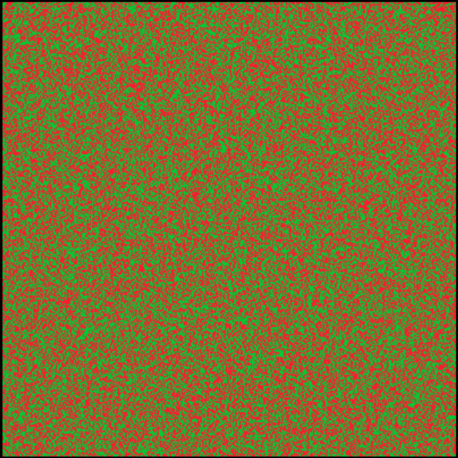
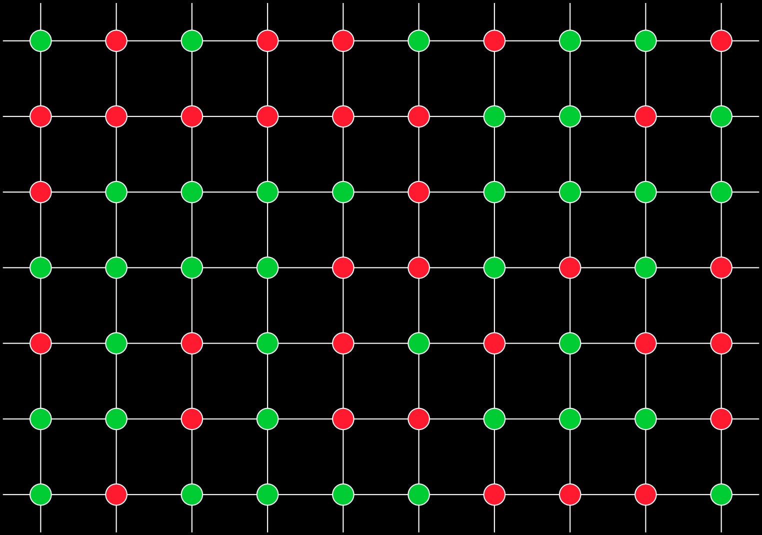
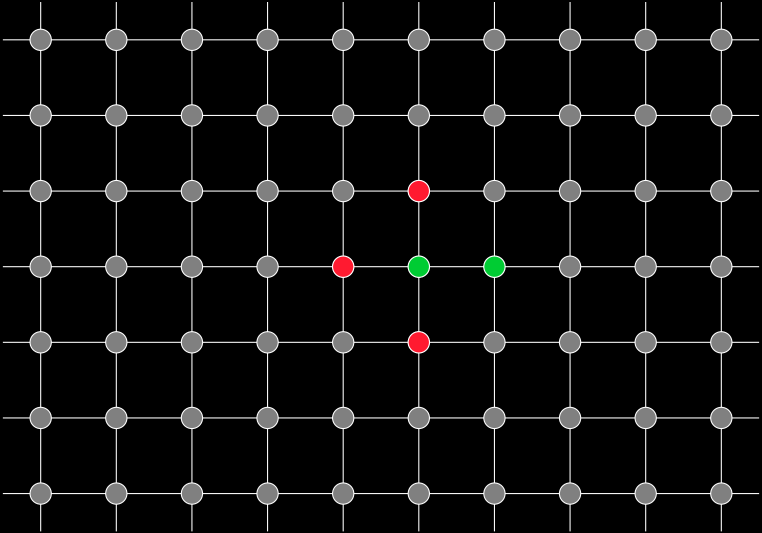
ONSAGER'S SOLUTION OF THE ISING MODEL COULD HAVE BEEN GUESSED

Manuel Kauers · Institute for Algebra · JKU

Joint work with Doron Zeilberger







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- If the flip decreases the energy, it is performed unconditionally
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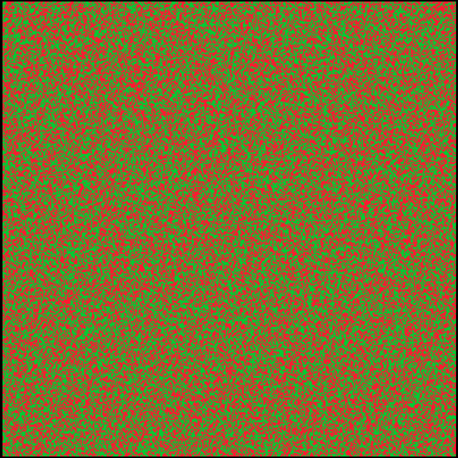
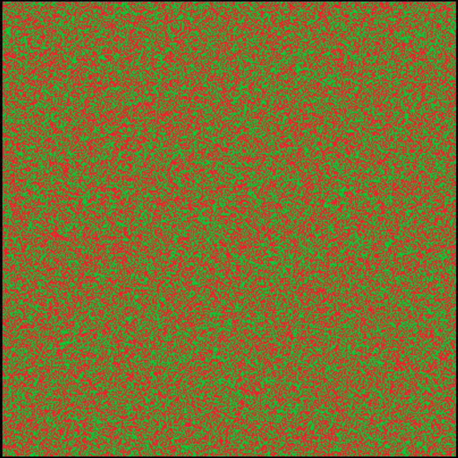
- Note: $p \to 1$ for $T \to \infty$ and $p \to 0$ for $T \to 0$



High temperature

Low temperature



Medium temperature

Eventually, the probability of observing a certain configuration s is

$$\frac{e^{-E(s)/T}}{\sum_c e^{-E(c)/T}}$$

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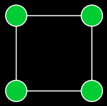
$E($  $) = \frac{1}{2}(0 - 4) = -2$

$$P = x^{-2}$$

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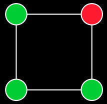
$E($  $) = \frac{1}{2}(2 - 2) = 0$

$$P = 1 + x^{-2}$$

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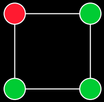
$E($  $) = \frac{1}{2}(2 - 2) = 0$

$$P = 2 + x^{-2}$$

The denominator

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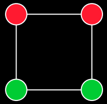
$E\big($  $\big) = \frac{1}{2}(2-2) = 0$

$$P = 3 + x^{-2}$$

The denominator

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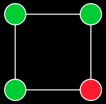
$E($  $) = \frac{1}{2}(2 - 2) = 0$

$$P = 4 + x^{-2}$$

The denominator

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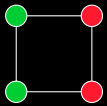
$E($  $) = \frac{1}{2}(2 - 2) = 0$

$$P = 5 + x^{-2}$$

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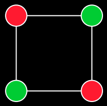
$E($  $) = \frac{1}{2}(4 - 0) = 2$

$$P = 5 + x^{-2} + x^{2}$$

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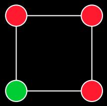
$E($  $) = \frac{1}{2}(2 - 2) = 0$

$$P = 6 + x^{-2} + x^2$$

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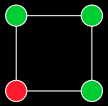
$E($  $) = \frac{1}{2}(2 - 2) = 0$

$$P = 7 + x^{-2} + x^{2}$$

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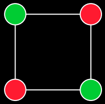
$E($  $) = \frac{1}{2}(4 - 0) = 2$

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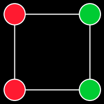
$E\Big($  $\Big) = \frac{1}{2}(2 - 2) = 0$

$$P = 8 + x^{-2} + 2x^2$$

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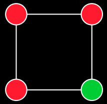
$E($  $) = \frac{1}{2}(2 - 2) = 0$

$$P = 9 + x^{-2} + 2x^2$$

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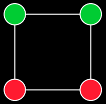
$E($  $) = \frac{1}{2}(2-2) = 0$

$$P = 10 + x^{-2} + 2x^2$$

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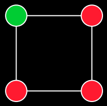
$E($  $) = \frac{1}{2}(2 - 2) = 0$

$$P = 11 + x^{-2} + 2x^2$$

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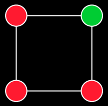
$E($  $) = \frac{1}{2}(2-2) = 0$

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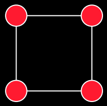
$E($  $) = \frac{1}{2}(0 - 4) = -2$

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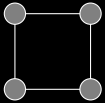

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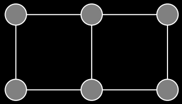
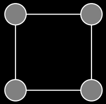


$$P = 24 + 4x^{-2} + 16x^{-1} + 16x + 4x^2$$



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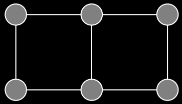

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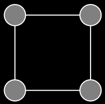

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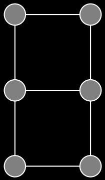

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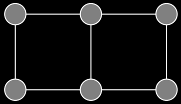

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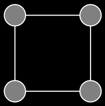

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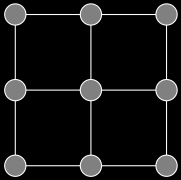
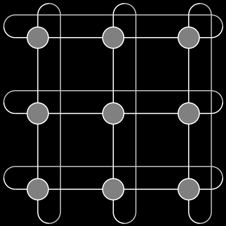
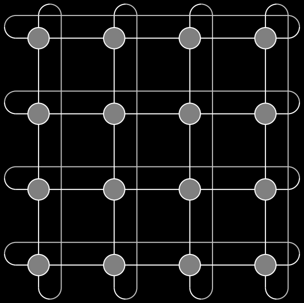


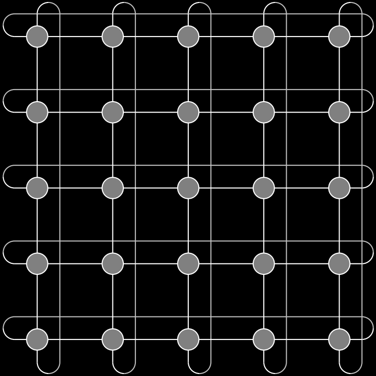
$$P = 152 + 4x^{-4} + 16x^{-3} + 48x^{-2} + 112x^{-1} + 112x + 48x^2 + 16x^3 + 4x^4$$



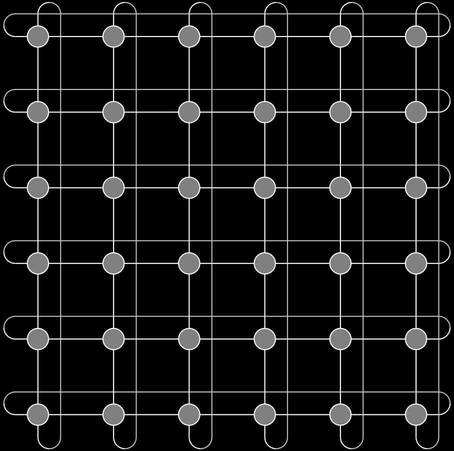
$$P = 102x^{-3} + 144x^{-1} + 198x + 48x^{3} + 18x^{5} + 2x^{9}$$



$$P = 20524+2x^{-16}+32x^{-12}+64x^{-10}+424x^{-8}+1728x^{-6}+6688x^{-4}+ 13568x^{-2}+13568x^{2}+6688x^{4}+1728x^{6}+424x^{8}+64x^{10}+32x^{12}+2x^{16}$$



$$P = 2470x^{-15} + 14800x^{-13} + 82750x^{-11} + 314300x^{-9} + 1024150x^{-7} + 2645740x^{-5} + 5276500x^{-3} + 7413900x^{-1} + 7431800x + 5230300x^{3} + 2696080x^{5} + 1014900x^{7} + 311800x^{9} + 74500x^{11} + 16300x^{13} + 3140x^{15} + 850x^{17} + 100x^{19} + 50x^{21} + 2x^{25}$$



$$P = 13172279424 + 2x^{-36} + 72x^{-32} + 144x^{-30} + 1620x^{-28} + 6048x^{-26} + 35148x^{-24} + 159840x^{-22} + 804078x^{-20} + 3846576x^{-18} + 17569080x^{-16} + 71789328x^{-14} + 260434986x^{-12} + 808871328x^{-10} + 2122173684x^{-8} + 4616013408x^{-6} + 8196905106x^{-4} + 11674988208x^{-2} + 11674988208x^{2} + 8196905106x^{4} + 4616013408x^{6} + 2122173684x^{8} + 808871328x^{10} + 260434986x^{12} + 71789328x^{14} + 17569080x^{16} + 3846576x^{18} + 804078x^{20} + 159840x^{22} + 35148x^{24} + 6048x^{26} + 1620x^{28} + 144x^{30} + 72x^{32} + 2x^{36}$$

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$$f(x) := \lim_{n,m\to\infty} \frac{\log(P_{n,m})}{nm}$$

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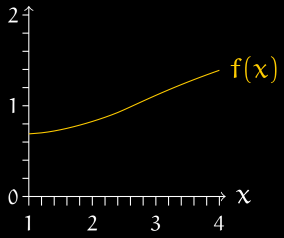

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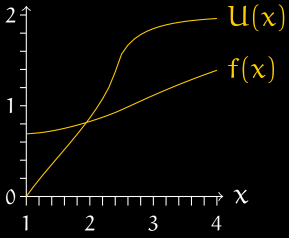

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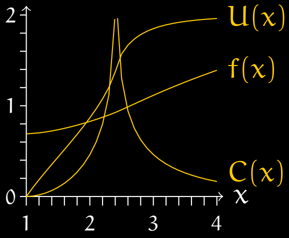

Theorem (Onsager 1944):

$$f(x) = \log(x + x^{-1}) - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n}{n}^2\left(\frac{x - x^{-1}}{(x + x^{-1})^2}\right)^{2n}$$

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Note: $f(x)$ is D-finite, so we are in business!

$$(x^{28} - 12x^{26} + 34x^{24} + 36x^{22} - 145x^{20} - 24x^{18} + 220x^{16} - 24x^{14} - 145x^{12} + 36x^{10} + 34x^{8} - 12x^{6} + x^{4}) f^{(5)}(x) + (16x^{27} - 172x^{25} + 380x^{23} + 748x^{21} - 1548x^{19} - 1144x^{17} + 2200x^{15} + 664x^{13} - 1352x^{11} - 28x^{9} + 300x^{7} - 68x^{5} + 4x^{3}) f^{(4)}(x) + (69x^{26} - 660x^{24} + 770x^{22} + 4972x^{20} - 6973x^{18} - 7720x^{16} + 11644x^{14} + 3128x^{12} - 5797x^{10} + 316x^{8} + 290x^{6} - 36x^{4} - 3x^{2}) f^{(3)}(x) + (81x^{25} - 672x^{23} - 554x^{21} + 8216x^{19} - 6021x^{17} - 22816x^{15} + 21732x^{13} + 11920x^{11} - 14889x^{9} + 3328x^{7} - 346x^{5} + 24x^{3} - 3x) f''(x) + (15x^{24} - 96x^{22} - 630x^{20} + 3048x^{18} - 6075x^{16} + 8736x^{14} - 12068x^{12} + 32624x^{10} - 16119x^{8} + 2816x^{6} + 58x^{4} - 24x^{2} + 3) f'(x) = 0$$

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(order 5, degree 28)

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Van der Waerden's change of variables (1941)

Write

$$P_{n,m}(x) = \left(\frac{x + 2 + x^{-1}}{2}\right)^{nm} Z_{n,m}(w)$$

with

$$w = \frac{x - 1}{x + 1} \ ,$$

and translate everything from P and x to Z and w.

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$$f(x) = \log\Big(\frac{2}{1-w^2}\Big) + \lim_{n\to\infty} \frac{\log(Z_{n,n}(w))}{n^2}$$

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and translate everything from P and x to Z and w. Note:

$$g(w) := \lim_{n\to\infty} \frac{\log(Z_{n,n}(w))}{n^2}$$

$$
\begin{aligned}
&(14w^{36} - 131w^{34} - 18w^{32} + 2487w^{30} - 4184w^{28} + 517w^{26} - \\
&390w^{24} - 521w^{22} + 9480w^{20} - 8561w^{18} - 1182w^{16} + 4957w^{14} - \\
&3584w^{12} + 1263w^{10} - 138w^{8} - 11w^{6} + 2w^{4})g^{(5)}(w) + (224w^{35} - \\
&1798w^{33} - 2296w^{31} + 39766w^{29} - 63952w^{27} + 65362w^{25} - \\
&132112w^{23} + 52462w^{21} + 117672w^{19} - 134738w^{17} + 84904w^{15} - \\
&24158w^{13} - 4416w^{11} + 3206w^{9} - 32w^{7} - 102w^{5} + 8w^{3})g^{(4)}(w) + \\
&(966w^{34} - 6543w^{32} - 21066w^{30} + 183603w^{28} - 304248w^{26} + \\
&481689w^{24} - 1009950w^{22} + 603411w^{20} + 125400w^{18} - \\
&410805w^{16} + 324858w^{14} - 132495w^{12} + 22176w^{10} - 5973w^{8} + \\
&1710w^{6} - 183w^{4} - 6w^{2})g^{(3)}(w) + (1134w^{33} - 6177w^{31} - \\
&43482w^{29} + 222213w^{27} - 388776w^{25} + 967263w^{23} - 2351094w^{21} + \\
&1447773w^{19} - 240672w^{17} - 406155w^{15} + 482682w^{13} - 99801w^{11} - \\
&39264w^{9} + 13005w^{7} - 1002w^{5} - 9w^{3} - 6w)g''(w) + (210w^{32} - \\
&735w^{30} - 14694w^{28} + 40827w^{26} - 98904w^{24} + 419745w^{22} - \\
&970122w^{20} + 572835w^{18} - 12960w^{16} - 192117w^{14} + 226374w^{12} - \\
&134823w^{10} + 11232w^{8} + 6963w^{6} - 1302w^{4} + 9w^{2} + 6)g'(w) = 0
\end{aligned}
$$

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(order 5, degree 36)

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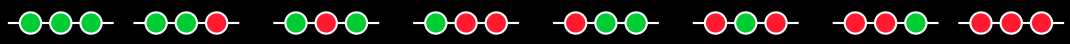

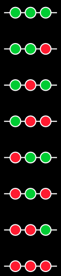

$$\begin{pmatrix}
x^3 & 1 & 1 & x^{-1} & 1 & x^{-1} & x^{-1} & 1 \\
x^2 & x & x^{-1} & 1 & x^{-1} & 1 & x^{-2} & x \\
x^2 & x^{-1} & x & 1 & x^{-1} & x^{-2} & 1 & x \\
x & 1 & 1 & x & x^{-2} & x^{-1} & x^{-1} & x^2 \\
x^2 & x^{-1} & x^{-1} & x^{-2} & x & 1 & 1 & x \\
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$$T = \begin{pmatrix} x & x^{-1} \\ x^{-1} & x \end{pmatrix}^{\otimes n} \mathrm{diag}(\dots)$$

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- This approach is feasible for $n \leq 12$, which is not enough

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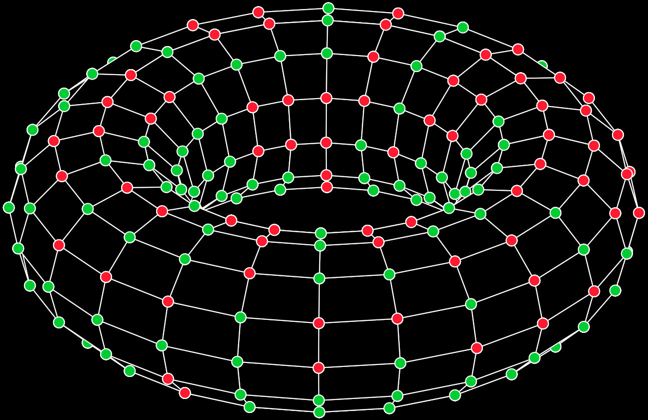

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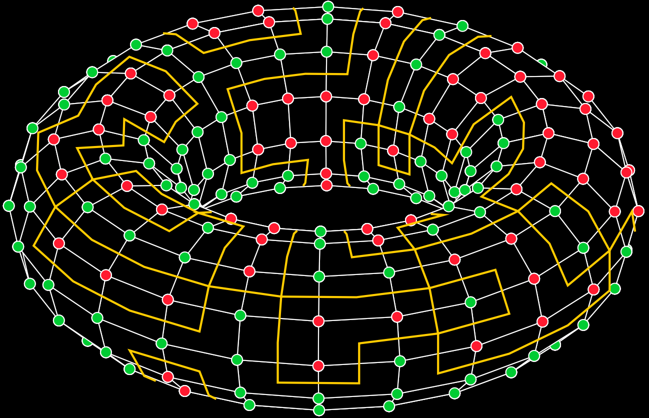

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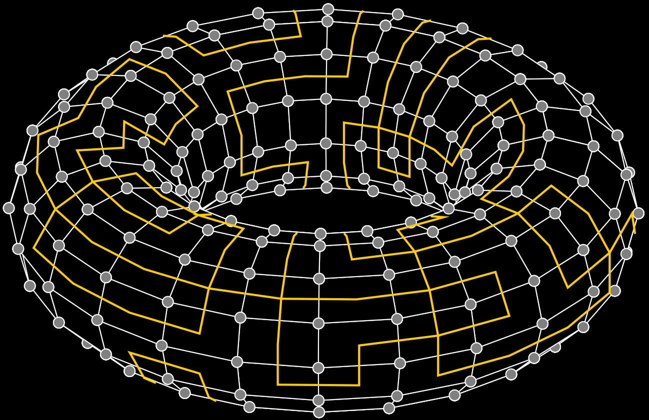

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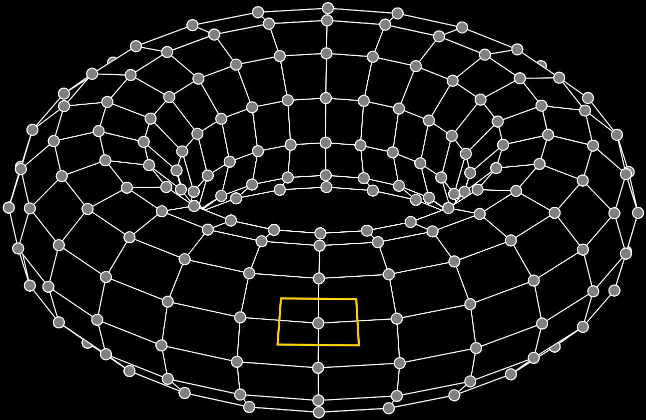

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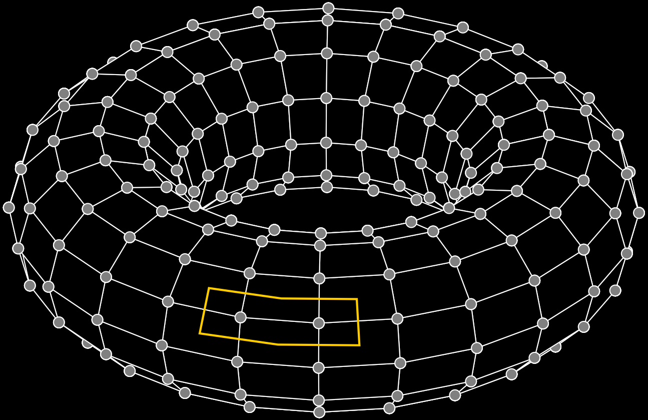

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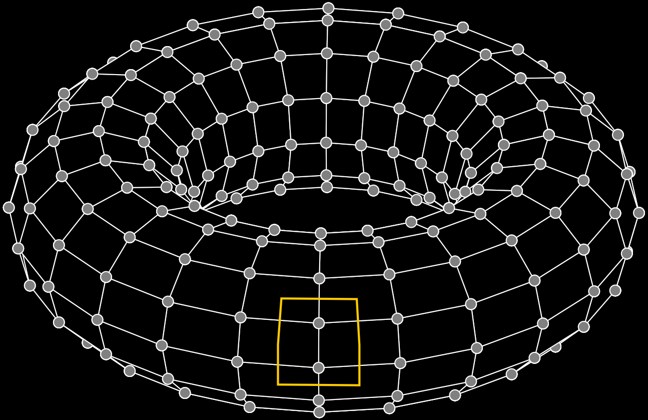

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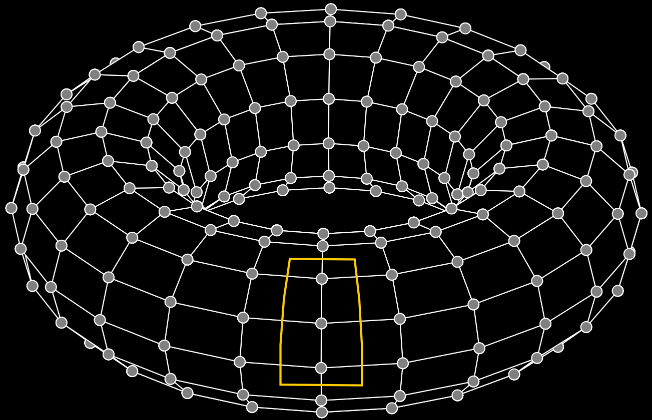

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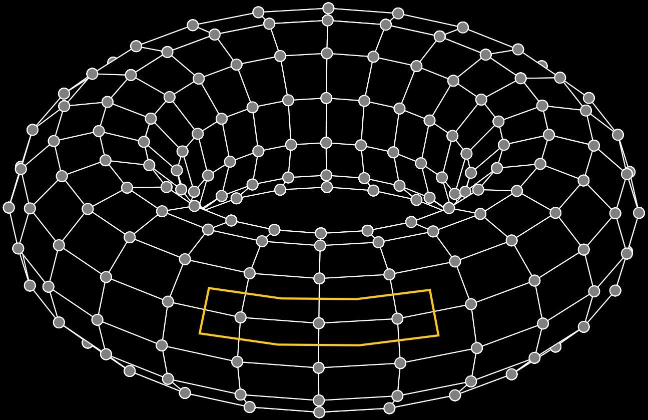

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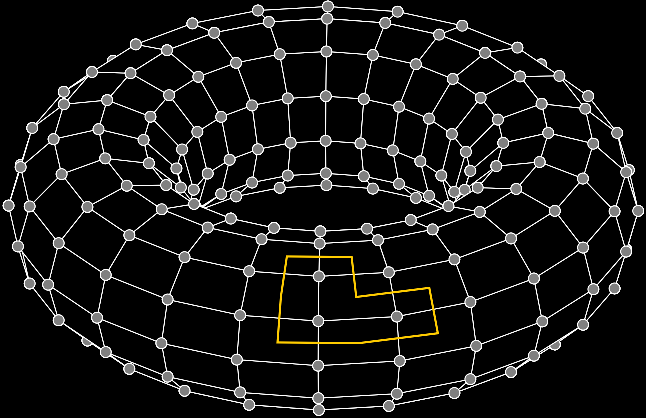

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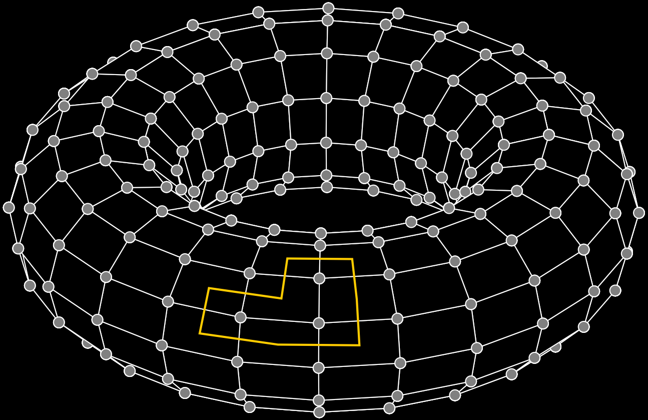

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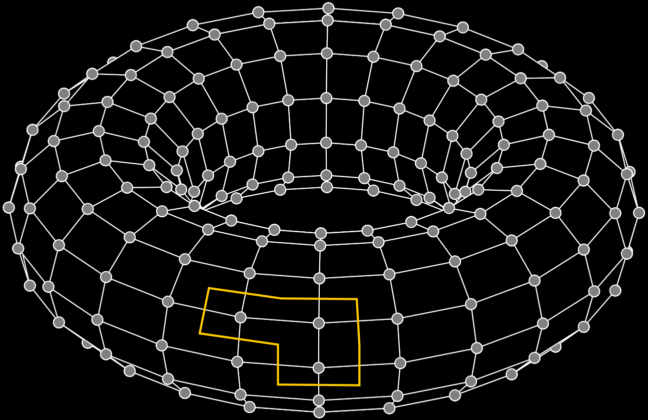

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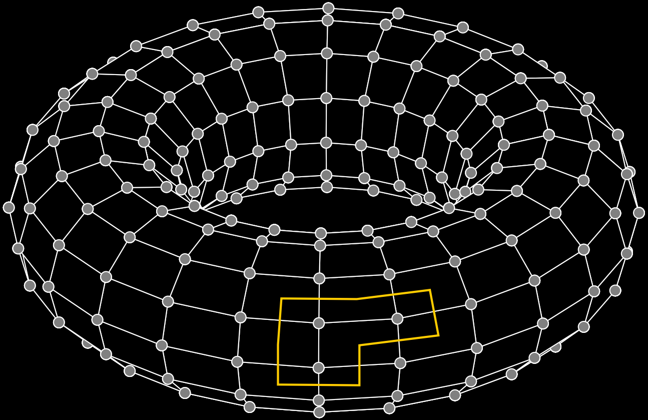

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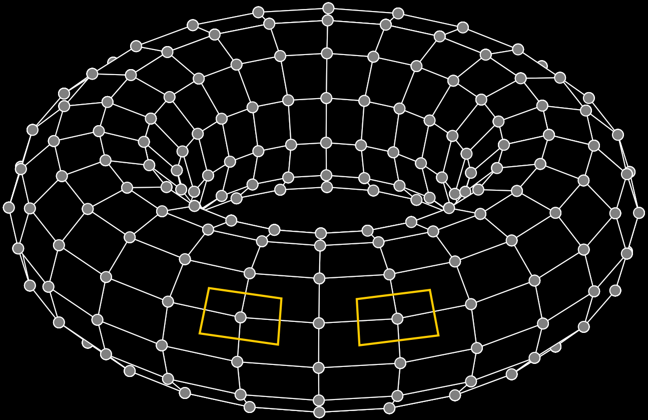

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We were able to compute poly_k for all $k \leq 32$.

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Unfortunately, 32 terms of $g(w)$ are still not enough.

Problem 1: $f(x)$ is not a (formal) power series

- Apply a change of variables proposed in 1941 by van der Waerden
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Problem 2: We cannot compute enough terms to guess $f(x)$

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Theorem (Kramers-Wannier 1941):

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Idea 2: change to a new variable which is invariant under $x \leftrightarrow x^*$

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in accordance with Onsager's formula.

And now?

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In physical terms y measures the "external field".

If we define

$$f(x, y) := \lim_{n,m \to \infty} \frac{\log(P_{n,m})}{nm}$$

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The function $f(x, y)$ knows some additional features about the physical system, in particular:

- "Magnetization": $M(x, y) = y\frac{d}{dy}f(x, y)$

Onsager announced (without proof) the formula

$$M(x,1) = \begin{cases} 0 & \text{if } x < 1+\sqrt{2} \\ \left(\frac{(x^2+1)(x^2-2x-1)(x^2+2x-1)}{(x-1)^4(x+1)^4}\right)^{1/8} & \text{if } x \geq 1+\sqrt{2} \end{cases}$$

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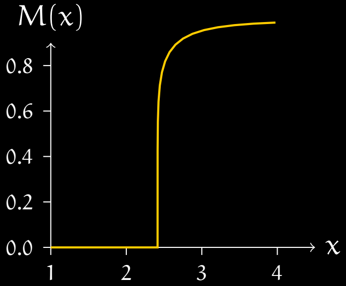

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For numerical values x, y, the limit $f(x,y)$ can be obtained numerically from the largest eigenvalue of the transfer matrix.

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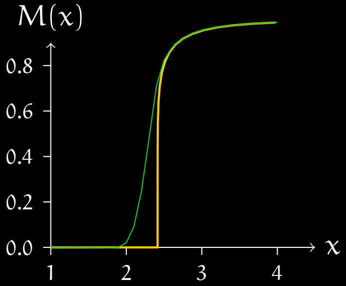
- We still can compute $P_{n,m}(x,y)$ by the transfer matrix method
- But van der Waerden and Kramers-Wannier break down

For numerical values x, y, the limit $f(x,y)$ can be obtained numerically from the largest eigenvalue of the transfer matrix.

Numerical differentiation gives approximations for $M(x)$.

Onsager announced (without proof) the formula

$$M(x,1) = \begin{cases} 0 & \text{if } x < 1+\sqrt{2} \\ \left(\frac{(x^2+1)(x^2-2x-1)(x^2+2x-1)}{(x-1)^4(x+1)^4}\right)^{1/8} & \text{if } x \geq 1+\sqrt{2} \end{cases}$$



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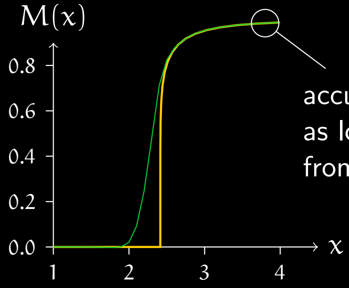

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$$(a_0 + a_1 x + \cdots + a_{10} x^{10}) M(x) + (b_0 + b_1 x + \cdots + b_{10} x^{10}) M'(x) = 0$$

with undetermined integer coefficients a_i, b_i.

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Unfortunately, our accuracy is not enough to find the equation.