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The Integer Decomposition Property in Smooth Polytopes

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Abstract. Tadao Oda conjectured that every smooth polytope, in any dimension, has the Integer Decomposition Property. In this paper, we show this result for subclasses of polytopes: smooth combinatorial cubes of any dimension and 3-dimensional pseudo-prisms.

Keywords: polytope, smooth, IDP

1 Introduction

For lattice polytopes P and Q, we say that (P, Q) has the *Integer Decomposition Property*, or that it *is IDP*, if every lattice point in the Minkowski sum $P + Q = \{p + q : p \in P, q \in Q\}$ can be written as the sum of a lattice point in P and a lattice point in Q. For a single polytope P, we say that P is *IDP* when (P, kP) is IDP for all positive integers k. IDP polytopes are directly related to Ehrhart theory, and are of great interest in commutative algebra and the study of toric varieties, as well as being of use in integer programming. In general, it is an open question to characterize when a polytope is IDP.

While being IDP is a global property of a polytope, we are interested in its relationship to the more local notion of smoothness. First, a d-dimensional polytope is *simple* if each vertex is contained in exactly d edges (and so also exactly d facets). We define the *primitive edge directions* of a vertex to be the smallest lattice directions along all its adjacent edges, and then say that a d-dimensional polytope is *smooth* if it is simple and if the primitive edge directions at each vertex form a basis for the integer lattice $\mathbb{Z}^d$.

In 1997, Tadao Oda made the following conjecture, documented in [13], which remains unproven.

Conjecture 1.1 (Oda's Conjecture)**.** All smooth polytopes are IDP.

This problem is motivated by its relationship to the study of toric varieties, as there is a correspondence between smooth polytopes and ample divisors of smooth toric varieties. Suppose that X_Σ is a smooth projective toric variety, and that $\mathcal{L}$ is an ample line

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bundle on it. Oda's Conjecture is equivalent to the statement that the embedding of X_Σ given by $\mathcal{L}$ is projectively normal, or in other words, that the multiplication map

$$H^0(X_\Sigma, \mathcal{L}) \otimes ... \otimes H^0(X_\Sigma, \mathcal{L}) \to H^0(X_\Sigma, \mathcal{L}^{\otimes k}),$$

is surjective ([12]).

The interest in the consequences of smoothness is not limited to the IDP, but includes stronger properties such as the existence of a unimodular covering or triangulation. Indeed, there is a hierarchy of properties, cataloged in [9], of which the IDP is the weakest. And despite attracting considerable interest, including as the subject of an Oberwolfach mini-workshop in 2007, Oda's conjecture remains open, even in three dimensions. As such, even partial or computational results relating to any such properties are of interest, as in [6, 5, 11, 2, 7].

In particular, recent progress was made towards proving the conjecture in [1], where Beck et al. showed that 3-dimensional, centrally symmetric, smooth polytopes are IDP. In this paper, we contribute other sufficient conditions for smooth polytopes to be IDP.

We define a d-dimensional *combinatorial cube* to be a polytope whose face poset is in bijection with the face poset of the unit cube, $[0,1]^d$. Then, we prove the following.

Theorem 4.8. Smooth combinatorial cubes of any dimension are IDP.

In the process of proving this, we also define a *pseudo-prism*, a more general class of polytopes with specific characteristics reminiscent of a prism, and show the following.

Theorem 4.6. Every smooth, 3-dimensional pseudo-prism is IDP.

2 Preliminaries

2.1 Properties of smooth and IDP polytopes

In this paper, we will explore the structure imposed on polytopes when they are smooth; in particular, we will be concerned with whether two faces of a polytope are parallel. In general, every k-face F of a polytope lies in a unique k-dimensional affine subspace, which we denote with $\mathrm{aff}(F)$. We use $\mathrm{lin}(F)$ to denote the linear subspace parallel to $\mathrm{aff}(F)$, and then two faces F and G of a polytope are *parallel* when $\mathrm{lin}(F) = \mathrm{lin}(G)$.

This additional structure that we find in certain classes of smooth polytopes will allow us to consider the integer decomposition property. We will employ the following well-known facts.

Proposition 2.1. *Basic IDP properties:*

(a) Let P be a d-dimensional polytope. Then, (P, kP) is IDP for all integers $k \geq d-1$.

(b) All polygons are IDP.

Proof. A proof of (a) can be found in [4, Theorem 2.2.12], and was originally proved in [3]. The result of (b) follows from (a), or alternatively is shown in [10]. □

We also have the following useful characterization of being IDP, which we use repeatedly.

Proposition 2.2 (IDP Equivalence)**.** *Let P and Q be polytopes. For a lattice point $a \in \mathbb{Z}^d$, define*

$$R_a = P \cap (a + (-Q)).$$

Then, (P, Q) is IDP if and only if for all a, R_a contains a lattice point whenever it is non-empty.

In this paper, we rely heavily on the use of *unimodular transformations*, which are linear maps $\mathbb{R}^d \to \mathbb{R}^d$ which send the lattice $\mathbb{Z}^d$ to itself. Equivalently, a linear transformation is unimodular if and only if the determinant of the matrix representing it is ± 1. Importantly, for every lattice basis, there exists a unimodular transformation which sends it to the standard basis, and follows that unimodular transformations preserve the IDP.

It will be relevant to consider when two polytopes are *Minkowski equivalent*, and we will use the following characterization of this property.

Proposition 2.3. *Polytopes P and Q are Minkowski equivalent if and only if there is a bijection between their face posets such that all corresponding facets are parallel to each other.*

Since linear transformations preserve the parallelism of affine subspaces, it follows that they also preserve Minkowski equivalence. We will particularly apply this fact to unimodular transformations.

There are many geometric implications of being Minkowski equivalent; in particular, we have the following.

Lemma 2.4. *Suppose polytopes P and $-Q$ in $\mathbb{R}^d$ are Minkowski equivalent. If P and Q are disjoint, then there is a hyperplane which separates them that is parallel to a facet of P.*

2.2 Combinatorial cubes

Consider C, an arbitrary smooth d-dimensional combinatorial cube. Importantly, each face of a combinatorial cube is itself a combinatorial cube of smaller dimension.

In order to discuss the faces of C, we will employ the bijection between the face posets of C and the d-dimensional unit cube, $[0,1]^d$. We can label the faces of the unit cube as the F_I^J, where I and J are every pair of disjoint subsets of $[d]$. Then we define

$$F_I^J := \left\{ (x_1, x_2, ..., x_d) \in [0,1]^d : x_k = \begin{cases} 0 & \text{if } k \in I \\ 1 & \text{if } k \in J \end{cases} \right\},$$

so that F_I^J is a $(d - (|I| + |J|))$-dimensional face of the unit cube. Via the bijection, we reuse this face labeling for the *faces of* C, so that the F_I^J uniquely identify every face of C.

With this notation, we can also easily recover the following relationships between the faces of a cube.

Proposition 2.5. *Basic properties of faces of cubes.*

(a) The face $F_{I_1}^{J_1}$ contains the face $F_{I_2}^{J_2}$ if and only if $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$.

(b) The intersection of two faces $F_{I_1}^{J_1}$ and $F_{I_2}^{J_2}$ is the face $F_{I_1 \cup I_2}^{J_1 \cup J_2}$.

However, when it is appropriate, we prefer a more concise notation for F_I^J: we denote elements of J with a bar, rather than using the superscript. Instead of writing $F_{\{y\}}^{\{x,z\}}$, we may write $F_{\bar{x}y\bar{z}}$.

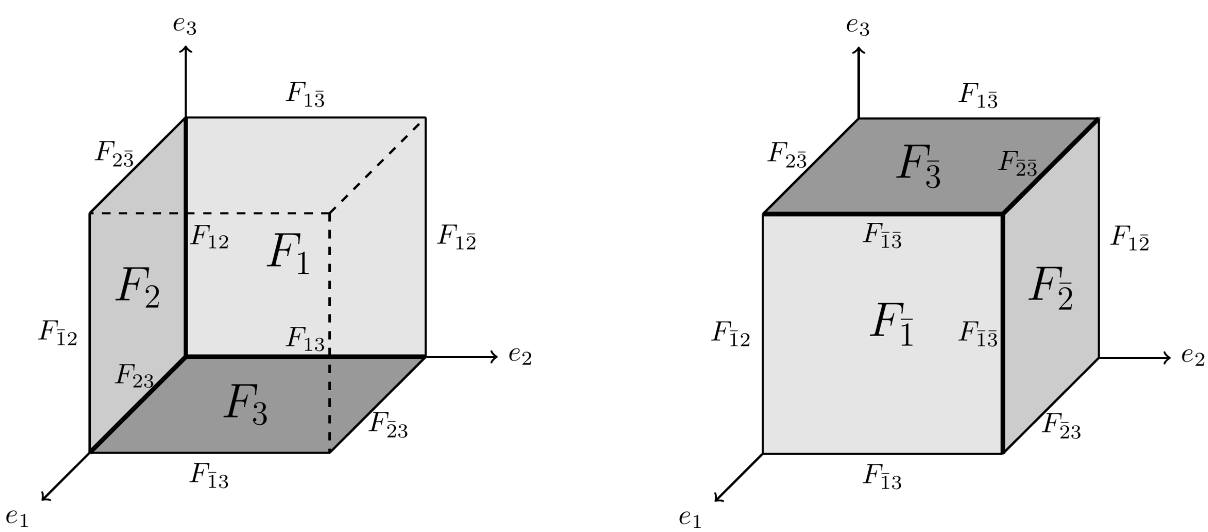


Figure 1: The 3-dimensional unit cube, viewed from the 'inside' and 'outside.'

For example, as in Figure 1, when C is 3-dimensional the six (2-dimensional) facets of C are F_1, $F_{\bar{1}}$, F_2, $F_{\bar{2}}$, F_3, $F_{\bar{3}}$. As in Proposition 2.5, F_1 contains the 1-dimensional faces $F_{12}, F_{13}, F_{1\bar{2}}, F_{1\bar{3}}$, and the intersection of the 2-dimensional faces F_1 and $F_{\bar{2}}$ is the 1-dimensional face $F_{1\bar{2}}$.

In cubes of all dimensions there is a notion of opposite facets, and our notation indicates this: We say that two facets F_x and $F_{\bar{x}}$ of C are *opposite* each other. Since faces of combinatorial cubes are themselves cubes, we have that F_{xy} and $F_{x\bar{y}}$ are *opposite* each other within F_x. In a full-dimensional cube, two facets can be parallel only if they are opposite.

As C is smooth, the primitive edge directions at each vertex of C span the integer lattice, and there always exists a unimodular transformation which sends these directions to the standard basis vectors. Thus, since unimodular transformations and translations preserve the IDP and subspace parallelism, we may assume that one corner of C lies at the origin and has primitive edge directions along the coordinate axes, $e_1, e_2, ..., e_d$. We note, however, that not all of the faces of C will inherit this placement.

3 Parallelism in combinatorial cubes

In general, a combinatorial cube may be irregular in shape, as in Figure 2.

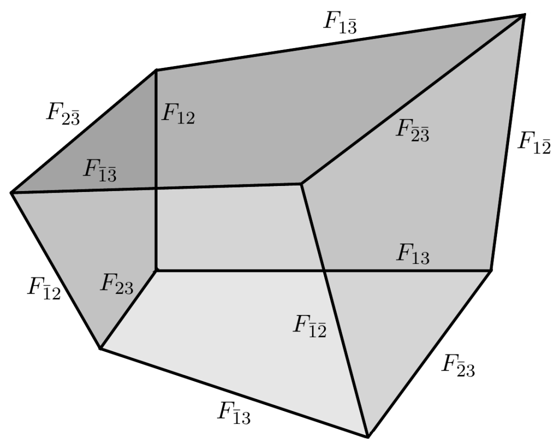


Figure 2: A 3-dimensional combinatorial cube with no parallel faces.

However, we will show that smoothness imposes restrictions on the relative positions of the facets of a cube, in the following theorem.

Theorem 3.1. *Let C be a d-dimensional smooth combinatorial cube in $\mathbb{R}^d$ with $d \geq 2$. Then, C has two parallel facets.*

To build intuition, let us observe the consequences of assuming smoothness of a cube in two dimensions: Suppose that C is a smooth quadrilateral.

As above, we can take one corner of C to be at the origin, with primitive edge directions $\{(1, 0), (0, 1)\}$. Then, since C is smooth, the primitive edge directions at each of the four vertices must span the integer lattice $\mathbb{Z}^2$, so they must be of the form given in Figure 3, for some integers m and n.

Then, since $\{(-1, -n), (-m, -1)\}$ must span $\mathbb{Z}^2$, it must be that

$$\det \begin{pmatrix} -1 & -m \\ -n & -1 \end{pmatrix} = \pm 1.$$

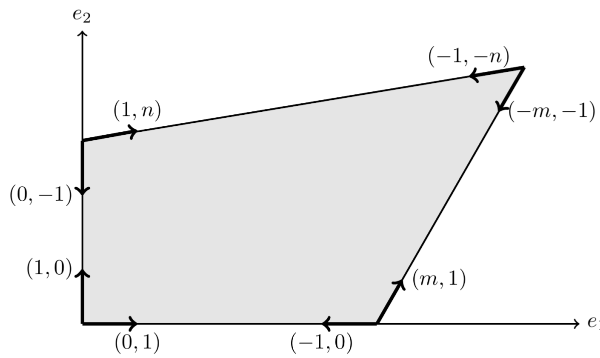


Figure 3: 2-dimensional cube, with primitive edge directions at each vertex.

It follows easily that one of n, m must be 0, resulting in either vertical or horizontal edges which are parallel.

The higher dimensional result follows inductively, with an examination by cases shown in three dimensions in Figure 4.

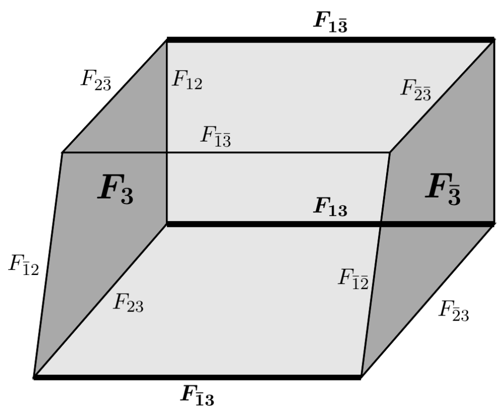
4 IDP in prisms, pseudo-prisms, and cubes

In the process of showing that combinatorial cubes of every dimension are IDP, we produce several intermediate results about a more general class of polytopes.

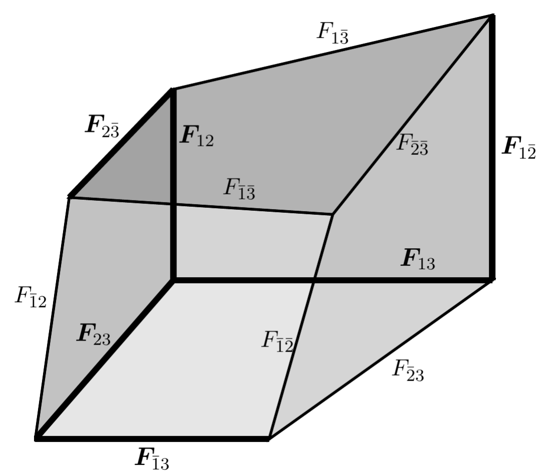
A d-dimensional *prism* is a polytope which is affinely equivalent to a polytope $Q \times [0,1]$ for some $(d-1)$-dimensional polytope Q. This yields *top* and *bottom* facets, $Q \times \{1\}$ and $Q \times \{0\}$, respectively, which are parallel to each other. Thus, there are edges between the corresponding vertices of the top and bottom, and so all other facets of a prism, the *side* facets, are themselves $(d-1)$-dimensional prisms, whose tops and bottoms are $(d-2)$-faces of the top and bottom of the original prism.

We define a *pseudo-prism* to be a polytope whose face lattice is isomorphic to that of a prism and whose corresponding top and bottom facets are parallel. Observe that this means that the top and bottom faces of a pseudo-prism have the same face poset, so in particular have the same number of vertices. Further note that by Theorem 3.1, every d-dimensional smooth combinatorial cube is a pseudo-prism, where Q is a $(d-1)$-dimensional smooth combinatorial cube.

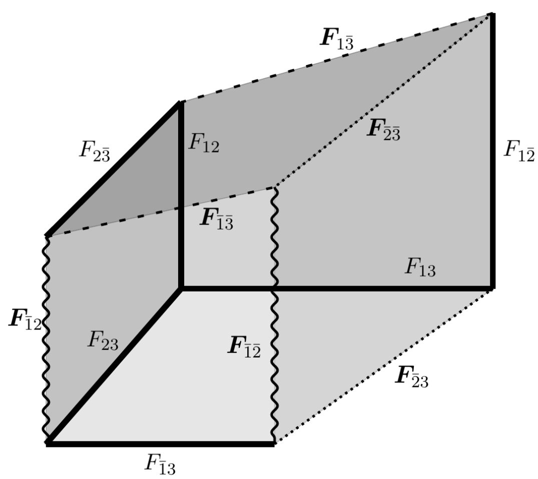
It is convenient to visualize pseudo-prisms whose top and bottom facets are parallel to the coordinate plane $\{(x_1, ..., x_d) : x_d = 0\}$. Again, since unimodular transformations and translations preserve the IDP, Minkowski equivalence, and subspace parallelism, we may assume that pseudo-prisms have tops and bottoms parallel to this coordinate plane, and further that smooth pseudo-prisms have one vertex at the origin, with primitive edge directions $e_1, e_2, ..., e_d$.



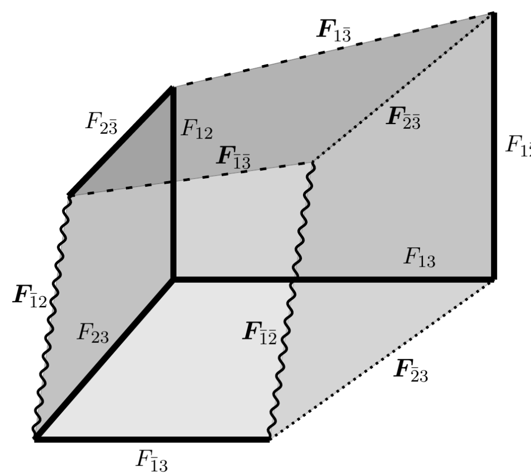
(A) The faces $F_{1\bar{3}}$, F_{13}, and $F_{\bar{1}3}$ are all parallel to each other.



(B) The pairs of faces $(F_{12}, F_{1\bar{2}})$, $(F_{13}, F_{1\bar{3}})$, and $(F_{23}, F_{2\bar{3}})$ are parallel.



(C) The pairs of faces $(F_{\bar{1}2}, F_{\bar{1}\bar{2}})$, $(F_{1\bar{3}}, F_{\bar{1}\bar{3}})$, and $(F_{\bar{2}3}, F_{\bar{2}\bar{3}})$ are also parallel, and $\text{lin}(F_{\bar{1}2}) = \text{span}(e_3)$.



(D) The pairs of faces $(F_{\bar{1}2}, F_{\bar{1}\bar{2}})$, $(F_{1\bar{3}}, F_{\bar{1}\bar{3}})$, and $(F_{\bar{2}3}, F_{\bar{2}\bar{3}})$ are also parallel, and $\text{lin}(F_{\bar{1}2}) \neq \text{span}(e_3)$.

Figure 4: A 3-dimensional combinatorial cube, with various parallel faces.

It is clear that the top and bottom facets of d-dimensional prisms are translations of the same $(d-1)$-dimensional polytope. While this is not true in general for pseudo-prisms, there is a strong relationship between their top and bottom facets, which we give in the following lemma.

Lemma 4.1. *The top and bottom of a pseudo-prism are Minkowski equivalent.*

Supposing that a pseudo-prism P has bottom which lies in $\{(x_1, ..., x_d) : x_d = b\}$ and top which lies in $\{(x_1, ..., x_d) : x_d = b + h\}$ for integers b and h, we define *slices* S_l of P as the nonempty intersections

$$S_l = P \cap \{(x_1, ..., x_d) : x_d = b + l\}$$

for heights $l \in [h]$. It is not hard to show that each slice of a smooth pseudo-prism has only integer vertices, and the following lemma is a natural consequence.

Lemma 4.2. *If P is a smooth pseudo-prism of dimension d, then every slice of P is an integer polytope of dimension $d-1$ and is Minkowski equivalent to its bottom (and so also its top).*

Oda's Conjecture has been proved in only two dimensions. However, another two dimensional result that doesn't require smoothness was proved by Hasse et al. in 2007.

Theorem 4.3 ([8]). *Let P and P' be Minkowski equivalent lattice polygons. Then, (P, P') is IDP.*

An analogous result for dimensions higher than two is known to be untrue in many cases, for instance, any pair (P, kP) where P is not IDP, such as the Reeve tetrahedron. However, the structure of a pseudo-prism lends itself to the following theorem.

Theorem 4.4. *Let P and P' be Minkowski equivalent, smooth pseudo-prisms which are d-dimensional. Let $\{S_l\}$ and $\{S'_m\}$ be the slices of P and P' respectively, and suppose that for every l and m, the pair (S_l, S'_m) is IDP. Then, (P, P') is IDP.*

Proof. Assume without loss of generality that the bottom B and top T of P lie in the respective hyperplanes $\{(x_1, ..., x_d) : x_d = 0\}$ and $\{(x_1, ..., x_d) : x_d = t\}$ for some integer t, while the bottom B' and top T' of P' lie in the hyperplanes $\{(x_1, ..., x_d) : x_d = b'\}$ and $\{(x_1, ..., x_d) : x_d = t'\}$ for integers $b' > t'$, respectively.

To show that (P, P') is IDP, we will use Proposition 2.2. Suppose that $R_a = P \cap (a + (-P'))$ is non-empty for some point $a = (a_1, ..., a_d) \in \mathbb{Z}^d$.

We see that $a + (-P')$ has top $a + (-B)$ which is contained in $\{(x_1, ..., x_d) : x_d = a_d - b'\}$ and bottom $a + (-T)$ which is contained in $\{(x_1, ..., x_d) : x_d = a_d - t'\}$. All of $a + (-P')$ lies between these two hyperplanes, and all of P lies between $\{(x_1, ..., x_d) : x_d = 0\}$ and $\{(x_1, ..., x_d) : x_d = t\}$.

In order for R_a to be non-empty, the top or bottom of at least one of P or $a + (-P')$ lies in a hyperplane which is vertically between the top and bottom hyperplanes of the other. So, suppose without loss of generality that B lies between the top and bottom hyperplanes of $a + (-P')$. This means that there is a slice S'_m of P' such that the slice $G := a + (-S'_m)$ of $a + (-P')$ lies in the hyperplane $\{(x_1, ..., x_d) : x_d = 0\}$.

Suppose towards contradiction that B itself does not intersect G. As B and G are both $(d-1)$-dimensional polytopes, consider them only in the subspace $\{(x_1, ..., x_d) : x_d = 0\}$. Since they are Minkowski equivalent, Lemma 2.4 gives that there is a $(d-2)$-dimensional hyperplane $\widetilde{H} \subseteq \{(x_1, ..., x_d) : x_d = 0\}$ which separates B and G. Further, this Proposition gives us that $\widetilde{H}$ is parallel to a $(d-2)$-face $\widetilde{B}$ of B and so is also parallel to the corresponding face $\widetilde{G}$ of G.

Since P is a pseudo-prism and $\widetilde{B}$ is a face of B, we know that P has a facet F such that $\widetilde{B} = B \cap F$, a $(d-2)$-face of P. As P' is Minkowski equivalent to P, it has corresponding facet F' which is parallel to F, and $\widetilde{G} = G \cap (a + (-F'))$.

In particular, we have that F and $a + (-F')$ are parallel and non-equal since $\widetilde{B}$ and $\widetilde{G}$ are disjoint. Thus, there is a hyperplane H which is parallel to both F and $a + (-F')$ which contains $\widetilde{H}$. However, this H must separate C and $a + (-P')$, contradicting our supposition that R_a is non-empty. Therefore, it must be that B intersects G, as in Figure 5.

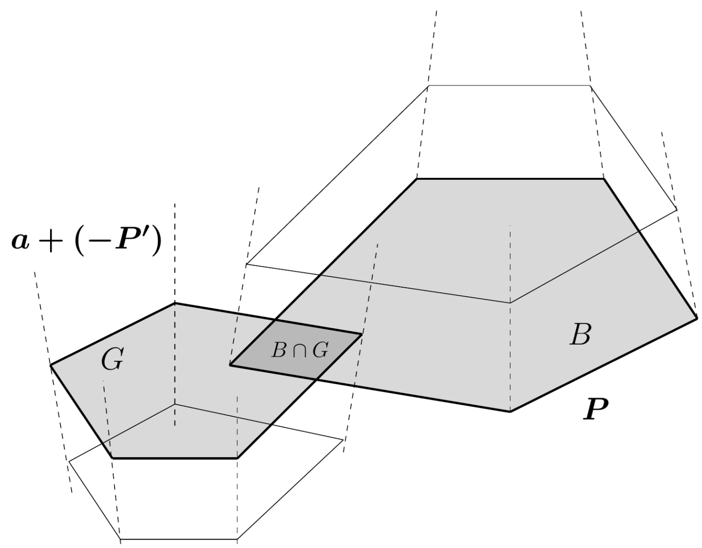


Figure 5: Slices B of P and G of $a + (-P')$.

By assumption, (B, S'_m) is IDP. By Proposition 2.2, since $B \cap G = B \cap (a + (-S'_m))$ is nonempty, it must contain a lattice point. Thus, R_a contains a lattice point, so we conclude that (P, P') is IDP. $\square$

By employing Theorem 4.3, we satisfy the required hypotheses of Theorem 4.4 when we restrict to 3-dimensional smooth pseudo-prisms, and we can conclude the following.

Theorem 4.5. *Let P and P' be Minkowski equivalent, smooth, 3-dimensional pseudo-prisms. Then, (P, P') is IDP.*

Theorem 4.6. *Every smooth, 3-dimensional pseudo-prism is IDP.*

Combinatorial cubes are highly structured polytopes, and as such are conducive to induction. This allows us to satisfy the hypotheses of Theorem 4.4 in any dimension, yielding the following results.

Theorem 4.7. *Suppose that C and C' are Minkowski equivalent smooth cubes of dimension $d \geq 2$. Then, (C, C') is IDP.*

Theorem 4.8. *Smooth combinatorial cubes of any dimension are IDP.*

5 Future work

Smooth polytopes have only been shown to be IDP under strong additional assumptions; in its full generality, Oda's Conjecture remains open. Following these results, it is natural to try to relax the structure of pseudo-prisms and examine polytopes with some weaker characteristic involving parallelism, or to more deeply explore the geometric consequences of smoothness. Alternatively, it is interesting to explore which assumption in [1], 3-dimensional or centrally symmetric, could be shown to be superfluous. Others have classified all smooth, 3-dimensional polytopes which have small numbers of internal lattice points: In [11], Lundman expanded on the work in [2], leaving open ample opportunity for exploration. There are stronger properties than the IDP, e.g. the existence of a unimodular triangulation, which might be associated to these or other classes of polytopes.

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