Creative microscoping (based on joint work with Victor Guo)

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Ramanujan's list

S. Ramanujan (1914) recorded a list of 17 (hypergeometric) series for $1/\pi$, excluding

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3}(4n+1)\cdot(-1)^n = \frac{2}{\pi}$$

(provided by G. Bauer in 1859) but including

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3}(21460n+1123)\cdot\frac{(-1)^n}{882^{2n+1}} = \frac{4}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3}(26390n+1103)\cdot\frac{1}{99^{4n+2}} = \frac{1}{2\pi\sqrt{2}}.$$

D. Chudnovsky and G. Chudnovsky (1989):

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n(\frac{1}{2})_n(\frac{5}{6})_n}{n!^3}(545140134n+13591409)\cdot\frac{(-1)^n}{53360^{3n+2}} = \frac{3}{2\pi\sqrt{10005}}.$$

Hypergeometric notation

Though there is no q in those formulas (and when it shows up at some stage it is always assumed to be inside the unit disc, $|q| < 1$), it is easy to give now the (standard!) hypergeometric notation. We define

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j);$$

the q-Pochhammer symbol and its non-q-version are given by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{j=0}^{n-1}(a+j)$$

for non-negative integers n, so that

$$\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n \quad \text{and} \quad \lim_{q\to 1} \frac{(q;q)_\infty (1-q)^{1-a}}{(q^a;q)_\infty} = \Gamma(a).$$

The related q-notation also includes the q-numbers $[n] = [n]_q = \dfrac{1-q^n}{1-q}$.

Ramanujan-type formulas for $1/\pi^2$

I do not discuss proofs of the formulas on Ramanujan's list (first appeared in the work of the Borweins and of the Chudnovskys in the late 1980s). Instead, I indicate generalizations of those due to J. Guillera:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{n!^5}(20n^2+8n+1)\frac{(-1)^n}{2^{2n}} = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{n!^5}(820n^2+180n+13)\frac{(-1)^n}{2^{10n}} = \frac{128}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3(\frac{1}{4})_n(\frac{3}{4})_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}} = \frac{32}{\pi^2};$$

and there are several other instances found experimentally by Guillera and others. The only available proofs are "highly hypergeometric", with no "modular" argument known like for the classical formulas for $1/\pi$. There is also one formula of this shape for $1/\pi^3$ (with $n!^7$ in the denominator) and two for $1/\pi^4$ (with $n!^9$), all experimental.

Ramanujan-type supercongruences

A *supercongruence* is a p-adic congruence which happens to hold not just modulo a prime p but a higher power of p.

L. Van Hamme (1997) observed (and E. Mortenson proved in 2008) that

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^3}{n!^3}(4n+1)(-1)^n \equiv \left(\frac{-1}{p}\right) p \pmod{p^3} \quad \text{for } p > 2.$$

In 2009, I tested the entries on Ramanujan's list to find out (experimentally) that Van Hamme's pattern extends to all them; for example,

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(21460n+1123)\frac{(-1)^n}{882^{2n}} \equiv 1123\left(\frac{-1}{p}\right) p \pmod{p^3},$$

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(26390n+1103)\frac{1}{99^{4n}} \equiv 1103\left(\frac{-2}{p}\right) p \pmod{p^3}$$

for $p > 11$. Here $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Proofs of supercongruences

All Guillera's formulae for $1/\pi^2$ transform to congruences $\pmod{p^5}$; the formula for $1/\pi^3$ becomes a congruence $\pmod{p^7}$, while the formulas for $1/\pi^4$ turn into the congruences $\pmod{p^9}$.
My observations in 2009 were supported by proofs of the Van Hamme–Mortenson instance and also of

$$\sum_{n=0}^{p-1} \frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(20n+3)\frac{(-1)^n}{2^{2n}} \equiv 3\left(\frac{-1}{p}\right)p \pmod{p^3},$$

$$\sum_{n=0}^{p-1} \frac{(\frac12)_n^3(\frac14)_n(\frac34)_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}} \equiv 3p^2 \pmod{p^5}$$

for $p > 2$, using the algorithm of creative telescoping (due to Gosper, Wilf and Zeilberger), namely some suitable WZ pairs already found by Guillera. Those proofs were quite technical but worked well for several other WZ-supported examples.

q-supercongruences

Quite recently, and not surprisingly, the congruence story received a q-development. Victor Guo managed to extend the argument of those WZ-proofs (including the WZ pairs) to q-settings and proved several q-analogues of Ramanujan supercongruences. In his joint work with J.-C. Liu (2018), he also managed to construct two q-analogues of Ramanujan formulas for $1/\pi$, recycling the WZ pairs used for the congruences.

This was the moment for me to become interested in the development. Though my enthusiasm towards the discovery of the q-analogues for $1/\pi$ lasted short, with Guo we constructed other q-instances. Some of them were based on the (non-q) ideas of Guillera, and others on what we have learnt from Christian Krattenthaler. More specifically, Christian demonstrated that such identities are specialisation of cubic and quartic summations, very classical stuff from the q-Bible.

A particular example

One particluar case of our production with Guo is the identity

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n^2(q;q^2)_{2n}}{(q^2;q^2)_{2n}(q^6;q^6)_n^2}[8n+1] = \frac{(q^3;q^2)_\infty(q^3;q^6)_\infty}{(q^2;q^2)_\infty(q^6;q^6)_\infty},$$

which is a q-analogue of

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^2}{2^{8n}3^{2n}}(8n+1) = \sum_{n=0}^{\infty} \frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3\,9^n}(8n+1) = \frac{2\sqrt{3}}{\pi}.$$

We also checked experimentally the following q-supercongruence

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k^2(q;q^2)_{2k}}{(q^2;q^2)_{2k}(q^6;q^6)_k^2}[8k+1]q^{2k^2} \equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right) \pmod{[n]\Phi_n(q)^2}$$

for any n coprime with 6, where $\Phi_n(q) = \prod_{\substack{j=1\\(j,n)=1}}^{n}(1-e^{2\pi ij/n}q) \in \mathbb{Z}[q]$ are cyclotomic polynomials. This is now a theorem.

A more general congruence

In fact, our general congruence reads

$$\sum_{k=0}^{n-1} \frac{(aq;q^2)_k (q/a;q^2)_k (q;q^2)_{2k}}{(q^2;q^2)_{2k} (aq^6;q^6)_k (q^6/a;q^6)_k} [8k+1] q^{2k^2} \equiv q^{-(n-1)/2} [n] \left(\frac{-3}{n}\right),$$

modulo $[n](1-aq^n)(a-q^n)$, where both a and q are indeterminates. The denominators of the summand related to a are the factors $(aq^6;q^6)_{n-1}(q^6/a;q^6)_{n-1}$; their limits as $a \to 1$ are relatively prime to $\Phi_n(q)$, since n is coprime with 6. On the other hand, the limit of $(1-aq^n)(a-q^n)$ as $a \to 1$ has the factor $\Phi_n(q)^2$. Thus, the limiting case $a \to 1$ implies

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k^2 (q;q^2)_{2k}}{(q^2;q^2)_{2k} (q^6;q^6)_k^2} [8k+1] q^{2k^2} \equiv q^{-(n-1)/2} [n] \left(\frac{-3}{n}\right) \pmod{[n]\Phi_n(q)^2}$$

for any n coprime with 6.

A cubic summmation

To see the truth of the a-congruence modulo $a - q^n$ and $1 - aq^n$, in the summation formula

$$\sum_{k=0}^{\infty} \frac{(1-acq^{4k})(a;q)_k(q/a;q)_k(ac;q)_{2k}}{(1-ac)(cq^3;q^3)_k(a^2cq^2;q^3)_k(q;q)_{2k}}\, q^{k^2} = \frac{(acq^2;q^3)_\infty(acq^3;q^3)_\infty(aq;q^3)_\infty(q^2/a;q^3)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty(a^2cq^2;q^3)_\infty(cq^3;q^3)_\infty}$$

replace q with q^2, take $c = q/a$ and then aq for a:

$$\sum_{k=0}^{\infty} \frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}\,[8k+1]\, q^{2k^2} = \frac{(q^5;q^6)_\infty(q^7;q^6)_\infty(aq^3;q^6)_\infty(q^3/a;q^6)_\infty}{(q^2;q^6)_\infty(q^4;q^6)_\infty(aq^6;q^6)_\infty(q^6/a;q^6)_\infty}.$$

A partial congruence

We substitute $a = q^n$ (or $a = q^{-n}$) into this finding

$$\sum_{k=0}^{\infty} \frac{(aq;q^2)_k (q/a;q^2)_k (q;q^2)_{2k}}{(q^2;q^2)_{2k} (aq^6;q^6)_k (q^6/a;q^6)_k} [8k+1] q^{2k^2} = \frac{(q^5;q^6)_\infty (q^7;q^6)_\infty (aq^3;q^6)_\infty (q^3/a;q^6)_\infty}{(q^2;q^6)_\infty (q^4;q^6)_\infty (aq^6;q^6)_\infty (q^6/a;q^6)_\infty}.$$

Then the left-hand side terminates at $k = (n-1)/2$ and the terms for $(n-1)/2 < k \le n-1$ vanish, while the right-hand side assumes the form $q^{-(n-1)/2}[n]$ if $n \equiv 1 \pmod 3$, $-q^{-(n-1)/2}[n]$ if $n \equiv 2 \pmod 3$, and 0 if $3 \mid n$.

This means that the difference

$$\sum_{k=0}^{n-1} \frac{(aq;q^2)_k (q/a;q^2)_k (q;q^2)_{2k}}{(q^2;q^2)_{2k} (aq^6;q^6)_k (q^6/a;q^6)_k} [8k+1] q^{2k^2} - q^{-(n-1)/2}[n] \left(\frac{-3}{n}\right)$$

is divisible by $a - q^n$ and $a - q^{-n}$ in $\mathbb{Z}(q)$.

Thus, we have the congruence modulo $(a - q^n)(1 - aq^n)$.

Asymptotical analysis

On the other hand, we can also analyse the radial-asymptotics behaviour of

$$\sum_{k=0}^{\infty} \frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}\,[8k+1]\,q^{2k^2}$$
$$= \frac{(q^5;q^6)_\infty(q^7;q^6)_\infty(aq^3;q^6)_\infty(q^3/a;q^6)_\infty}{(q^2;q^6)_\infty(q^4;q^6)_\infty(aq^6;q^6)_\infty(q^6/a;q^6)_\infty}$$

at a primitive d-th root of unity $\zeta \neq 1$, for any $d \mid n$. Since $\gcd(n,6)=1$, so that $\gcd(d,6)=1$ as well, the right-hand side is seen to be *bounded* as $q \to \zeta$. Furthermore, denote by

$$c_q(k) = [8k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}\,q^{2k^2}$$

the k-th term of the sum on the left-hand side, and write the latter as

$$\sum_{\ell=0}^{\infty} c_q(\ell d)\sum_{k=0}^{d-1}\frac{c_q(\ell d+k)}{c_q(\ell d)}.$$

Radial limits

Consider the limit of

$$\sum_{\ell=0}^{\infty} c_q(\ell d) \sum_{k=0}^{d-1} \frac{c_q(\ell d+k)}{c_q(\ell d)}$$

as $q \to \zeta$ radially. For the terms of the internal sum,

$$\lim_{q\to\zeta} \frac{c_q(\ell d+k)}{c_q(\ell d)} = \frac{c_\zeta(\ell d+k)}{c_\zeta(\ell d)} = c_\zeta(k).$$

A simple analysis also shows that

$$\lim_{q\to\zeta} c_q(\ell d) = \frac{1}{2^{4\ell}} \binom{4\ell}{2\ell} \frac{(a\zeta;\zeta^2)_{\ell d}(\zeta/a;\zeta^2)_{\ell d}}{(a\zeta^6;\zeta^6)_{\ell d}(\zeta^6/a;\zeta^6)_{\ell d}} = \frac{1}{2^{4\ell}} \binom{4\ell}{2\ell}.$$

Since $\binom{4\ell}{2\ell} \sim 2^{4\ell}/\sqrt{2\pi\ell}$ as $\ell \to \infty$, hence

$$\sum_{\ell=0}^{\infty} \frac{1}{2^{4\ell}} \binom{4\ell}{2\ell} = \infty.$$

A microscopic behaviour

On the other hand, the limit of

$$\sum_{\ell=0}^{\infty} c_q(\ell d) \sum_{k=0}^{d-1} \frac{c_q(\ell d + k)}{c_q(\ell d)}$$

remains bounded as $q \to \zeta$. We conclude that

$$\sum_{k=0}^{d-1} c_\zeta(k) = 0 \quad \text{implying} \quad \sum_{k=0}^{n-1} c_\zeta(k) = \frac{n}{d} \sum_{k=0}^{d-1} c_\zeta(k) = 0.$$

In other words,

$$\sum_{k=0}^{n-1} c_q(k) \equiv 0 \pmod{\Phi_d(q)}$$

for any $d \mid n$.

The finale of proof

Since

$$\sum_{k=0}^{n-1} c_q(k) \equiv 0 \pmod{\Phi_d(q)}$$

is true for any $d \mid n$, and $\prod_{d|n} \Phi_d(q) = [n]$, we finally arrive at

$$\sum_{k=0}^{n-1} c_q(k) \equiv 0 \pmod{[n]} \equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right).$$

Thus, the desired congruence

$$\sum_{k=0}^{n-1} \frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}[8k+1]q^{2k^2} \equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right)$$

is also true modulo $[n]$.

A philosophy

The underlying q-extension

$$\sum_{k=0}^{\infty} \frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}\,[8k+1]\,q^{2k^2} = \frac{(q^5;q^6)_\infty(q^7;q^6)_\infty(aq^3;q^6)_\infty(q^3/a;q^6)_\infty}{(q^2;q^6)_\infty(q^4;q^6)_\infty(aq^6;q^6)_\infty(q^6/a;q^6)_\infty}.$$

of Ramanujan's formula for $1/\pi$ not only implies the latter identity

$$\sum_{n=0}^{\infty} \frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3\,9^n}\,(8n+1) = \frac{2\sqrt3}{\pi}$$

(when we take $a=1$ and $q\to1$) but also the supercongruences

$$\sum_{n=0}^{p-1} \frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3\,9^n}\,(8n+1) \equiv p\left(\frac{-3}{p}\right) \pmod{p^3}$$

for $p>3$ (when we take a to make the sum terminating and let q tend to any root of unity different from 1).

The same argument works to truncate the hypergeometric sum at $(p-1)/2$.

A microscope vs. telescope

Surprisingly enough, many (really many!) other q-hypergeometric identities from the q-Bible correspond to some meaningful (and, sometimes, unexpected) congruences from the radial limits. We have dubbed analysing this limiting behaviour as "microscoping", especially after observing that the creative telescoping fails to produce the required congruences (even when WZ pairs exist).

Here are two (out of many) instances that were conjectures for a substantial period of time, which are now established with our method with Guo:

$$\sum_{k=0}^{(p-1)/2} (3k+1)\,\frac{(\frac12)_k^5}{(1)_k^3(\frac54)_k^2} \equiv 0 \pmod{p^3} \quad \text{if } p\equiv 3 \pmod 4,$$

$$\sum_{k=0}^{p^s-1} (-1)^k(2dk+1)\frac{(\frac1d)_k^3}{k!^3} \equiv p^s(-1)^{(p^s-1)/d} \pmod{p^{s+2}} \quad \text{if } p^s\equiv 1 \pmod d,$$

$s\ge 1$ an integer.

Applications of the q-microscope

Examples of q-supercongruences coming out from Andrews' q-analogue of Gauss' ${}_2F_1(-1)$ sum, in the form

$$\sum_{k=0}^{\infty}\frac{(aq;q^2)_k(bq;q^2)_kq^{k^2+k}}{(q^2;q^2)_k(abq^4;q^4)_k}=\frac{(aq^3;q^4)_\infty(bq^3;q^4)_\infty}{(q^2;q^4)_\infty(abq^4;q^4)_\infty},$$

are

$$\sum_{k=0}^{m}\frac{(aq;q^2)_k(bq;q^2)_kq^{k^2+k}}{(q^2;q^2)_k(abq^4;q^4)_k}\equiv 0\ (\mathrm{mod}(1-aq^n)(1-bq^n))$$

and (after the rearrangement $a\to a^{-1}$, $b\to b^{-1}$ and $q\to q^{-1}$)

$$\sum_{k=0}^{m}\frac{(aq;q^2)_k(bq;q^2)_kq^{2k}}{(q^2;q^2)_k(abq^4;q^4)_k}\equiv 0\ (\mathrm{mod}(1-aq^n)(1-bq^n));$$

which are valid for any $n\equiv 3 \pmod 4$ and $m=n-1$ or $(n-1)/2$. In particular,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k}\equiv 0\ (\mathrm{mod}\,\Phi_n(q)^2).$$

for any such n.

Hyper(geometric) and super(congruence) wishes

Happy celebration, Christian!