward, downward, or leftward (but not rightward) when moving from the right edge to the bottom edge;
- no two pipes cross more than once.

The last tile is a 2×1 domino tile, which occupies two vertically adjacent empty squares in the grid. A (non-quantum) *bumpless pipe dream* (BPD) (as defined in [9]), is a QBPD in which the last three tiles above (, , and the domino tile) do not appear, so in a BPD pipes only move downward and leftward.

Example 3.2. *Figure 2 shows a non-example of a QBPD. The pipe starting on row 3 moves rightward in the tiles $(2,2)$ and $(2,3)$, which violates Definition 3.1. See Figure 3 for examples of valid QBPD.*

Definition 3.3. The *binomial weight* for a QBPD P, denoted $\mathrm{bwt}(P)$, is the product of weights contributed by the following tiles:

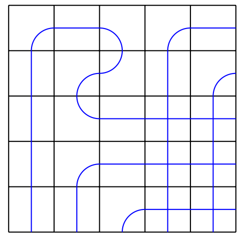


Figure 2: A non-example of a QBPD

- An empty tile on row i and column j contributes $x_i - y_j$;
- A domino tile whose upper cell is on row i contributes q_i;
- A cross tile on row i where the vertical strand moves upwards contributes q_i;
- A southwest elbow on row i contributes $-q_i$;
- A vertical tile on row i where the strand moves upward contributes $-q_i$.

In other words, let $P(i,j)$ denote the cell on row i and column j, and let

$$\begin{aligned} E(P) &:= \{(i,j) : P(i,j) \text{ is a single empty cell}\}, \\ Q(P) &:= \{(i,j) : P(i,j) \text{ is the upper cell of a domino or a tile} \\ &\qquad \text{in which the vertical strand moves upward}\}, \\ NQ(P) &:= \{(i,j) : P(i,j) \text{ is a tile or a tile} \\ &\qquad \text{in which the strand moves upward}\}. \end{aligned}$$

Then,

$$\begin{aligned} \operatorname{bwt}(P) &:= \prod_{(i,j)\in E(P)} (x_i - y_j) \prod_{(i,j)\in Q(P)} q_i \prod_{(i,j)\in NQ(P)} (-q_i) \\ &= (-1)^{|NQ(P)|} \prod_{(i,j)\in E(P)} (x_i - y_j) \prod_{(i,j)\in Q(P)\cup NQ(P)} q_i \,. \end{aligned}$$

A QBPD P is said to be associated with a permutation w if the pipe starting on the right on row i ends up on column $w(i)$.

Let QBPD(w) denote the set of QBPDs associated to w. Our main result is the following:

Theorem 3.4. *The quantum double Schubert polynomial indexed by $w \in S_n$ is the sum of binomial weights of all QBPDs associated to w:*

$$\mathfrak{S}^q_w(x,y) = \sum_{P\in \mathrm{QBPD}(w)} \operatorname{bwt}(P)\,.$$

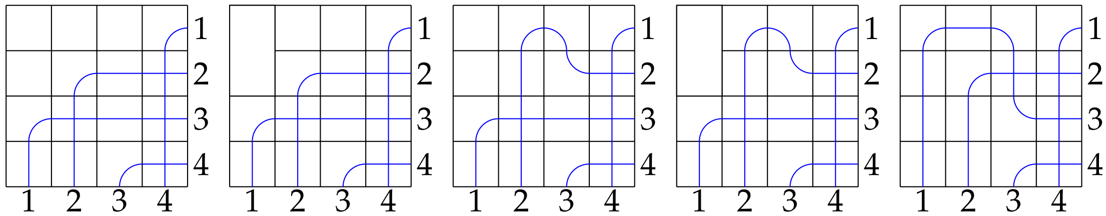


Figure 3: QBPDs for 4213

Example 3.5. *From Figure 3, we have*

$$\mathfrak{S}^q_{4213}(x,y) = (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1) + q_1(x_1-y_2)(x_1-y_3) \\ + (x_1-y_1)(x_2-y_1)(-q_1) + q_1(-q_1) + (-q_1)q_2\,.$$

As a corollary, we have the following formula.

Corollary 3.6. *The quantum Schubert polynomial indexed by $w \in S_n$ is the sum of monomial weights of all QBPDs associated to w:*

$$\mathfrak{S}^q_w(x) = \sum_{P \in \mathrm{QBPD}(w)} \mathrm{wt}(P)$$

where

$$\mathrm{wt}(P) := \prod_{(i,j)\in E(P)} x_i \prod_{(i,j)\in Q(P)} q_i \prod_{(i,j)\in NQ(P)} (-q_i)\,.$$

3.1 Droop moves and lift moves

Droop moves on bumpless pipe dreams are defined in [9]. They are moves of the form illustrated in Figure 4. Droop moves can be extended to the QBPD setting. We allow a droop move only if the result is a valid and reduced QBPD.

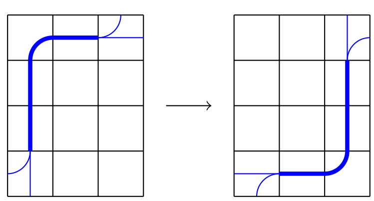


Figure 4: A droop move (light color indicates possibilities)

In [9], Lam, Lee, and Shimozono proved that any bumpless pipe dream of a given permutation can be obtained from the Rothe diagram by a sequence of droops. To generate all QBPDs, we introduce other moves called *lift moves*.

The lift moves are moves in which a horizontal segment of a strand is "lifted up" into a detour that goes up, moves left, and then moves back down. Figure 5 shows an example of a lift move. There might be other unpictured pipes in the picture as long as the result is a valid and reduced QBPD.

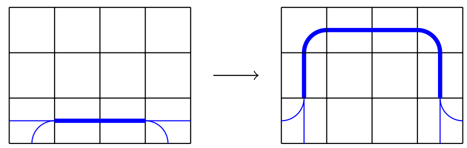


Figure 5: A lift move (light color indicates possibilities)

A QBPD is said to be *unpaired* if it has no domino tile. To generate all QBPDs, we can generate all unpaired QBPDs and find all ways to pair empty boxes into dominos. Note that droops moves and lift moves preserve the permutation associated to the QBPD.

Lemma 3.7. *All unpaired QBPDs can be generated from the Rothe diagram using a sequence of droop and lift moves.*

3.2 Stability

Given a QBPD of $w \in S_n$, we can think of w as being in S_{n+1}. In terms of QBPDs, we can extend an $n \times n$ QBPD P to a $(n+1) \times (n+1)$ QBPD as in Figure 6.

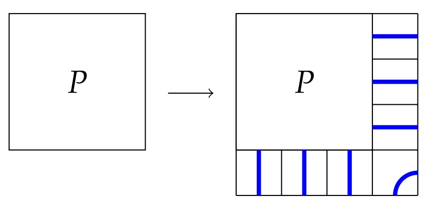


Figure 6: Extending a $n \times n$ QBPD to a $(n+1) \times (n+1)$ QBPD

The reverse is also possible:

Lemma 3.8. *Given $w \in S_{n+1}$ such that $w(n+1) = n+1$, for any $(n+1) \times (n+1)$ QBPD of w, we can restrict it to an $n \times n$ QBPD of w restricted to S_n.*

3.3 Overview of the proof of Theorem 3.4

We prove Theorem 3.4 by showing the quantity $\sum_{P \in \mathrm{QBPD}(w)} \mathrm{bwt}(P)$ satisfies a defining transition equation of the quantum double Schubert polynomials obtained by taking appropriate special cases of Theorem 2.1. For example, the transition equation for 3241 is as follows:$\mathfrak{S}_{3241} = (x_3 - y_1)\mathfrak{S}_{3214} + q_2\mathfrak{S}_{3124} + q_1q_2\mathfrak{S}_{1234}$.

In general, the transition equation has four families of terms on the right hand side. For details see [12, Proposition 3.14]. We show the quantity satisfies the transition equation by splitting the QBPDs of the permutations on the left hand side into four sets, and constructing four bijections between those four sets with the four families on the right hand side. However, portions of the last the family on the right hand side will remain, but they cancel out using another family of bijections. These bijections are constructed for permutations π with $\pi(n) \neq n$ which we can assume without loss of generality by Lemma 3.8.

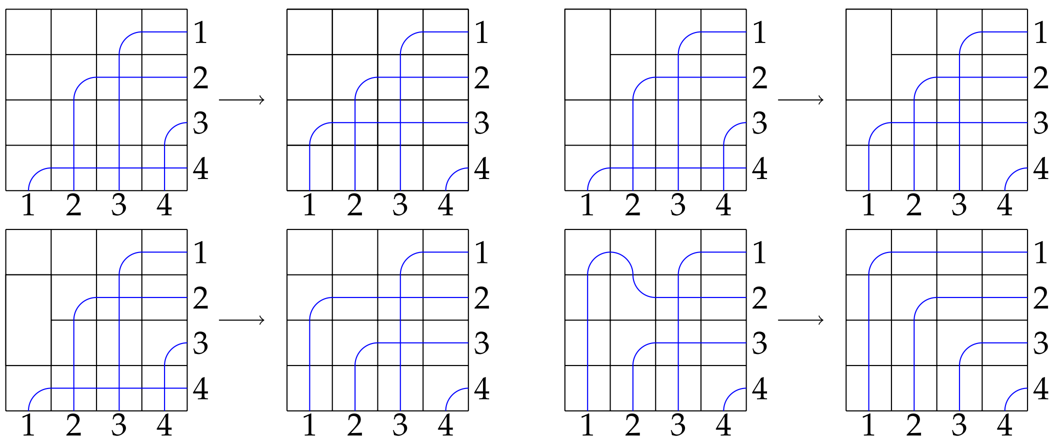


Figure 7: Bijections for the transition equations of 3241.

For example, the bijections for 3241 are illustrated in Figure 7. The top row (top left and top right correspondence) is one bijection that sends two QBPDs of 3241 to the two QBPDs of 3214 which corresponds to the $(x_3 - y_1)\mathfrak{S}_{3214}$ term on the right hand side. The bottom left correspondence is one bijection that sends the one remaining QBPD of 3241 to one QBPD of 3124, so the remaining part of the left hand side corresponds to part of the $q_2\mathfrak{S}_{3124}$ term on the right hand side. Finally, the bottom right correspondence sends the one remaining QBPD of 3124 to the one QBPD of 1234, which means the remaining parts of the $q_2\mathfrak{S}_{3124}$ term on the right hand side cancel outs with the $q_1q_2\mathfrak{S}_{1234}$ term.

In general, these bijections are more complicated, as some cases do not show up in the example above. The full details of these bijections can be found in [12, Section 3.3]. By showing how these bijections change the binomial weight, we establish that the quantum bumpless pipe dreams formula satisfies the transition equation and thus establish Theorem 3.4.

4 Cancellation Analysis

There is no cancellation in the bumpless pipe dream formula for the double Schubert polynomials. Quantum bumpless pipe dreams provide a combinatorial formula for the monomial expansion of the quantum double Schubert polynomial, but this formula is not cancellation-free. In this section, we give some analysis of how much and what kind of cancellation occurs.

Table 1 lists all the permutations in S_4 for which the QBPD formula gives cancellation when computing the quantum double Schubert polynomials.

Permutation	Monomials	QBPD Monomials	Cancellations	Number of QBPDs
$[4,1,3,2]$	50	54	2	9
$[3,1,4,2]$	18	20	1	4
$[1,4,3,2]$	46	48	1	9
$[2,1,4,3]$	12	14	1	5

Table 1: Nonzero cancellations for QBPDs in S_4, when considering the generated quantum double Schubert polynomial.

Example 4.1. *Permutation* 615432 *has* 97032 *monomials in its quantum double Schubert polynomial, while the total number of monomials generated from QBPDs is* 140052. 21510 *pairs of monomials cancel out, and the number of QBPDs is* 1038. *This is the permutation with the most cancellation in* S_6.

	Total	Average per Permutation	Permutation of Max	Max
S_3	0	–	–	–
S_4	5	0.208	$[4,1,3,2]$	2
S_5	1350	11.25	$[5,1,4,3,2]$	153
S_6	570549	792.43	$[6,1,5,4,3,2]$	21510

Table 2: Cancellations in S_n for $n = 3,4,5,6$

As shown in Table 2, the number of cancellations per permutation grows larger with greater n. In particular, we can observe that there are several ways that cancellations occur. Two QBPDs could completely cancel each other out in both the single and double quantum Schubert polynomial case, as in Figure 8, or they could partially cancel. In the case of partial cancellation, Figure 9 illustrates the cancellation of binomial terms and no cancellation in monomial terms, while Figure 10 illustrates the cancellation of monomial terms but not binomial terms.

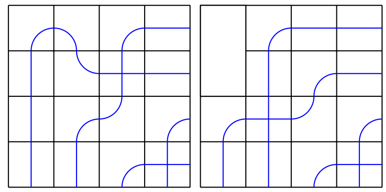


Figure 8: Two QBPDs for 2143 whose binomial weights cancel each other out completely. The left contributes $-q_1$, and the right contributes q_1.

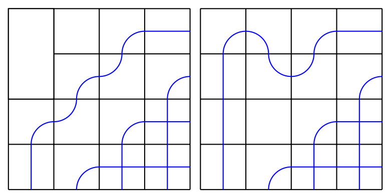


Figure 9: Two QBPDs for 1432 whose monomial weights do not cancel out, but binomial weights partially cancel out. The left QBPD contributes $x_1 q_1 - y_2 q_1$, and the right QBPD contributes $-x_3 q_1 + y_2 q_1$.

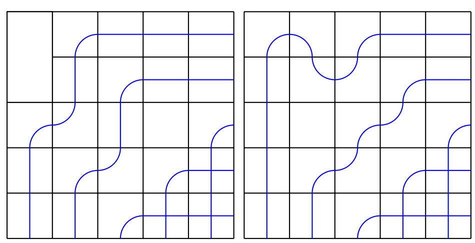


Figure 10: Two QBPDs for 12543 whose monomial weights cancel each other out, but binomial weights do not cancel completely. The left QBPD contributes $x_3 q_1 - y_4 q_1$, and the right contributes $-x_3 q_1 + y_2 q_1$.

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