Some multivariate master polynomials for permutations, set partitions, and perfect matchings, and their continued fractions [a]

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[a] Based on joint work with Alan Sokal

Plan of the talk

Introduction

If $(a_n)_{n\geq 0}$ is a sequence of combinatorial numbers or polynomials with $a_0 = 1$, it is often fruitful to seek to express its ordinary generating function as a continued fraction of either Stieltjes (S) type,

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}},$$

or Jacobi (J) type,

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \cdots}}}.$$

Both sides are interpreted as formal power series in t.

Contraction formulae of an S-fraction to a J-fraction

$$\cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{\cdots}}} = \cfrac{1}{1-\alpha_1 x-\cfrac{\alpha_1\alpha_2 x^2}{1-(\alpha_2+\alpha_3)x-\cfrac{\alpha_3\alpha_4 x^2}{\cdots}}}.$$

i.e., the above S-fraction and J-fraction are equal if

$$\begin{aligned}
\gamma_0 &= \alpha_1 \\
\gamma_n &= \alpha_{2n} + \alpha_{2n+1} \quad \text{for} \quad n \geq 1 \\
\beta_n &= \alpha_{2n-1}\alpha_{2n}.
\end{aligned}$$

This line of investigation, i.e.

$$(a_n) \mapsto (\alpha_n) \ (or \ ((\gamma_n), (\beta_n))),$$

goes back at least to Euler, but it gained impetus following Flajolet's seminal discovery that any S-type (resp. J-type) continued fraction can be interpreted combinatorially as a generating function of Dyck (resp. Motzkin) paths with suitable weights for each rise and fall (resp. each rise, fall and level step).

Our approach will be (in part) to run this program in reverse: we start from a continued fraction in which the coefficients α (or γ and β) contain indeterminates in a nice pattern, and we attempt to find a combinatorial interpretation for the resulting polynomials a_n – namely, as enumerating permutations, set partitions or perfect matchings to some natural maultivariate statistics.
We call our a_n "master polynomials" because our CF will contain the maximum number of independent inderterminates consistent with the given pattern.

Permutations: S-fraction

Euler

$$\sum_{n\geq 0} n!\, x^n = \cfrac{1}{1 - \cfrac{1\, x}{1 - \cfrac{1\, x}{1 - \cfrac{2\, x}{1 - \cfrac{2\, x}{\cdots}}}}}$$

$$= \cfrac{1}{1 - x - \cfrac{1^2\, x^2}{1 - 3\, x - \cfrac{2^2\, x^2}{\cdots}}}$$

with coefficients $\alpha_{2k-1} = k$, $\alpha_{2k} = k$.

A four-variable generalization

Introduce the polynomials $P_n(x, y, u, v)$ by the following CF

$$\sum_{n\geq 0} P_n(x,y,u,v)t^n = \cfrac{1}{1 - \cfrac{x\,t}{1 - \cfrac{y\,t}{1 - \cfrac{(x+u)\,t}{1 - \cfrac{(y+v)\,t}{1 - \cdots}}}}}.$$

with coefficients

$$\alpha_{2k-1} = x + (k-1)u \quad \alpha_{2k} = y + (k-1)v.$$

Clearly $P_n(x, y, u, v)$ is a homogeneous polynomial of degree n and $P_n(1, 1, 1, 1) = n!$.

Record classification

Given a permutation $\mathfrak{S}_n$, an index $i \in [n]$ (or value $\sigma(i) \in [n]$) is called a

- *record* (rec) (or *left-to-right maximum*) if $\sigma(j) < \sigma(i)$ for all $j < i$;
- *antirecord* (arec) (or *right-to-left minimum*) if $\sigma(j) > \sigma(i)$ for all $j > i$;

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- *antirecord* (arec) (or *right-to-left minimum*) if $\sigma(j) > \sigma(i)$ for all $j > i$;

- *exclusive record* (erec) if it is a record and not also an antirecord;
- *exclusive antirecord* (earec) if it is an antirecord and not also a record;
- *record-antirecord* (rar) if it is both a record and an antirecord;
- *neither-record-antirecord* (nrar) if it is neither a record nor an antirecord.

Cycle classification

We say that an index $i \in [n]$ is a

- *cycle peak* (cpeak) if $\sigma^{-1}(i) < i > \sigma(i)$;
- *cycle valley* (cval) if $\sigma^{-1}(i) > i < \sigma(i)$;
- *cycle double rise* (cdrise) if $\sigma^{-1}(i) < i < \sigma(i)$;
- *cycle double fall* (cdfall) if $\sigma^{-1}(i) > i > \sigma(i)$;
- *fixed point* (fix) if $\sigma^{-1}(i) = i = \sigma(i)$.

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- *fixed point* (fix) if $\sigma^{-1}(i) = i = \sigma(i)$.

We denote the number of cycles, records, antirecords, ... in σ by cyc(σ), rec(σ), arec(σ), ..., respectively.
A rougher classification is that an index $i \in [n]$ (or value $\sigma(i)$) is an

- *excedance* (exc) if $\sigma(i) > i$;
- *anti-excedance* (aexc) if $\sigma(i) < i$;
- *fixed point* (fix) if $\sigma(i) = i$.

Two combinatorial interpretations

Theorem 1 (S-fraction for permutations)

The polynomials defined by the S-fraction have the combinatorial interpretations

$$P_n(x, y, u, v) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} u^{n-\mathrm{exc}(\sigma)-\mathrm{arec}(\sigma)} v^{\mathrm{exc}(\sigma)-\mathrm{erec}(\sigma)} \quad (1)$$

and

$$P_n(x, y, u, v) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{erec}(\sigma)} u^{n-\mathrm{exc}(\sigma)-\mathrm{cyc}(\sigma)} v^{\mathrm{exc}(\sigma)-\mathrm{erec}(\sigma)}. \quad (2)$$

Special cases (1)

$$P_n(x, yv, 1, v) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} v^{\mathrm{exc}(\sigma)}$$
$$= \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{erec}(\sigma)} v^{\mathrm{exc}(\sigma)}.$$

The triple statistics (arec, erec, exc) and (cyc, erec, exc) are equidistributed on $\mathfrak{S}_n$.

Special cases (2)

- The Stirling cycle polynomials

$$P_n(x,1,1,1) = \sum_{k=0}^{n} s(n,k)x^k = x(x+1)\dots(x+n-1).$$

or their homogenized version

$$P_n(x,y,y,y) = \sum_{k=0}^{n} s(n,k)x^k y^{n-k} = x(x+y)\dots(x+(n-1)y).$$

- The Eulerian polynomials

$$P_n(1,y,1,y) = A_n(y) = \sum_{k=0}^{n} A(n,k)y^k$$

or

$$P_n(x,y,x,y) = A_n(x,y) = \sum_{k=0}^{n} A(n,k)x^{n-k}y^k.$$

Special cases (3): Dumont-Kreweras 1988

The record-antirecord permutation polynomials

$$P_n(a,b,1,1) = \sum_{\sigma\in\mathfrak{S}_n} a^{\mathrm{arec}(\sigma)} b^{\mathrm{erec}(\sigma)}$$

or

$$P_n(a,b,c,c) = \sum_{\sigma\in\mathfrak{S}_n} a^{\mathrm{arec}(\sigma)} b^{\mathrm{erec}(\sigma)} c^{n-\mathrm{arec}(\sigma)-\mathrm{erec}(\sigma)}.$$

Note that

$$\sum_{n=0}^{\infty} P_n(a,b,1,1)t^n = \frac{\sum_{n\geq 0}(a)_n(b)_n t^n/n!}{\sum_{n\geq 0}(a)_n(b-1)_n t^n/n!},$$

where $(a)_n = a(a+1)\dots(a+n-1)$.

Special cases (4)

The polynomials [sequence A145879/A202992]

$$P_n(x,x,u,u) = \sum_{\sigma\in\mathfrak{S}_n} x^{n-\mathrm{nrar}(\sigma)} u^{\mathrm{nrar}(\sigma)} = \sum_{k=0}^{n} T(n,k) x^{n-k} u^k$$

where $T(n,k)$ is the number of permutations in $\mathfrak{S}_n$ having exactly k indices that are the middle point of a pattern 321 (or 123). In particular $T(n,0)$ is the number of 123-avoiding permutations, which equals the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. So the polynomials interpolate between C_n and $n!$.

Special cases (5): Narayana polynomials

$$P_n(x,y,0,0) = \sum_{\sigma\in\mathfrak{S}_n(321)} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)}$$
$$= \sum_{\sigma\in\mathfrak{S}_n(321)} x^{\mathrm{arec}(\sigma)} y^{\mathrm{exc}(\sigma)}$$
$$= \sum_{k=0}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} x^k y^{n-k}.$$

These combinatorial interpretations of Narayana numbers were found by Vella'03 and Elisalde'04.

Record and cycle classifications: First J-fraction

We have classified indices in a permutation according to their record status:
exlusive record, exclusive antirecord, record-antirecord or neither-record-antirecord;
and aslo according to their cycle status:
cycle peak, cycle valley, cycle double rise, cycle double fall or fixed point.
Applying now both classifications simultaneously, we obtain 10 disjoint categories.
Note that if an index i is an erec (resp. earec) then i must be an exc (resp. anti-excedance).

Record-cycle classifications: 10 classes

- ereccval: exclusive records that are also cycle valleys;
- erecdrise: exclusive records that are also cycle double rises;
- eareccpeak: exclusive antirecords that are also cycle peaks;
- eareccdfall: exclusive antirecords that are also cycle double falls;
- rar: record-antirecords (that are always fixed points);
- nrcpeak: neither-record-antirecords that are also cycle peaks;
- nrcval: neither-record-antirecords that are also cycle valleys;
- nrcdrise: neither-record-antirecords that are also cycle double falls;
- nrcfall: neither-record-antirecords that are also cycle falls;
- nrfix: neither-record-antirecords that are also fixed points.

First J-fraction

$$Q_n(x_1, x_2, y_1, y_2, z, u_1, u_2, v_1, v_2, w) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{earccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

But we can go farther!
If i is a fixed point of σ, we define its level by

$$\mathrm{lev}(i, \sigma) := \#\{j < i : \sigma(j) > i\} = \#\{j > i : \sigma(j) < i\}.$$

Clearly, a fixed point i is a record-antirecord if its level is 0, and a neither-record-antirecord if its level is ≥ 1.

First J-fraction

For $\sigma \in \mathfrak{S}_n$ and $\ell \geq 0$ we define

$$\mathrm{fix}(\sigma, \ell) := \#\{i \in [n] : \sigma(i) = i \text{ and } \mathrm{lev}(i, \sigma) = \ell\}.$$

Introduce indeterminates $\mathbf{w} = (w_\ell)_{\ell \geq 0}$ and write

$$\mathbf{w}^{\mathbf{fix}(\sigma)} := \prod_{\ell=0}^{\infty} w_\ell^{\mathrm{fix}(\sigma,\ell)} = \prod_{i \in \mathrm{Fix}(\sigma)} w_{\mathrm{lev}(i,\sigma)}.$$

The master polynomial encoding all these (now infinitely many) statistics is

$$Q_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, \mathbf{w}) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{earccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \mathbf{w}^{\mathbf{fix}(\sigma)}$$

Theorem 2 (First J-fraction for permutations)

The OGF of the polynomials Q_n has the J-type continued fraction

$$\sum_{n=0}^{\infty} Q_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, \mathbf{w}) t^n =$$

$$\cfrac{1}{1 - w_0 t - \cfrac{x_1 y_1 t^2}{1 - (x_2 + y_2 + w_1)t - \cfrac{(x_1 + u_1)(y_1 + v_1)t^2}{1 - \cdots}}},$$

with coefficients $\gamma_0 = w_0$,

$$\gamma_n = [x_2 + (n-1)u_2] + [y_2 + (n-1)v_2] + w_n \quad \textit{for} \quad n \geq 1$$
$$\beta_n = [x_1 + (n-1)u_1][y_1 + (n-1)v_1].$$

Second J-fraction (with cyc)

Define the polynomial

$$\hat{Q}_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, \mathbf{w}, \lambda) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{earccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} \mathbf{w}^{\mathbf{fix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}.$$

This generalization is less satisfying, because cyc does not seem to mesh with the record classification: even the three-variable polynomials

$$\hat{P}_n(x, y, \lambda) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}$$

do not have a J-fraction with polynomial coefficients.

Second J-fraction

Theorem 3 ($v_1 = y_1$ and $v_2 = y_2$)

The OGF of the polynomials Q_n has the J-type continued fraction

$$\sum_{n=0}^{\infty} \hat{Q}_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, y_2, \mathbf{w}, \lambda) t^n =$$

$$\cfrac{1}{1 - \lambda w_0 t - \cfrac{\lambda x_1 y_1 t^2}{1 - (x_2 + y_2 + \lambda w_1) t - \cfrac{(\lambda + 1)(x_1 + u_1) t^2}{1 - \cdots}}},$$

with coefficients $\gamma_0 = \lambda w_0$,

$$\gamma_n = [x_2 + (n-1)u_2] + n y_2 + \lambda w_n \quad \textit{for} \quad n \geq 1$$
$$\beta_n = (\lambda + n - 1)[x_1 + (n-1)u_1] y_1.$$

The two J-fractions generalize (1) and (2) in Theorem 1

Comparing Theorem 1 (1) with the first J-fraction the polynomial Q_n reduces to $P_n(x, y, u, v)$ if we set

$$x_1 = x_2 = x,\ y_1 = y_2 = y,\ w_0 = xz$$
$$u_1 = u_2 = w_\ell = 1\ (\ell \geq 1),\ v_1 = v_2 = v.$$

The weight function reduces to

$$w(\sigma) = x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} v^{\mathrm{exc}(\sigma)} z^{\mathrm{rar}(\sigma)}.$$

Comparing Theorem 1 (2) with the second J-fraction the polynomial $\hat{Q}_n$ reduces to $P_n(x, y, u, v)$ if we set

$$x_1 = x_2 = y,\ u_1 = u_2 = v,\ w_0 = z$$
$$y_1 = y_2 = v_1 = v_2 = w_\ell = 1(\ell \geq 1),\ \lambda = x.$$

The weight function reduces to

$$\hat{w}(\sigma) = x^{\mathrm{cyc}(\sigma)} y^{\mathrm{earec}(\sigma)} v^{\mathrm{aexc}(\sigma)} z^{\mathrm{rar}(\sigma)}.$$

Statistics on permutations

We have the following equidistribution:

$$(\mathrm{arec}, \mathrm{erec}, \mathrm{exc}, \mathrm{rar}) \sim (\mathrm{cyc}, \mathrm{earec}, \mathrm{exc}, \mathrm{rar}).$$

Note that $\mathrm{rec} = \mathrm{erec} + \mathrm{rar}$. We derive

$$(\mathrm{arec}, \mathrm{rec}, \mathrm{exc}) \sim (\mathrm{cyc}, \mathrm{arec}, \mathrm{exc}).$$

- Cori (2008) and Foata-Han (2009) : $(\mathrm{arec}, \mathrm{rec}) \sim (\mathrm{cyc}, \mathrm{arec})$ on $\mathfrak{S}_n$ and the distribution of $(\mathrm{cyc}, \mathrm{arec})$ is symmetric.

A symmetric continued fraction expansion

In Theorem 3 if we set

$$x_1 = x_2 = y,\ u_1 = u_2 = 1,\ w_0 = y$$
$$y_1 = y_2 = v_1 = v_2 = z,\ w_\ell = 1(\ell \geq 1),\ \lambda = x,$$

we have the symmetric J-CF expansion

$$\sum_{n=0}^{\infty} \left(\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{arec}(\sigma)} z^{\mathrm{exc}(\sigma)} \right) t^n =$$

$$\cfrac{1}{1 - xy\, t - \cfrac{xyz\, t}{1 - (x + y + z)\, t - \cfrac{(x+1)(y+1)z\ t}{1 - \cdots}}}$$

with $\gamma_0 = xy$,

$$\gamma_n = x + y + n - 1 + nz$$
$$\beta_n = (x + n - 1)(y + n - 1)z, \quad \text{for} \quad n \geq 1.$$

p,q-generalizations of Euler's continued fractions

Define the p, q-analog of n:

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = \sum_{j=0}^{n-1} p^j q^{n-1-j}.$$

Foata-Zeilberger (1990), Biane (1993),
De Médicis-Viennot (1994), Simion-Stanton(1994, 1996),
Clarke-Steingrimsson-Z. (1997), Randrianarivony (1998).
Postnikov (2001?), Williams (2006), Corteel (2007), Josuat-Vergès (2010), ...
Permutation tableaux, TASEP, PASEP.

Crossings and nestings

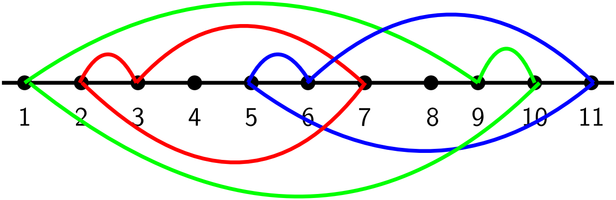
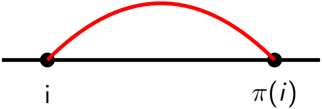
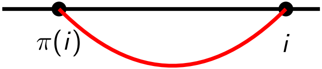


Figure: Pictorial representation of the permutation
$\pi = 9\,3\,7\,4\,6\,11\,2\,8\,10\,1\,5 = (1, 9, 10)\,(2, 3, 7)\,(4)\,(5, 6, 11)\,(8)$

We draw an upper (resp. lower) arc from i to $\pi(i)$ if $i < \pi(i)$ (resp. $i > \pi(i)$):

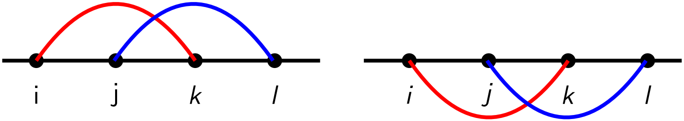




Upper and lower crossings

We say that a quadruple $i < j < k < l$ forms an

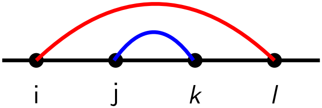
- *upper crossing* (ucross) if $k = \sigma(i)$ and $l = \sigma(j)$;
- *lower crossing* (lcross) if $i = \sigma(k)$ and $j = \sigma(l)$.

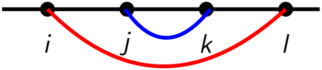


Upper and nestings

We say that a quadruple $i < j < k < l$ forms an

- *upper nesting* (unest) if $l = \sigma(i)$ and $k = \sigma(j)$;
- *lower nesting* (lnest) if $i = \sigma(l)$ and $j = \sigma(k)$.

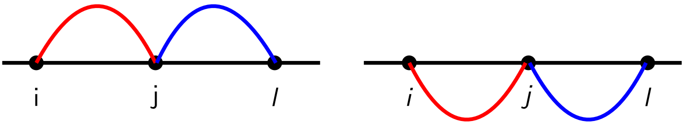




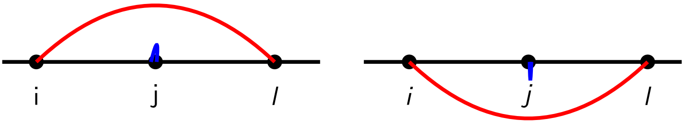
Upper and lower joining

We consider also some "degenerate" cases with $j = k$, by saying a triplet $i < j < l$ forms an

- *upper joining* (ujoin) if $\sigma(i) = j$ and $\sigma(j) = l$;
- *lower joining* (ljoin) if $i = \sigma(j)$ and $j = \sigma(l)$;



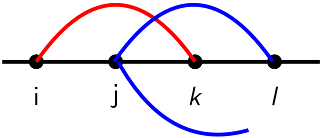
- *upper pseudo-nesting* (upsnest) if $l = \sigma(i)$ and $j = \sigma(j)$;
- *lower pseudo-nesting* (lpsnest) if $i = \sigma(l)$ and $j = \sigma(j)$.

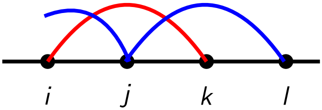


Refined categories of upper crossing

We say that a quadruplet $i < j < k < l$ forms an

- *upper crossing of type cval* (ucrosscval) if $k = \sigma(i)$ and $l = \sigma(j)$ and $\sigma^{-1}(j) > j$;
- *upper crossing of type cdrise* (ucrosscdrise) if $k = \sigma(i)$ and $l = \sigma(j)$ and $\sigma^{-1}(j) < j$;

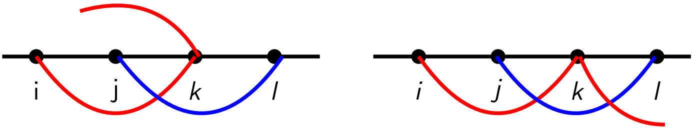




Refined categories of lower crossing

We say that a quadruplet $i < j < k < l$ forms an

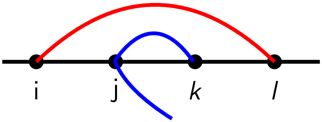
- *lower crossing of type cpeak* (lcrosscpeak) if $i = \sigma(k)$ and $j = \sigma(l)$ and $\sigma^{-1}(k) < k$;
- *lower crossing of type cdfall* (lcrosscdfall) if $i = \sigma(k)$ and $j = \sigma(l)$ and $\sigma^{-1}(k) > k$;

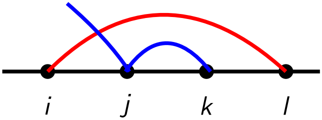


Refined categories of upper nesting

We say that a quadruplet $i < {\color{red}j} < k < l$ forms an

- *upper nesting of type cval* (unestcval) if $l = \sigma(i)$ and $k = \sigma(j)$ and $\sigma^{-1}(j) > j$;
- *upper nesting of type cdrise* (unestcdrise) if $l = \sigma(i)$ and $k = \sigma(j)$ and $\sigma^{-1}(j) < j$;

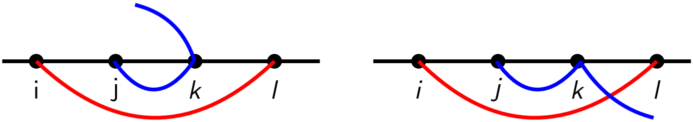




Refined categories of lower nesting

We say that a quadruplet $i < j < k < l$ forms an

- *lower nesting of type cpeak* (lnestcdpeak) if $l = \sigma(i)$ and $j = \sigma(j)$ and $\sigma^{-1}(k) < k$;
- *lower nesting of type cdfall* (lnestcdfall) if $i = \sigma(l)$ and $j = \sigma(j)$ and $\sigma^{-1}(k) > k$.



First J-fraction for permutations

Define the polynomial

$$Q_n(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{p},\mathbf{q},s) := Q_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,\mathbf{w}, \\ p_{+1},p_{+2},p_{+2},p_{-1},p_{-2},q_{+1},q_{+2},q_{-1},q_{-2},s) =$$

$$\sum_{\sigma\in\mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)}\, x_2^{\mathrm{earccdfall}(\sigma)}\, y_1^{\mathrm{ereccval}(\sigma)}\, y_2^{\mathrm{ereccdrise}(\sigma)} \times \\ u_1^{\mathrm{nrcpeak}(\sigma)}\, u_2^{\mathrm{nrcdfall}(\sigma)}\, v_1^{\mathrm{nrcval}(\sigma)}\, v_2^{\mathrm{nrcdrise}(\sigma)}\, \mathbf{w}^{\mathbf{fix}(\sigma)} \times \\ p_{+1}^{\mathrm{ucrosscval}(\sigma)}\, p_{+2}^{\mathrm{ucrosscdrise}(\sigma)}\, p_{-1}^{\mathrm{lcrosscpeak}(\sigma)}\, p_{-2}^{\mathrm{lcrosscdfall}(\sigma)} \times \\ q_{+1}^{\mathrm{unestcval}(\sigma)}\, q_{+2}^{\mathrm{unestcdrise}(\sigma)}\, q_{-1}^{\mathrm{lnestcpeak}(\sigma)}\, q_{-2}^{\mathrm{inestcdfall}(\sigma)}\, s^{\mathrm{psnest}(\sigma)}.$$

First J-fraction for permutations

$$\sum_{n=0}^{\infty} Q_n(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{p},\mathbf{q},s)t^n =$$

$$\cfrac{1}{1 - w_0 t - \cfrac{x_1 y_1 t^2}{1-(x_2+y_2+sw_1)t - \cfrac{(p_1 x_1 + q_{-1}u_1)(p_{+1}y_1+q_{+1}v_1)t^2}{1-\cdots}}}$$

with coeffiicents $\gamma_0 = w_0$ and for $n \geq 1$,

$$\gamma_n = (p_{-2}^{n-1}x_2 + q_{-2}[n-1]_{p_{-2},q_{-2}}u_2) + (p_{+2}^{n-1}y_2 + q_{+2}[n-1]_{p_{+2},q_{+2}}v_2) + s^n w_n$$

$$\beta_n = (p_{-1}^{n-1}x_1 + q_{-1}[n-1]_{p_{-1},q_{-1}}u_1)(p_{+1}^{n-1}y_1 + q_{+1}[n-1]_{p_{+1},q_{+1}}v_1).$$

First master J-fraction (1)

Rather than counting the total numbers of nestings, we should instead count the number of upper (resp. lower) crossings or nestings that use a particular vertex j (resp. k) in second (resp. third) position, and then attribute weights to the vertex j (resp. k) depending on these values.

$$\begin{aligned}
\mathrm{ucross}(j,\sigma) &= \#\{i < j < k < l:\ k = \sigma(i) \text{ and } l = \sigma(j)\} \\
\mathrm{unest}(j,\sigma) &= \#\{i < j < k < l:\ k = \sigma(j) \text{ and } l = \sigma(i)\} \\
\mathrm{lcross}(k,\sigma) &= \#\{i < j < k < l:\ i = \sigma(k) \text{ and } j = \sigma(l)\} \\
\mathrm{lnest}(k,\sigma) &= \#\{i < j < k < l:\ i = \sigma(l) \text{ and } j = \sigma(k)\}.
\end{aligned}$$

First master J-fraction (2)

N.B. $\text{ucross}(j, \sigma)$ and $\text{unest}(j, \sigma)$ can be nonzero only when j is a cycle valley or a cycle double rise, while $\text{lcross}(k, \sigma)$ and $\text{lnest}(k, \sigma)$ can be nonzero only when k is a cycle peak or a cycle double fall. And obviously we have

$$\text{ucrosscval}(\sigma) = \sum_{j \in \text{cval}} \text{ucross}(j, \sigma)$$

and analogously for the other seven crossing/nesting quantities.

First master J-fraction (3)

We now introduce five infinite families of indeterminates $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ where $\mathbf{x} = (x_{\ell,\ell'})_{\ell,\ell' \geq 0}$ and $\mathbf{w} = (w_\ell)_{\ell \geq 0}$, and define the polynomial

$$Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{w}) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i \in \mathrm{cval}} a_{\mathrm{ucross}(i,\sigma),\mathrm{unest}(i,\sigma)} \prod_{i \in \mathrm{cpeak}} b_{\mathrm{lcross}(i,\sigma),\mathrm{lnest}(i,\sigma)} \times \prod_{i \in \mathrm{cdfall}} c_{\mathrm{lcross}(i,\sigma),\mathrm{lnest}(i,\sigma)} \prod_{i \in \mathrm{cdrise}} d_{\mathrm{ucross}(i,\sigma),\mathrm{unest}(i,\sigma)} \prod_{i \in \mathrm{fix}} w_{\mathrm{lev}(i,\sigma)}$$

These polynomials then have a beautiful J-fraction.

First master J-fraction (4)

Theorem 4 (First master J-fraction for permutations)

The OGF of the polynomials $Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{w})$ *has the J-type continued fraction*

$$\sum_{n=0}^{\infty} Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{w}) t^n =$$

$$\cfrac{1}{1 - w_0 t - \cfrac{a_{00} b_{00} t^2}{1 - (c_{00} + d_{00} + w_1) t - \cfrac{(a_{00} + a_{10})(b_{01} + b_{10}) t^2}{1 - \cdots}}}$$

with coefficients $\gamma_n = c^*_{n-1} + d^*_{n-1} + w_n$ *and* $\beta_n = a^*_{n-1} b^*_{n-1}$*, where* $a^*_{n-1} := \sum_{\ell=0}^{n-1} a_{\ell, n-1-\ell} = a_{0,n-1} + a_{1,n-2} + \ldots + a_{n-1,0}$.

Second master J-fraction (with cyc)

We again introduce five infinite families of indeterminates $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ where $\mathbf{a} = (a_\ell)_{\ell \ge 0}$, $\mathbf{b} = (b_{\ell,\ell'})_{\ell,\ell' \ge 0}$, $\mathbf{c} = (c_{\ell,\ell'})_{\ell,\ell' \ge 0}$, $\mathbf{d} = (d_{\ell,\ell'})_{\ell,\ell' \ge 0}$, and $\mathbf{e} = (e_\ell)_{\ell \ge 0}$, and define the polynomial

$$
\begin{aligned}
\hat{Q}_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \lambda) = \\
&\sum_{\sigma \in \mathfrak{S}_n} \lambda^{\mathrm{cyc}(\sigma)} \prod_{i \in \mathrm{cval}} a_{\mathrm{ucross}(i,\sigma)+\mathrm{unest}(i,\sigma)} \prod_{i \in \mathrm{cpeak}} b_{\mathrm{lcross}(i,\sigma),\mathrm{lnest}(i,\sigma)} \times \\
&\prod_{i \in \mathrm{cdfall}} c_{\mathrm{lcross}(i,\sigma),\mathrm{lnest}(i,\sigma)} \prod_{i \in \mathrm{cdrise}} d_{\mathrm{ucross}(i,\sigma)+\mathrm{unest}(i,\sigma),\mathrm{unest}(\sigma^{-1}(i),\sigma)} \\
&\prod_{i \in \mathrm{fix}} e_{\mathrm{lev}(i,\sigma)}
\end{aligned}
$$

These polynomials then have a beautiful J-fraction.

Theorem 5 (Second master J-fraction for permutations)

The OGF of the polynomials $Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \lambda)$ has the J-type continued fraction

$$\sum_{n=0}^{\infty} \hat{Q}_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \lambda) t^n =$$

$$\cfrac{1}{1 - \lambda e_0 t - \cfrac{\lambda a_0 b_{00} t^2}{1 - (c_{00} + d_{00} + \lambda e_1) t - \cfrac{(\lambda+1) a_1 (b_{00} + b_{10}) t^2}{1 - \cdots}}}$$

with coefficients $\gamma_n = c^*_{n-1} + d^\natural_{n-1} + \lambda e_n$ *and* $\beta_n = (\lambda + n - 1) a_{n-1} b^*_{n-1}$, *where* $b^*_{n-1} := \sum_{\ell=0}^{n-1} b_{\ell, n-1-\ell}$, $c^*_{n-1} := \sum_{\ell=0}^{n-1} c_{\ell, n-1-\ell}$, $d^\natural_{n-1} := \sum_{\ell=0}^{n-1} d_{n-1, \ell}$.

A remark on the inversion statistic

A inversion of a permutation $\sigma \in \mathfrak{S}_n$ is a pair $i, j \in [n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

Lemma 1

We have

$$\begin{aligned} \mathrm{inv} = cval + \mathrm{cdrise} + \mathrm{cdfall} + \mathrm{ucross} + \mathrm{lcross} \\ + 2(\mathrm{unest} + \mathrm{lnest} + \mathrm{psnest}). \end{aligned}$$

de Médicis and Viennot, Clarke-Steingrimsson-Z., ...

Set partitions: S-fraction

The Bell number B_n is the number of partitions of an n-element set into nonempty blocks with $B_0 = 1$.

$$\sum_{n=0}^{\infty} B_n\, t^n = \cfrac{1}{1 - \cfrac{1\,t}{1 - \cfrac{1\,t}{1 - \cfrac{1\,t}{1 - \cfrac{2\,t}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = 1$, $\alpha_{2k} = k$.

$$\sum_{n=0}^{\infty} B_n(x,y,v)\, t^n = \cfrac{1}{1 - \cfrac{x\,t}{1 - \cfrac{y\,t}{1 - \cfrac{x\,t}{1 - \cfrac{(y+2v)\,t}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = x$, $\alpha_{2k} = y + (k-1)v$.
Clearly $B_n(x,y,v)$ is a homogeneous polynomial of degree n; it has three truly independent variables.

Theorem 6 (S-fraction for set)

The polynomials $B_n(x,y,v)$ have the combinatorial interpretation

$$B_n(x,y,v) = \sum_{\pi \in \Pi_n} x^{|\pi|} y^{\mathrm{erec}(\pi)} v^{n-|\pi|-\mathrm{erec}(\pi)}$$

where $|\pi|$ (resp. $\mathrm{erec}(\pi)$) denotes the number of blocks (resp. exclusive records) in π.

Given $\pi \in \Pi_n$, for $i \in [n]$, we define $\sigma'(i)$ to be the next-larger element after i in its block, if i is not the largest element in its block, and 0 otherwise. Then $\mathrm{erec}(\pi) := \mathrm{erec}(\sigma')$. For example, if $\pi = \{1,5\} - \{2,3,7\} - \{4\} - \{6\}$, then $\sigma' = 5370000$.

Given a partition π of $[n]$, we say that an element $i \in [n]$ is

- an *opener* if it is the smallest element of a block of size ≥ 2;
- a *colser* if it is the largest element of a block of size ≥ 2;
- an *insider* if it is a non-opener non-closer element of a block of size ≥ 3;
- a *singleton* if it is the sole element of a block of size 1.

Clearly every element $i \in [n]$ belongs to precisely one of these four classes.

J-fraction

We can refine the polynomial $B_n(x, y, v)$ by distinguishing between singletons and blocks of size ≥ 2; in addition, we can distinguish between exclusive records that are openers and those that are insiders. Define

$$B_n(x_1, x_2, y_1, y_2, v) = \sum_{\pi \in \Pi_n} x_1^{m_1(\pi)} x_2^{m_{\geq 2}(\pi)} \times y_1^{\mathrm{erecin}(\pi)} y_2^{\mathrm{erecop}(\pi)} v^{n-|\pi|-\mathrm{erec}(\pi)},$$

where $m_1(\pi)$ is the number of singletons in π, $m_{\geq 2}(\pi)$ is the number of non-singletons blocks, $\mathrm{erecin}(\pi)$ is the number of exclusive records that are insiders, and $\mathrm{erecop}(\pi)$ is the number of exclusive records that are openers.

Theorem 7 (J-fraction for set partitions)

$$\sum_{n=0}^{\infty} B_n(x_1, x_2, y_1, y_2, v)t^n =$$

$$\cfrac{1}{1 - x_1 t - \cfrac{x_2 y_2 t^2}{1 - (x_1 + y_1)t - \cfrac{x_2(y_2 + v)t^2}{1 - \cdots}}}$$

with coefficients $\gamma_0 = x_1$,

$$\gamma_n = x_1 + y_1 + (n-1)v \quad \textit{for } n \geq 1$$
$$\beta_n = x_2[y_2 + (n-1)v].$$

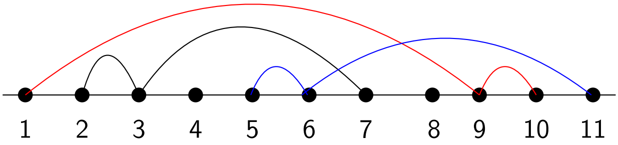
Graph of a partition

Let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a partition of $[n]$. We associate a graph $\mathcal{G}_\pi$ with vertex set $[n]$ such that i, j are joined by an edge if and only if they are consecutive elements within the same block. We then say that a quadruplet $i < j < k < l$ forms a

- *crossing* (cr) if $(i, k) \in \mathcal{G}_\pi$ and $(j, l) \in \mathcal{G}_\pi$;
- *nesting* (ne) if $(i, l) \in \mathcal{G}_\pi$ and $(j, k) \in \mathcal{G}_\pi$.

We also say that a triplet $i < k < l$ forms a

- *pseudo-nesting* (psne) if $(i, l) \in \mathcal{G}_\pi$.



$$\pi = \{\{1, 9, 10\},\ \{2, 3, 7\},\ \{4\},\ \{5, 6, 11\},\ \{8\}\}.$$

First $p, q-$generalization

We now introduce a (p, q)-generalization of previous polynomial:

$$B_n(x_1, x_2, y_1, y_2, v, p, q, r) = \sum_{\pi \in \Pi_n} x_1^{m_1(\pi)} x_2^{m_{\geq 2}(\pi)} y_1^{\mathrm{erecin}(\pi)} y_2^{\mathrm{erecop}(\pi)} \times$$
$$v^{n-|\pi|-\mathrm{erecop}(\pi)} p^{\mathrm{cr}(\pi)} q^{\mathrm{ne}(\pi)} r^{\mathrm{psne}(\pi)}.$$

Theorem 8

$$\sum_{n=0}^{\infty} B_n(x_1, x_2, y_1, y_2, v, p, q, r) t^n = \cfrac{1}{1 - x_1 t - \cfrac{x_2 y_2 t^2}{1 - \cdots}}$$

with coefficients $\gamma_0 = x_1$,

$$\gamma = r^n x_1 + p^{n-1} y_1 + q[n-1]_{p,q} v \quad \textit{for } n \geq 1$$
$$\beta_n = x_2(p^{n-1} y_2 + q[n-1]_{p,q} v).$$

First master J-fraction

Rather that counting the *total* numbers of quadrauplets $i < j < k < l$ that form crossings or nestings, we should instead count the number of crossings or nestings that use a particular vertex k in third (or sometimes second) position, and then attribute weights to the vertex k depending on those values. We define

$$\begin{aligned} \mathrm{cr}(k,\pi) &= \#\{i < j < k < l \colon\ (i,k) \in \mathcal{G}_\pi \text{ and } (j,l) \in \mathcal{G}_\pi\} \\ \mathrm{ne}(k,\pi) &= \#\{i < j < k < l \colon\ (i,l) \in \mathcal{G}_\pi \text{ and } (j,k) \in \mathcal{G}_\pi\}. \end{aligned}$$

In addition we define the *quasi-nesting* of the vertex k:

$$\mathrm{qne}(k,\pi) = \#\{i < k < l \colon\ (i,l) \in \mathcal{G}_\pi\}$$

First master J-fraction

Note that $\mathrm{cr}(k,\pi)$ and $\mathrm{ne}(k,\pi)$ can be nonzero only when k is either an insider or a closer; and we obviously have

$$\mathrm{cr}(\pi) = \sum_{k \in \mathrm{insiders} \cap \mathrm{closers}} \mathrm{cr}(k,\pi)$$

$$\mathrm{ne}(\pi) = \sum_{k \in \mathrm{insiders} \cap \mathrm{closers}} \mathrm{ne}(k,\pi)$$

$$\mathrm{psne}(\pi) = \sum_{k \in \mathrm{singletons}} \mathrm{qne}(k,\pi).$$

First master J-fraction

We now introduce four infinite families of indterminates $\mathbf{a} = (a_\ell)_{\ell \geq 0}$, $\mathbf{b} = (a_{\ell,\ell'})_{\ell,\ell' \geq 0}$, $\mathbf{c} = (c_{\ell,\ell'})_{\ell,\ell' \geq 0}$, $\mathbf{e} = (e_\ell)_{\ell \geq 0}$ and define the polynomials $B_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e})$ by

$$B_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}) = \sum_{\pi \in \Pi_n} \prod_{i \in \text{openers}} \mathbf{a}_{\text{qne}(i,\pi)} \prod_{i \in \text{closers}} \mathbf{b}_{\text{cr}(i,\pi),\text{ne}(i,\pi)} \prod_{i \in \text{insiders}} \mathbf{c}_{\text{cr}(i,\pi),\text{ne}(i,\pi)} \prod_{i \in \text{singletons}} \mathbf{e}_{\text{psne}(i,\pi)}$$

Theorem 9 (Master J-fraction for set partitions)

The OGF of the polynomials $B_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e})$ *has the J-type CF*

$$\sum_{n=0}^{\infty} B_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}) t^n = \cfrac{1}{1 - e_0 t - \cfrac{a_0 b_{00} t^2}{1 - (c_{00} + e_1) t - \cfrac{a_1 (b_{01} + b_{10}) t^2}{1 - \cdots}}}$$

with coefficients

$$\gamma_n = \sum_{\ell=0}^{n-1} c_{\ell, n-1-\ell} + e_n, \quad \beta_n = a_{n-1} \sum_{\ell=0}^{n-1} b_{\ell, n-1-\ell}.$$

- We have also a second master J-fraction using the notion of *overlapping* and *covering*.

Perfect matchings

Euler:

$$\sum_{n=0}^{\infty}(2n-1)!!t^n = \cfrac{1}{1-\cfrac{1\,t}{1-\cfrac{2\,t}{1-\cfrac{3\,t}{1-\cdots}}}}$$

We introduce the polytnomials $M_n(x,y,u,v)$ by

$$\sum_{n=0}^{\infty} M_n(x,y,u,v)t^n = \cfrac{1}{1-\cfrac{x\,t}{1-\cfrac{(x+v)\,t}{1-\cfrac{(x+2u)\,t}{1-\cdots}}}}$$

with coefficients $\alpha_{2k-1} = x+(2k-2)u$, $\alpha_{2k} = y+(2k-1)v$

Master S-fraction

We can regard a perfect matching either as a special type of partition (namely, one in which all blocks are of size 2) or as a special type of permutation (namely, a fixed-point-free involution). We now introduce four infinite families of indterminates $\mathbf{a} = (a_\ell)_{\ell \geq 0}$, $\mathbf{b} = (a_{\ell,\ell'})_{\ell,\ell' \geq 0}$,and define the polynomials $M_n(\mathbf{a},\mathbf{b})$ by

$$M_n(\mathbf{a},\mathbf{b}) = \sum_{\pi \in \mathcal{M}_{2n}} \prod_{i \in \text{openers}} \mathbf{a}_{\text{qne}(i,\pi)} \prod_{i \in \text{closers}} \mathbf{b}_{\text{cr}(i,\pi),\text{ne}(i,\pi)}.$$

Of course, we have $M_n(\mathbf{a},\mathbf{b}) = B_{2n}(\mathbf{a},\mathbf{b},\mathbf{0},\mathbf{0})$.

Theorem 10 (Master S-fraction for perfect matchings)

The OGF of the polynomials $B_n(\mathbf{a}, \mathbf{b})$ *has the S-type CF*

$$\sum_{n=0}^{\infty} M_n(\mathbf{a}, \mathbf{b}) t^n = \cfrac{1}{1 - \cfrac{a_0 b_{00} t^2}{1 - \cfrac{a_1 (b_{01} + b_{10}) t^2}{1 - \cdots}}}$$

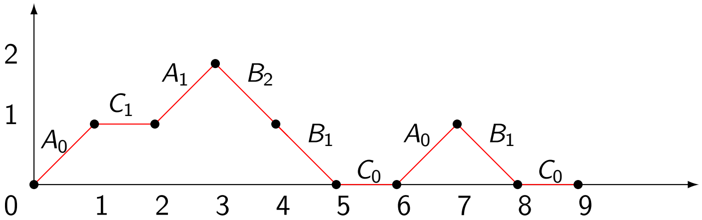
with coefficients $\alpha_n = a_{n-1} b^*_{n-1}$, *where*

$$b^*_{n-1} = \textstyle\sum_{\ell=0}^{n-1} b_{\ell, n-1-\ell}.$$

- Unfortunately we treat openers and closers asymmetrically.

Prelininaries: Flajolet's fondamental lemma

Consider the following Motzkin path γ :



The weight is

$$w(\gamma) = A_0^2 A_1 B_2 B_1^2 C_1 C_0^2.$$

Let $\mathfrak{M}_n$ be the set of Motzkin paths of length $n \geq 1$. Then

$$1 + \sum_{n\geq 1} \sum_{\gamma \in \mathfrak{M}_n} w(\gamma) x^n = \cfrac{1}{1 - C_0 x - \cfrac{A_0 B_1 x^2}{1 - C_1 x - \cfrac{A_1 B_2 x^2}{\cdots}}}. \qquad (3)$$

Labelled Dyck and Motzkin paths

Let $\mathbf{A} = (A_k)_{k \geq 0}$, $\mathbf{B} = (B_k)_{k \geq 1}$ and $\mathbf{C} = (C_k)_{k \geq 0}$ be sequences of nonnegative integers. An $(\mathbf{A}, \mathbf{B}, \mathbf{C})$-labelled Motzkin path of length n is a pair (ω, ξ) where $\omega = (\omega_0, \ldots, \omega_n)$ is a Motzkin path of length n, and $\xi = (\xi_1, \ldots, \xi_n)$ is a sequence of integres satisfying

$$1 \leq \xi_i \leq \begin{cases} A(h_{i-1}) & \text{if } h_i = h_{i-1} + 1 \text{ (i.e. step } i \text{ is a rise)} \\ B(h_{i-1}) & \text{if } h_i = h_{i-1} - 1 \text{ (i.e. step } i \text{ is a fall)} \\ C(h_{i-1}) & \text{if } h_i = h_{i-1} \text{ (i.e. step } i \text{ is a level step)} \end{cases}$$

where h_i is the height of the Motzkin path after step i, i.e. $\omega_i = (i, h_i)$ and

$$A_k = k + 1 \ (k \geq 0), \quad B_k = k \ (k \geq 1), \quad C_k = 2k + 1 \ (k \geq 0).$$

It is convenient to divide the level steps into three types: Let $C_k = C_k^{(1)} + C_k^{(2)} + C_k^{(3)}$ with

$$C_k^{(1)} = k, \quad C_k^{(2)} = k, \quad C_k^{(3)} = 1 \ (k \geq 0).$$

Thus, Euler's CF expansion for $\sum_{n \geq 0} n! x^n$ is equivalent to say that the number of $(\mathbf{A}, \mathbf{B}, \mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{(3)})$-labelled 3-colored Motzkin paths of length n is $n!$.

We then use a variant of the Foata-Zeilberger bijection for the first J-fraction. To prove the second fraction we need to construct a bijection that will allow us to count the number of cycles (cyc), which is a global variable. We will employ a variant of Biane's bijection.

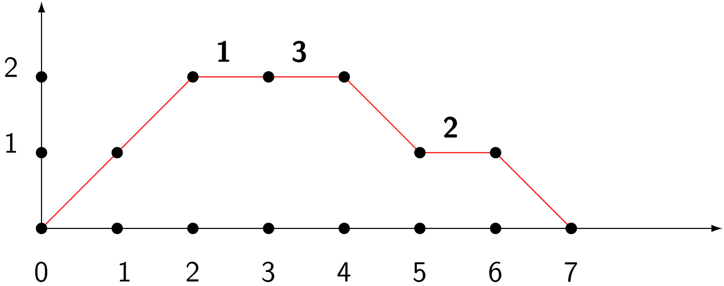
Permutations: Outline of Proofs

1. Definition of the Motzkin path.
2. Definition of the labels ξ_i.
3. Proof of bijection.
4. Translation of the statistics.
5. Computation of the weights.

Definition of the Motzkin path

Given a permutation $\sigma \in \mathfrak{S}_n$, we classify the indices $i \in [n]$ in the usual way as cycle peak, cycle valley, cycle doube rise, cycle double fall or fixed point. We then define a path $\omega = (\omega_0, \ldots, \omega_n)$ strating at $\omega_0 = (0, 0)$ and ending at $\omega_n = (n, 0)$, with steps $s_1, \ldots, s_n$ as follows:

- If i is a cycle valley, then s_i is a rise.
- If i is a cycle peak, then s_i is a fall.
- If i is a cycle double fall, then s_i is a level step of type 1.
- If i is a cycle double rise, then s_i is a level step of type 2.
- If i is a fixed point, then s_i is a level step of type 3.



The 3-coloreed Motzkin path corresponding to the permutation $\sigma = (1, 5, 2, 6, 7, 3)(4)$

We then need to explain how the labels ξ are defined; next we will prove that the mapping is indeed a bijection; next we will translate the various statistics from $\mathfrak{S}_n$ to our labelled Motzkin paths; and finally we will sum over labels ξ to obtain the weight $W(\omega)$ associated to a Motzkin path ω, which upon applying Flajolet's result will yield our Theorem.

Joyeux anniversaire, Christian!