Lattice Path Combinatorics and Interactions

Editors:
Cyril Banderier, Christian Krattenthaler, Michael Wallner

Séminaire Lotharingien de Combinatoire, volume 87B

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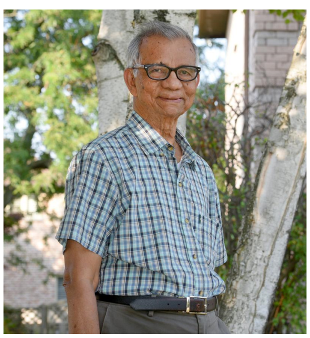
Preface, 10pp. *on Lattice Path Combinatorics and Applications*

PREFACE TO THE SPECIAL ISSUE FOR THE 2021 LATTICE PATH CONFERENCE

CYRIL BANDERIER, CHRISTIAN KRATTENTHALER, AND MICHAEL WALLNER

1. The Lattice Path Conference

1.1. **History of the conference.** Following an interdisciplinary interest for lattice paths after the publication in 1979 of the two books *Lattice Path Combinatorics with Statistical Applications* by Tadepalli Venkata Narayana and *Lattice Path Counting and Applications* by Sri Gopal Mohanty, Mohanty initiated a series of conferences on this topic.



Sri Gopal Mohanty, father of the Lattice Path Conference.

The first two events were held at McMaster University in 1984 and 1990, and then at the University of Delhi in 1994, University of Vienna in 1998, University of Athens in 2002, East Tennessee State University in 2007, University of Siena in 2010, Cal Poly Pomona University in 2015, Centre International de Rencontres Mathématiques in 2021. A new initiative started by dedicating the fourth conference in Wien to the memory of Germain Kreweras (1918–1998) and Tadepalli Venkata Narayana (1930–1987), both of whom made a significant contribution to the field. In the same spirit, the 2002 conference was dedicated to the memory of István Vincze (1912–1999). The 2015 conference was dedicated to Shreeram Shankar Abhyankar (1930–2013), Philippe Flajolet (1948–2011), and Lajos Takács (1924–2015).

For each conference there is the tradition to have a journal special issue where anyone could submit work related to lattice paths.[1] We shall come back to the current special issue in Section 2. Before, let us say a few words about the event itself.

1.2. **The 2021 Lattice Path Conference.** The 9th Lattice Path Conference was held in hybrid form (given the circumstances of the global Covid pandemic) during 21–25 June 2021 at the Centre International de Rencontres Mathématiques (CIRM) in Luminy, France. This event attracted 227 registered participants with affiliations from the "four corners of the world" from 34 different countries of which 24 attended in person. Moreover, about 40 more people attended casually online without being registered (we made the event fully open), and some colleagues even organized some watch parties in their university, so the real number of participants in the event will remain unknown.

[1]See the website of the Lattice Path Conference for links to these special issues and further information.

As visible via some details in the above photos,
the 2021 Lattice Path Conference was a hybrid conference.

The conference consisted of 27 invited talks and 22 accepted posters, leaving also time for scientific discussions. The presentations covered a wide range of topics, including

- algebraic combinatorics (Young tableaux, representation theory, symmetric polynomials, representation theory, ...),
- analytic combinatorics (kernel method, symbolic method, asymptotics, ...),
- enumerative combinatorics (walks in convex and non-convex cones, alternating sign matrices, generating functions, bijections, ...),
- probability theory (Brownian motion, Markov chains, random trees, ...),
- computer algebra (orthogonal polynomials, holonomy theory, ...),
- theoretical computer science (random generations, context-free grammars, ...),
- number theory (integer partitions, continued fractions, ...),
- theoretical physics (Liouville quantum gravity, square ice, ...).

The conference was a great success, with many stimulating talks and discussions. We would like to thank all of the participants, including the invited speakers and the attendees, for their contributions to the conference. We would also like to extend our gratitude to the staff at CIRM, who, like always, was warmly welcoming, even in this difficult period: CIRM reopened for our meeting after many months of closure due to the Covid containment and the interdiction of meetings. Finally, we would like to thank Kilian Raschel who was the main sponsor of this event via his ERC starting grant COMBINEPIC, which allowed us to help CIRM who financially suffered a lot of so many months of closure.

This event successfully brought together leading experts in the field and contributed to the development of new research directions. Several conjectures were even solved during the event! We hope that our special issue dedicated to this conference will serve as a valuable resource for researchers in combinatorics, probability theory, and statistical physics, and that they will inspire further research on lattice paths.

Let us now give the lists of organizers, speakers, and participants of this meeting.

1.3. **Scientific and organizing committee.**

- Cyril Banderier (CNRS, Université Sorbonne Paris Nord)
- Jehanne Dousse (CNRS, Université Lyon 1)
- Enrica Duchi (Université Paris Diderot)
- Christian Krattenthaler (Universität Wien)
- Greta Panova (University of Southern California)
- Kilian Raschel (CNRS, Université de Tours)
- Michael Wallner (TU Wien)

1.4. Invited talks.

George Andrews (Pennsylvania State University):
Schmidt type partitions and partition analysis

Andrei Asinowski (Alpen-Adria-Universität Klagenfurt):
Vectorial kernel method and lattice paths with patterns

Philippe Biane (CNRS, Université Paris-Est):
Mating of discrete trees and walks in the quarter-plane

Alin Bostan (INRIA, Saclay):
How to prove or disprove the algebraicity of a generating function using a computer

Mireille Bousquet-Mélou (CNRS, Université de Bordeaux):
Invariants for walks avoiding a quadrant

Timothy Budd (Radboud University):
Winding of simple walks on the square lattice

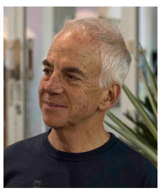
Philippe Di Francesco (University of Illinois at Urbana-Champaign and IPhT Saclay):
Triangular ice: combinatorics and limit shapes

Sergi Elizalde (Dartmouth College):
Counting lattice paths by the number of crossings and major index

Ilse Fischer (University of Vienna):
The alternating sign matrices/descending plane partitions relation: n+3 pairs of equivalent statistics

Ira Gessel (Brandeis University):
Redundant generating functions in lattice path enumeration

Vadim Gorin (MIT):
Addition of matrices at high temperature



Tony Guttmann (Melbourne University):
Extracting asymptotics from series coefficients

Nina Holden (ETH Zürich):
Random triangulations and bijective paths to Liouville quantum gravity

Mourad E.H. Ismail (University of Central Florida):
Orthogonal polynomials, moments, and continued fractions

Satya Majumdar (CNRS, Université Paris Sud):
Nonintersecting Brownian bridges in the flat-to-flat geometry

Olya Mandelshtam (University of Waterloo):
A Markov chain on tableaux that projects to a multispecies totally asymmetric zero range process

Irène Marcovici (Université de Lorraine):
Bijections between walks inside a triangular domain and Motzkin paths of bounded amplitude

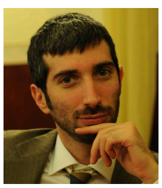
Stephen Melczer (University of Pennsylvania):
Lattice walks and analytic combinatorics in several variables

Robin Pemantle (University of Pennsylvania):
Generating function technologies: applications to lattice paths

Bruno Salvy (INRIA / ENS Lyon):
Computation of tight enclosures for Laplacian eigenvalues

Michael Singer (North Carolina State University):
Differentially algebraic generating series for walks in the quarter plane

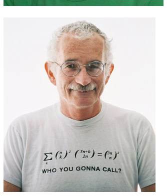
Perla Sousi (University of Cambridge):
The uniform spanning tree in 4 dimensions



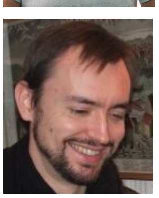
Andrea Sportiello (CNRS, Université Paris Nord):
Boltzmann sampling in linear time: irreducible context-free structures

Xavier Viennot (CNRS, Université de Bordeaux):
Heaps and lattice paths

Karen Yeats (University of Waterloo):
Łukasiewicz walks and generalized tandem walks



Doron Zeilberger (Rutgers University):
Using symbolic dynamical programming in lattice paths combinatorics



Paul Zinn-Justin (Melbourne University):
Generalized pipe dreams and lower-upper scheme

A recording of all the talks can be found at `https://lipn.fr/~banderier/LPC/2021/`.

1.5. Accepted posters.

- Ault Shaun, Charles Kicey: *From lattice paths to standard Young tableaux*
- Cyril Banderier, Marie-Louise Lackner, Michael Wallner: *Latticepathology and symmetric functions*
- Nicholas Beaton: *Walks obeying two-step rules on the square lattice*
- Swee Hong Chan, Igor Pak, Greta Panova: *Log-concavity in posets and random walks*
- Sergey Dovgal, Mohamed Lamine Lamali, Philippe Duchon: *A phase transition in non-deterministic walks with two or more variables*
- Andrew Elvey Price: *Enumeration of walks with small steps by winding angle*
- Rigoberto Flórez, José L. Ramírez, Fabio A. Velandia, Diego Villamiza: *Restricted Dyck paths*
- Xi Chen, Bishal Deb, Alexander Dyachenko, Tomack Gilmore, Alan Sokal: *Coefficientwise total positivity of some matrices defined by linear recurrences*
- Hans Höngesberg: *Weight-preserving bijections between integer partitions and a class of alternating sign trapezoids*
- Heba Ayeda, David Beecher, Alan Krinik, Jeremy J. Lin, David Perez, Thuy Vu Dieu Lu, Weizhong Wong: *Lattice paths with alternating probabilities*
- Josef Küstner, Michael Schlosser, Meesue Yoo: *Lattice paths and negatively indexed weight-dependent binomial coefficients*
- Florian Lehner, Christian Lindorfer, Wolfgang Woess: *The language of self-avoiding walks*
- Satya Majumdar, Francesco Mori, Gregory Schehr: *Distribution of the time between maximum and minimum of random walks*
- Stéphane Ouvry, Alexios Polychronakos, Shuang Wu: *Algebraic area counting for lattice closed random walks*
- Alan Krinik, Gerardo Rubino: *The exponential-dual matrix method: applications to Markov chain analysis*
- Andrei Asinowski, Benjamin Hackl, Sarah Selkirk: *Down-step statistics in generalized Dyck paths*
- Myrto Kallipoliti, Robin Sulzgruber, Elini Tzanaki: *Patterns in Shi tableaux and Dyck paths*
- Malvina Vamvakari: *On q-order statistics*
- Florian Aigner, Gabriel Frieden: *qRSt: A probabilistic Robinson–Schensted correspondence for Macdonald polynomials*
- Quang-Nhat Le, Sinai Robins, Christophe Vignat, Tanay Wakhare: *A continuous analogue of lattice path enumeration*
- Jisun Huh, Sun-Young Nam, Meesue Yoo: *LLT polynomials in a nutshell: on Schur expansion of LLT polynomials*
- Benjamin De Bruyne, Satya Majumdar, Gregory Schehr: *Generating discrete-time constrained random walks.*

The poster session was organized on the online platform Gather.town, where all the posters can still be perused.

1.6. **The Hotel Latticepathologia escape game.** The image below shows a part of the "Hotel Latticepathologia" (a virtual location designed by Cyril Banderier, Jehanne Dousse, and Michael Wallner), where the poster session was taking place.

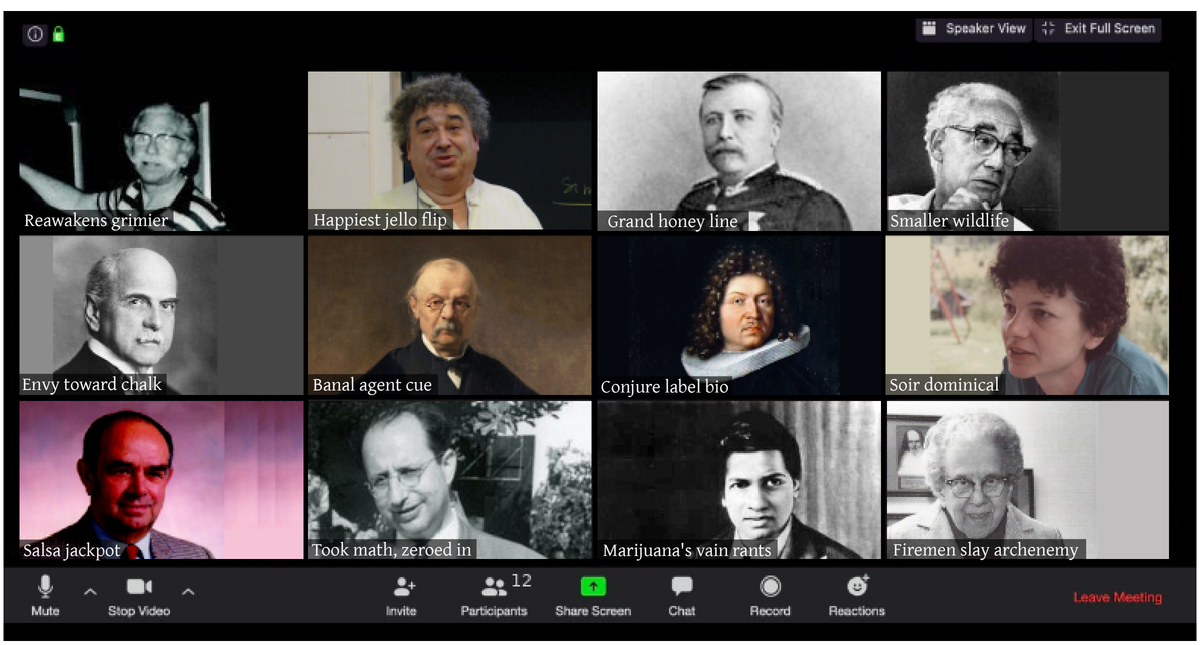
This virtual place gave the online participants the opportunity to socialize and to participate in an escape game! In its first part, you have to identify a mysterious mathematician in each room in the Hotel Latticepathologia (`https://tinyurl.com/s6njbsku`), using clues scattered throughout. Will you succeed?

In its second part, you have to solve a collection of puzzles designed in collaboration with Vivien Ripoll; see `https://lipn.fr/~cb/LPC/2021/Puzzles/`. It includes a musical concert by 9 conference participants. Don't miss it!

Below is a sample puzzle from the escape game. Can you solve it?

Confusing Dream

*I had the strangest dream last night: a Zoom meeting with some of our mathematical heroes. Luckily I was able to take a screenshot before waking up! They look utterly **confused** by the situation, though I note they all have one extraordinary feature.*



They were discussing the discovery of a new lattice path. I managed to copy it there: **https://□□.□□/□□□□□□□□**

1.7. **Registered participants.** Most of the 227 registered participants were online. A few of them had the opportunity to enjoy CIRM's beautiful surroundings.

List of registered participants: David Adame-Carrillo, Mohammed Ageel, Florian Aigner, Marie Albenque, Seamus Albion, Ian Alevy, Irha Ali, George Andrews, Omer Angel, Margaret Archibald, David Ash, Andrei Asinowski, Shaun Ault, Jean-Christophe Aval, Heba Ayeda, Arvind Ayyer, Beáta Bényi, Cyril Banderier, Josaphat Baolahy, Elena Barcucci, Jean-Luc Baril, Erik Bates, Nicholas Beaton, David Beecher, Chiheb Ben Bechir, Sudip Bera, Olivier Bernardi, Antonio Bernini, David Bevan, Philippe Biane, Arthur Blanc-Renaudie, Aubrey Blecher, Alin Bostan, Mireille Bousquet-Mélou, Cédric Boutillier, Jérémie Bouttier, Timothy Budd, Théophile Buffière, Ariane Carrance, Giulio Cerbai, Swee Hong Chan, Linxiao Chen, Shaoshi Chen, Frédéric Chyzak, Lapo Cioni, Alice Contat, Michael Coopman, Sylvie Corteel, Logan Crew, Cesar Cuenca, Nicolas Curien, Stéphane Dartois, Benjamin De Bruyne, Bishal Deb, Nachum Dershowitz, Hiranya Kishore Dey, Philippe Di Francesco, Lucia Di Vizio, Ruiwen Dong, Robert Donley, Jehanne Dousse, Sergey Dovgal, Thomas Dreyfus, Michael Drmota, Enrica Duchi, Dennis Eichhorn, Sergi Elizalde, Andrew Elvey Price, Sen-Peng Eu, Wenjie Fang, Valentin Féray, Luca Ferrari, Ilse Fischer, Rigoberto Flórez, Luis Fredes, Gabriel Frieden, Éric Fusy, Ira Gessel, Sudhir Ghorpade, Juan Gil, Tomack Gilmore, Vadim Gorin, Adam Gregory, Tony Guttmann, Hans Höngesberg, Benjamin Hackl, Aliakbar Haghighi, Eva-Maria Hainzl, Charlotte Hardouin, Kilian Hermann, Clemens Heuberger, Pawel Hitczenko, Hung Hoang, Nina Holden, Sam Hopkins, Yueyun Hu, Justin Hua, Mourad E. H. Ismail, Svante Janson, Helen Jenne, Frédéric Jouhet, Josef Küstner, Wonwoo Kang, Manuel Kauers, Rinat Kedem, Ghizlane Kettani, Mikhail Khristoforov, Charles Kicey, Donghyun Kim, Sergey Kirgizov, Victor Kleptsyn, Arnold Knopfmacher, Isaac Konan, Irina Kourkova, Christian Krattenthaler, Alan Krinik, Nishu Kumari, Raunak Kundagrami, Florian Lehner, Helder Lima, Zhicong Lin, Christian Lindorfer, Martin Loebl, Baptiste Louf, Torsten Mütze, Satya Majumdar, Pritam Majumder, Olya Mandelshtam, Jean-Francois Marckert, Irène Marcovici, Barbara Margolius, Hana Melánová, Stephen Melczer, Laurent Menard, Sri Gopal Mohanty, Derrick Mohlala, Francesco Mori, Lukas Nabergall, Philippe Nadeau, Victor Nador, Mehdi Naima, Hiroshi Naruse, Andreas Nessmann, David Nguyen, Hadrien Notarantonio, Soichi Okada, Stéphane Ouvry, J. E. Paguyo, Nimisha Pahuja, Greta Panova, Jay Pantone, Eveliina Peltola, Robin Pemantle, Karol Penson, Leonid Petrov, Renzo Pinzani, Thomas Prellberg, Helmut Prodinger, Sanjay Ramassamy, José Ramírez, Kilian Raschel, Vivien Ripoll, Tom Roby, Martin Rubey, Gerardo Rubino, Tiadora Ruza, Nasser Saad, Bruce Sagan, Manjil Pratim Saikia, Bruno Salvy, Yoshio Sano, Gilles Schaeffer, Gregory Schehr, Michael Schlosser, Jeanne Scott, Blair Seidler, Sarah Selkirk, Timo Seppalainen, Michael F. Singer, Alexandros Singh, Erik Slivken, Rebecca Smith, Alan Sokal, U-Keun Song, Perla Sousi, Andrea Sportiello, Richard Stanley, Dennis Stanton, Benedikt Stufler, Adrian Tanasa, Benjamin Terlat, Vasu Tewari, Paul Thévenin, Mikhail Tikhonov, Jordan Tirrell, Jessica Tomasko, Joonas Turunen, Eleni Tzanaki, Malvina Vamvakari, Roger Van Peski, Zoé Varin, Ekaterina Vassilieva, Fabio Velandia, Xavier Viennot, Christophe Vignat, Diego Villamizar, Michael Voit, Trung Vu, David Wahiche, Tanay Wakhare, Michael Wallner, Harriet Walsh, Guoliang Wang, Sebastian Wild, Mark Wilson, Peter Winkler, Wolfgang Woess, Elaine Wong, Zaidan Wu, Karen Yeats, Meesue Yoo, Sergey Yurkevich, Doron Zeilberger, Noam Zeilberger, Jiang Zeng, Yan Zhuang, Paul Zinn-Justin.

2. Our special issue in the "Séminaire Lotharingien de Combinatoire"

The "Séminaire Lotharingien de Combinatoire" is an international biannual seminar, cofounded in 1980 by Dominique Foata (Strasbourg), Adalbert Kerber (Aachen and Bayreuth), and Volker Strehl (Erlangen). The name of the seminar comes from the fact that these cities were almost covered by Lotharingia, a part of the Carolingian Empire. In 1994, an eponymous journal was launched and quickly gathered a wider international audience, welcoming also articles independent of any participation to the actual seminar. On some occasions, the journal has had special issues dedicated to Festschriften or conference proceedings. All volumes are freely accessible at `https://www.mat.univie.ac.at/~slc/`.

We are pleased to add another volume to this collection of the journal of the Séminaire Lotharingien de Combinatoire. Our volume, dedicated to the themes of the 2021 Lattice Path Conference, contains the following contributions:

1. Helmut Prodinger: *A walk in my lattice path garden*
2. Anthony J. Guttmann and Václav Kotěšovec: *A numerical study of L-convex polyominoes and 201-avoiding ascent sequences*
3. David W. Ash: *Introducing DASEP: the doubly asymmetric simple exclusion process*
4. Stéphane Ouvry and Alexios P. Polychronakos: *Signed area enumeration for lattice paths*
5. Aubrey Blecher and Arnold Knopfmacher: *Left-to-right maxima in Dyck paths*
6. Rigoberto Flórez, Toufik Mansour, José L. Ramírez, Fabio A. Velandia, and Diego Villamizar: *Restricted Dyck paths on valleys sequence*
7. Sergi Elizalde: *Counting lattice paths by crossings and major index II: tracking descents via two-rowed arrays*
8. Jang Soo Kim and Dennis Stanton: *Three families of q-Lommel polynomials*
9. Malvina Vamvakari: *On q-order statistics*
10. Thomas Dreyfus: *Differential algebraic generating series of weighted walks in the quarter plane*
11. Rodolphe Garbit and Kilian Raschel: *The generating function of the survival probabilities in a cone is not rational*
12. Rafik Aguech, Asma Althagafi, and Cyril Banderier: *Height of walks with resets, the Moran model, and the discrete Gumbel distribution*

We would like to thank all the authors for their patience during the interactions with the editors, which (hopefully!) resulted in a pleasantly polished volume... Last but not least, we would like to also thank all the referees for their excellent work. We hope that the reader will enjoy this volume and will eventually be motivated to contribute to the next Lattice Path Conference!

3. A panorama on lattice paths

Let us end this preface with two beautiful photos taken during a random walk in the proximity of the conference location (warm thanks to Andreas Nessmann and Sergey Dovgal for sharing them with us).

A path… what else could end our preface?

Prelude

A famous lattice path related to Bach. . .

Séminaire Lotharingien de Combinatoire **87B** (2023) *Special issue for the* 9th *International Conference*
Article #1, 49pp. *on Lattice Path Combinatorics and Applications*

Jackson Park, Windsor, Ontario
Wikimedia Commons

A WALK IN MY LATTICE PATH GARDEN

HELMUT PRODINGER[1]

[1]Stellenbosch University and NITheCS (National Institute for Theoretical and Computational Sciences), South Africa; https://www.math.tugraz.at/~prodinger

Abstract. Various lattice path models are reviewed. The enumeration is done using generating functions. A few bijective considerations are woven in as well. The kernel method is often used. Computer algebra was an essential tool. Some results are new, some have appeared before, but all are interesting.

The lattice path models considered are Hoppy walks and several models involving skew Dyck paths, Schröder paths, hex-trees, decorated ordered trees, multi-edge trees, etc., related to the sequence A002212 in the On-line Encyclopedia of Integer Sequences (created by N. Sloane). Weighted unary-binary trees also occur and we there improve on our old paper on Horton–Strahler numbers [P. Flajolet and H. Prodinger, 1986], by using a different substitution. Some material on Motzkin numbers and paths is also discussed. Some new results on 'Deutsch paths' in a strip are included as well. During the Covid period, I spent much time with this beautiful concept that I dare to call Deutsch paths, since Emeric Deutsch stands at the beginning with a problem that he posted in the American Mathematical Monthly some 20 years ago. Peaks and valleys, studied by Rainer Kemp 40 years ago under the names MAX-turns and MIN-turns, are revisited with a more modern approach, streamlining the analysis, relying on the 'subcritical case' (named so by Philippe Flajolet), the adding a new slice technique and once again the kernel method.

Keywords: Skew Dyck paths, decorated Dyck paths, generating functions, Motzkin paths, kernel method.

Contents

1. Introduction

Around 20 years ago I published a collection of examples about applications of the kernel method in the present journal. The success of this enterprise was unexpected and came as a very pleasant surprise. My current plan is to present again a collection of subjects, loosely related as they all have a lattice path flavour (trees are also allowed in my private book when lattice paths are mentioned). The subjects cover my last 2 or 3 years of research; some results were only posted on `arXiv`, and some are completely new.

As in the predecessor paper, the kernel method plays a role again, but also analytic techniques like singularity analysis and Mellin transform, as well as bijective results.

Let me emphasize that this is *not* a survey about lattice path enumeration, but a *personal survey*, that is, a very *personal* account of some of *my* interests and recent activities. Of course, I hope that some people will like it, and that some readers will even contact me to further investigate the models presented in this article. It is *not* a long novel, I organized the material rather as a collection of short stories, roughly in the order as I worked on them.

While some methods might be intimidating for the uninitiated, I tried to provide an accessible introduction, also by explaining and simplifying older proofs and applying old and powerful tools to many beautiful and attractive up-to-date problems. Even with traditional bricks, traditional timber, traditional paint one can build a beautiful house, a beautiful home, a beautiful garden!

People who want to properly learn the subject (via different approaches all keeping some intimate links with enumerative combinatorics) can go to Christian Krattenthaler's survey [29], or the older books by Mohanty [31] and Narayana [32] or to many deep and sophisticated articles by Mireille Bousquet-Mélou and Philippe Flajolet. I apologize to all those that I did not mention although I should have.

The sections are arranged in roughly the form they were conceived. They have their own little introductions so that a casual reader can look at various parts at his/her leisure. Like a walk in a real garden, you can just cross it or you can explore, in butterfly style, various flowers and other beauties.

I spent probably a year of work on that project, and so it can be hoped that the interested enthusiast can learn something from it. In the first place, I would like to mention the large variety of combinatorial objects, perhaps not known to everybody. Then I mention the "right" substitution of auxiliary variables that one finds in various places of the paper. Without them, certain computations/considerations would be (almost) impossible, in particular since computer algebra systems are not good to deal with roots. There are a few asymptotic calculations woven in, both related to my old friends, the height of trees and the register function (Horton–Strahler numbers). As mentioned, I do not distinguish much (cum grano salis) between lattice paths and trees. There are also a few attractive bijections to be found. I spent myriads of hours with Maple to guess the quantities of interest, especially in the context of Deutsch paths in a strip and bounded marked trees. Many explicit formulæ were found with the kernel method, often in combination with the adding-a-new-slice procedure.

My motivation for this work, apart from numerous improvements related to the existing literature, is a personal credo about how to proceed in various instances of this personal survey. I like generating functions and explicit expressions and I am not so thrilled by bounds and estimates. I know that I open a can of worms with such a statement, but nobody is forced to proceed like me. But those who do can take home something beautiful. It is clear to me that many researchers will not agree with 'beautiful' and replace it with something else. Fortunately, different views are allowed, and I mention freely that Philippe Flajolet's work, especially his younger period, had a lasting effect on the way I consider things. I should also mention Emeric Deutsch here, who is responsible for skew Dyck paths, marked trees, Deutsch paths and more.

So I send this *swan song* into the world. May it encourage younger people to pick up some stones in the garden in the hope that they are actually rough diamonds.

2. Hoppy walks

This section (which is extending our unpublished preprint arXiv:2009.13474) can be seen as a warm-up, introducing all the typical techniques, before a much longer section.

Deng and Mansour [10] introduce a rabbit named Hoppy and let him move according to certain rules. While the story about Hoppy is charming and entertaining, we do not need this here and move straight ahead to the enumeration issues. Eventually, the enumeration problem is one about k-Dyck paths ($k \geq 1$). The up-steps are $(1, k)$ and the down-steps are $(1, -1)$. The model that has $(1, 1)$ as up-step and the down-step are $(1, -k)$ will also be called k-Dyck paths.

The question is about the length of the last sequence of down-steps (shown in red in Figure 1). Or, phrased differently, how many k-Dyck paths end on level j with an up-step. Note that such paths have length $n = m + km - j$. The recent paper [49] consider these paths, although without the restriction that the last step must be an up-step.

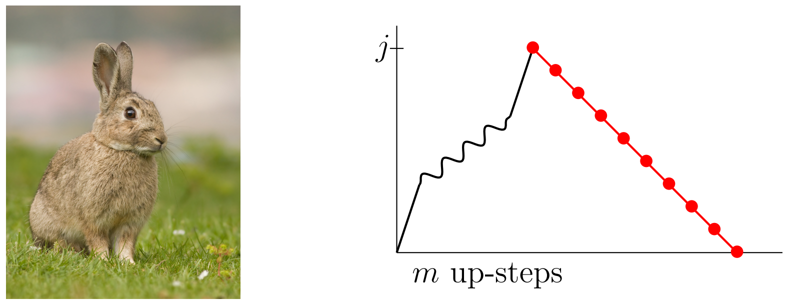


Figure 1. The rabbit Hoppy thinking in the lattice path garden…on the number of final down-steps in paths with steps $(1,-1)$ and $(1,k)$.

Source: https://en.wikipedia.org/wiki/File:Oryctolagus_cuniculus_Tasmania_2.jpg

The original description of Deng and Mansour is a reflection of Figure 1, with up-steps of size 1 and down-steps of size $-k$, but we prefer it as given here, since we are going to use the adding-a-new-slice method; see [18, 38]. A slice is here a run of down-steps, followed by an up-step. So, for each path, one begin with an up-step, and then $m-1$ new slices are added. We keep track of the level after each slice, using a variable u. The variable z is used to count the number of up-steps.

Deng and Mansour work out a formula which comprises $O(m)$ terms. For our walks, we obtain a more compact sum of only $O(j)$ terms (recall that j is the level of the last point).

We start with the following substitution which encodes that one adds a new slice

$$u^j \longrightarrow z \sum_{0\le h\le j} u^{h+k} = \frac{zu^k}{1-u}(1-u^{j+1}).$$

Now let $F_m(z,u)$ be the generating function of paths having m runs of down-steps. The substitution leads to

$$F_{m+1}(z,u) = \frac{zu^k}{1-u}F_m(z,1) - \frac{zu^{k+1}}{1-u}F_m(z,u), \quad F_0(z,u) = zu^k.$$

Let $F = \sum_{m\ge 0} F_m$, so that we do not care about the number m anymore; then

$$F(z,u) = zu^k + \frac{zu^k}{1-u}F(z,1) - \frac{zu^{k+1}}{1-u}F(z,u),$$

or, more conveniently,

$$F(z,u)(1-u+zu^{k+1}) = zu^k(1-u+F(z,1)).$$

The equation $1-u+zu^{k+1}=0$ (the so-called kernel equation) is famous when enumerating $(k+1)$-ary trees (or k-Dyck paths). Its relevant combinatorial solution (also the only one being analytic at the origin) is

$$\overline{u} = \sum_{\ell\ge 0} \frac{1}{1+\ell(k+1)}\binom{1+\ell(k+1)}{\ell} z^\ell.$$

Now, since $u := \overline{u}$ cancels the kernel and the left-hand side, $(u-\overline{u})$ must be a factor of the right-hand side (which is a polynomial in u); this gives

$$zu^k(1-u+F(z,1)) = -zu^k(u-\overline{u}).$$

Cancelling the kernel equation is thus a method which brings additional equations, allowing us to identify $F(z,1)$, and then $F(z,u)$ which is given by

$$F(z,u) = zu^k \frac{\overline{u}-u}{1-u+zu^{k+1}}.$$

The first factor has even a combinatorial interpretation, as a description of the first step of the path. It is also clear from this that the level reached is at least k after each slice. We do not care about the factor zu^k anymore, as it produces only a simple shift. The main interest is now how to get to the coefficients of

$$\frac{\overline{u}-u}{1-u+zu^{k+1}}$$

in an efficient way. First we deal with the denominators $(j \geq k+1)$

$$S_j := [u^j]\frac{1}{1-u+zu^{k+1}} = \sum_{0\leq m\leq j/k} (-1)^m \binom{j-km}{m} z^m.$$

One way to see this formula is to prove by induction that the sums S_j satisfy the recursion $S_j - S_{j-1} + zS_{j-k-1} = 0$ and initial conditions $S_0 = \cdots = S_k = 1$. In [49] such expressions also appear as determinants. Summarizing,

$$\frac{1}{1-u+zu^{k+1}} = \sum_{m\geq 0} (-1)^m z^m \sum_{j\geq km} \binom{j-km}{m} u^j.$$

Now we read off coefficients:

$$[u^j]\frac{\overline{u}}{1-u+zu^{k+1}} = \sum_{0\leq m\leq j/k} (-1)^m \binom{j-km}{m} z^m \sum_{\ell\geq 0} \frac{1}{1+\ell(k+1)} \binom{1+\ell(k+1)}{\ell} z^\ell$$

and further

$$[z^n][u^j]\frac{\overline{u}}{1-u+zu^{k+1}} = \sum_{0\leq m\leq j/k} \frac{(-1)^m}{1+(n-m)(k+1)} \binom{j-km}{m} \binom{1+(n-m)(k+1)}{n-m}.$$

The final answer to the Deng–Mansour enumeration (without the shift) is

$$\left(\sum_{0\leq m\leq j/k} \frac{(-1)^m}{1+(n-m)(k+1)} \binom{j-km}{m} \binom{1+(n-m)(k+1)}{n-m} \right) - (-1)^n \binom{j-1-kn}{n}. \tag{2.1}$$

If one wants to take care of the factor zu^k as well, one needs to do the replacements $n \to n+1$ and $j \to j+k$ in the formula just derived. That enumerates then the k-Dyck paths ending at level j after n up-steps, where the last step is an up-step.

The main contribution of this section is this equation (2.1); let us now discuss three variations about these walks.

An application. In [1] the authors considered the total number of down-steps of the last down-run in all k-Dyck paths. For $k=2,3,4$, this corresponds to the sequences A334680, A334682, A334719 in the OEIS[1], respectively. So, if the path ends on level j, the contribution to the total is j.

All we have to do here is to differentiate

$$F(z,u)=zu^k\frac{\overline{u}-u}{1-u+zu^{k+1}}$$

with respect to u, and then replace u by 1. The result is

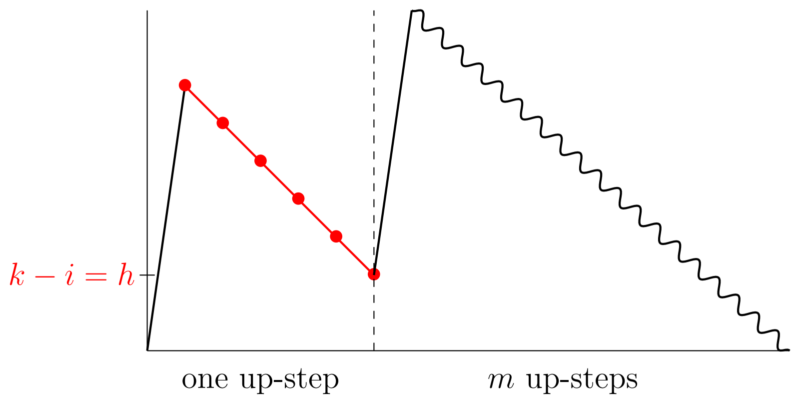
$$\frac{\overline{u}}{z}-\overline{u}-\frac{1}{z},$$

and the coefficient of z^m therein is

$$\frac{1}{1+(m+1)(k+1)}\binom{1+(m+1)(k+1)}{m+1}-\frac{1}{1+m(k+1)}\binom{1+m(k+1)}{m}.$$

The bivariate generating function does this enumeration cleanly and quickly.

Hoppy's early adventures. Now we investigate what Hoppy does after his first up-step; he might follow with $0,1,\dots,k$ down-steps. Eventually, we want to sum all these steps (red in the picture).



A new slice is now an up-step, followed by a sequence of down-steps. The substitution of interest is:

$$u^i\to z\sum_{0\le h\le i+k}u^h=\frac{z}{1-u}-\frac{zu^{i+k+1}}{1-u}.$$

Furthermore

$$F_{h+1}(z,u)=\frac{z}{1-u}F_h(z,1)-\frac{zu^{k+1}}{1-u}F_h(z,u),$$

and $F_0=u^h$, the starting level. We have

$$H(z,u)=\sum_{h\ge0}F_h(z,u)=u^h+\frac{z}{1-u}H(z,1)-\frac{zu^{k+1}}{1-u}H(z,u)$$

or

$$H(z,u)(1-u+zu^{k+1})=u^h(1-u)+zH(z,1).$$

[1]The On-line Encyclopedia of Integer Sequences (OEIS) is a database available at https://oeis.org/.

Plugging in $\overline{u}$ into the right-hand side gives 0, thus one has

$$zH(z,1) = -\overline{u}^h(1-\overline{u}),$$

which itself implies

$$H(z,u) = \frac{u^h(1-u) - \overline{u}^h(1-\overline{u})}{1-u+zu^{k+1}}.$$

But we only need $H(z,0)$, since we return to the x-axis at the end:

$$H(z,0) = [\![h=0]\!] + \overline{u}^{h+1} - \overline{u}^h.$$

(The Iverson notation $[\![P]\!]$ which is 1 when P is true and 0 otherwise is the notation of choice for combinatorialists, [21].) The total contribution of red steps is then

$$k + \sum_{h=0}^{k}(k-h)(\overline{u}^{h+1} - \overline{u}^h) = \sum_{h=1}^{k}\overline{u}^h;$$

the coefficient of z^m in this is the total contribution. Since $\overline{u} = 1 + z\overline{u}^{k+1}$, there is the further simplification

$$-1 + \frac{1}{z} + \frac{1}{1-\overline{u}} = \sum_{m\ge 1}\frac{k}{m+1}\binom{(k+1)m}{m}z^m.$$

Indeed, for $m \ge 1$, we have

$$\begin{aligned}
[z^m]\Big(-1 + \frac{1}{z} + \frac{1}{1-\overline{u}}\Big) &= -[z^m]\frac{1}{z\overline{u}^{k+1}} \\
&= -[z^{m+1}]\sum_{\ell\ge 0}\frac{-(k+1)}{(k+1)\ell-(k+1)}\binom{(k+1)\ell-(k+1)}{\ell}z^\ell \\
&= [z^{m+1}]\sum_{\ell\ge 0}\frac{(k+1)}{(k+1)(\ell-1)}\binom{(k+1)(\ell-1)}{\ell}z^\ell \\
&= \frac{(k+1)}{(k+1)m}\binom{(k+1)m}{m+1} = \frac{k}{m+1}\binom{(k+1)m}{m}.
\end{aligned}$$

We did not expect such a simple answer $\frac{k}{m+1}\binom{(k+1)m}{m}$ to this question about Hoppy's early adventures! This analysis of Hoppy's early adventures covers sequences `A007226`, `A007228`, `A124724` of [52], with references to [1].

Hoppy walks into negative territory. Hoppy is now adventurous and allows himself to go to level -1 as well, but not deeper. The setup with generating functions is the same, but the u-variable counts the level relative to the -1 level, so this has to be corrected later.

Hoppy, after some initial frustration discovers that he can now start with an up-step or a down-step. First, let us start Hoppy with an up-step:

$$F(z,u) = zu^{k+1} + \frac{zu^k}{1-u}F(z,1) - \frac{zu^{k+1}}{1-u}F(z,u),$$

which we conveniently rewrite as

$$F(z,u)(1-u+zu^{k+1}) = zu^{k+1}(1-u) + zu^kF(z,1).$$

Since the left-hand side cancels for $u = \overline{u}$, we get that $u = \overline{u}$ is also cancelling the right-hand side (which is a polynomial in u), and this implies that

$$\overline{u}(1-\overline{u}) + F(z,1) = 0.$$

This finally gives

$$F(z,u) = \frac{zu^k}{1-u+zu^{k+1}}\Big(u(1-u) - \overline{u}(1-\overline{u})\Big).$$

But Hoppy can also start with a downstep. So we have to add the result of the previous computation, and get finally

$$G(z,u) = \frac{zu^k}{1-u+zu^{k+1}}\Big(u(1-u) - \overline{u}(1-\overline{u})\Big) + \frac{zu^k}{1-u+zu^{k+1}}\big(\overline{u} - u\big)$$

or better

$$G(z,u) = \frac{zu^k}{1-u+zu^{k+1}}\big(\overline{u}^2 - u^2\big).$$

Now we need

$$\frac{\partial}{\partial u}G(z,1) - G(z,1).$$

This subtraction is necessary, since the contribution of u^j is not j as before but only $j-1$. The result is

$$\frac{\overline{u}^2}{z} - 2\overline{u}^2 - \frac{1}{z}.$$

Hoppy knows that $\overline{u}^d$ has beautiful coefficients:

$$\overline{u}^d = \sum_{\ell\ge 0}\binom{d-1+(k+1)\ell}{\ell}\frac{d}{k\ell+d}$$

and he inserts $k=2$ which gives A030983:

$$3z + 16z^2 + 83z^3 + 442z^4 + 2420z^5 + \cdots,$$

$k=3$ which gives A334608:

$$5z + 34z^2 + 236z^3 + 1714z^4 + 12922z^5 + \cdots,$$

$k=4$ which gives A334610:

$$7z + 58z^2 + 505z^3 + 4650z^4 + 44677z^5 + \cdots.$$

In general, we have

$$\frac{\overline{u}^2}{z} - 2\overline{u}^2 - \frac{1}{z} = \sum_{\ell\ge 0}\left[\binom{1+(k+1)(\ell+1)}{\ell+1}\frac{2}{k(\ell+1)+2} - 2\binom{1+(k+1)\ell}{\ell}\frac{2}{k\ell+2}\right]z^\ell.$$

Happy Hoppy decides to stop this line of computations here.

3. Combinatorics of the OEIS sequence A002212

The following (sub)sections give some (mostly new) results about the sequence

$$1, 1, 3, 10, 36, 137, 543, 2219, 9285, 39587, 171369, 751236, 3328218, 14878455, \dots,$$

which is A002212 in the OEIS. In the following four sections we consider different combinatorial structures enumerated by this sequence.

- Hex-trees are identified as weighted unary-binary trees, with weight one (see the article by Hana Kim and Richard Stanley [26]). Apart from left and right branches, as in binary trees, there are also unary branches, and they can come in different colours, here in just one colour. Unary-binary trees played a role in the present authors scientific development, as documented in [17], a paper written with the late and great Philippe Flajolet, about the register function (Horton–Strahler numbers) of unary-binary trees. In Section 4, we offer an improvement, using a "better" substitution than in [17]. The results can now be made fully explicit. As a by-product, this provides a definition and analysis of the Horton–Strahler numbers of hex-trees.

- Then we move to skew Dyck paths, as considered by Emeric Deutsch, Emanuele Munarini, and Simone Rinaldi in [12]. They are like Dyck paths, but allow for an extra step $(-1,-1)$, provided that the path does not intersect itself. An equivalent model, defined and described using a bijection, is from [12]: marked ordered trees; see Section 5. They are like ordered trees, with an additional feature, namely each rightmost edge (except for one that leads to a leaf) can be coloured with two colours. Since we find this class of trees to be interesting, we analyze two parameters of them: number of leaves and height. While the number of leaves for ordered trees is about $n/2$, it is only $n/10$ in the new model. For the height, the leading term $\sqrt{\pi n}$ drops to $\frac{2}{\sqrt{5}}\sqrt{\pi n}$. Of course, many more parameters of this new class of trees could be investigated, which we encourage to do.

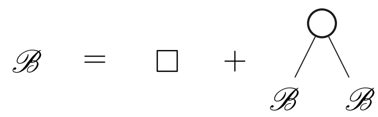
- The next two classes of structures are multi-edge trees; see Section 6. Our interest in them was already triggered in an earlier publication, together with Clemens Heuberger and Stephan Wagner [24]. They may be seen as ordered trees, but with weighted edges. The weights are integers ≥ 1, and a weight a may be interpreted as a parallel edges. The other class are 3-Motzkin paths. They are like Motzkin paths (Dyck paths plus horizontal steps); however, the horizontal steps come in three different colours. A bijection is described. Since 3-Motzkin paths and multi-edge trees are very much alike (using a variation of the classical rotation correspondence), all the structures that are discussed in this paper can be linked via bijections. Since these trees are not so common in the combinatorics community, details and examples are presented for the readers' benefit.

- Skew Dyck paths are finally discussed in more detail in Section 7.

4. Binary trees and Horton–Strahler numbers

This section is classical and serves as the basis of some new developments about weighted unary-binary trees. A full account can be found in [36].

Binary trees may be expressed by the following symbolic equation, which says that they include the empty tree and trees recursively built from a root followed by two subtrees, which are binary trees:

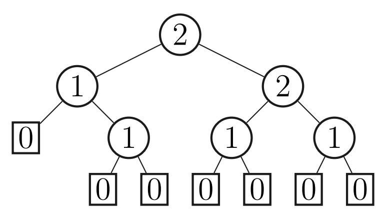


Binary trees are counted by Catalan numbers and there is an important parameter reg, which in Computer Science is called the register function. It associates to each binary tree (which is used to code an arithmetic expression, with data in the leaves and operators in the internal nodes) the minimal number of extra registers that is needed to evaluate the tree. The optimal strategy is to evaluate difficult subtrees first, and use one register to keep its value, which does not hurt, if the other subtree requires less registers. If both subtrees are equally difficult, then one more register is used, compared to the requirements of the subtrees. This natural parameter is among combinatorialists known as the Horton–Strahler numbers, and we will adopt this name throughout this paper.

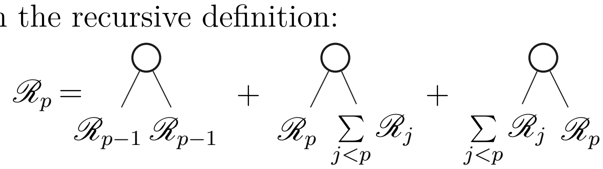
There is a recursive description of this function: $\mathsf{reg}(\square) = 0$, and if tree t has subtrees t_1 and t_2, then

$$\mathsf{reg}(t) = \begin{cases} \max\{\mathsf{reg}(t_1), \mathsf{reg}(t_2)\} & \text{if } \mathsf{reg}(t_1) \neq \mathsf{reg}(t_2), \\ 1 + \mathsf{reg}(t_1) & \text{otherwise.} \end{cases}$$

The recursive description attaches numbers to the nodes, starting with 0's at the leaves and then going up; the number appearing at the root is the Horton–Strahler number of the tree.



Let $\mathscr{R}_p$ denote the family of trees with Horton–Strahler number equal to p, then one gets immediately from the recursive definition:



In terms of generating functions, these equations are translated into

$$R_p(z) = zR_{p-1}^2(z) + 2zR_p(z)\sum_{j<p} R_j(z);$$

the variable z is used to mark the size (i.e., the number of internal nodes) of the binary tree. A historic account of these concepts, from the angle of Philippe Flajolet, who was one of the pioneers is [50]; compare also [48].

Amazingly, the recursion for the generating functions $R_p(z)$ can be solved explicitly! The substitution

$$z = \frac{u}{(1+u)^2}$$

that de Bruijn, Knuth, and Rice [9] also used, produces the nice expression

$$R_p(z) = \frac{1-u^2}{u} \frac{u^{2^p}}{1-u^{2^{p+1}}}.$$

Of course, once this is *known*, it can be proved by induction, using the recursive formula. For the readers benefit, this will be sketched now. We start with the auxiliary formula

$$\sum_{0 \le j < p} \frac{u^{2^j}}{1-u^{2^{j+1}}} = \frac{u}{1-u} - \frac{u^{2^p}}{1-u^{2^p}},$$

which we will prove now by induction: For $p = 0$, the formula $0 = \frac{u}{1-u} - \frac{u}{1-u}$ is correct, and then

$$\begin{aligned}\sum_{0 \le j < p+1} \frac{u^{2^j}}{1-u^{2^{j+1}}} &= \frac{u}{1-u} - \frac{u^{2^p}}{1-u^{2^p}} + \frac{u^{2^p}}{1-u^{2^{p+1}}} \\ &= \frac{u}{1-u} - \frac{u^{2^p}(1+u^{2^p})}{1-u^{2^{p+1}}} + \frac{u^{2^p}}{1-u^{2^{p+1}}} = \frac{u}{1-u} - \frac{u^{2^{p+1}}}{1-u^{2^{p+1}}}.\end{aligned}$$

Now the formula for $R_p(z)$ can also be proved by induction. First, $R_0(z) = \frac{1-u^2}{u}\frac{u}{1-u^2} = 1$, as it should, and

$$\begin{aligned}&zR_{p-1}^2(z) + 2zR_p(z)\sum_{j<p} R_j(z) \\ &\qquad = \frac{u}{(1+u)^2}\frac{(1-u^2)^2}{u^2}\frac{u^{2^p}}{(1-u^{2^p})^2} + \frac{2u}{(1+u)^2}R_p(z)\sum_{j<p}\frac{1-u^2}{u}\frac{u^{2^j}}{1-u^{2^{j+1}}} \\ &\qquad = \frac{u^{2^p-1}(1-u)^2}{(1-u^{2^p})^2} + \frac{2(1-u)}{(1+u)}R_p(z)\sum_{j<p}\frac{u^{2^j}}{1-u^{2^{j+1}}}.\end{aligned}$$

Solving

$$R_p(z) = \frac{u^{2^p-1}(1-u)^2}{(1-u^{2^p})^2} + \frac{2(1-u)}{(1+u)}R_p(z)\left[\frac{u}{1-u} - \frac{u^{2^p}}{1-u^{2^p}}\right]$$

leads to

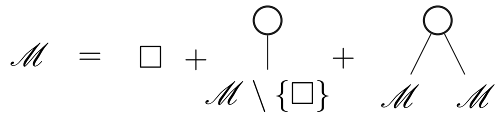
$$R_p(z)\frac{1-u}{1+u}\left[1 + 2\frac{u^{2^p}}{1-u^{2^p}}\right] = \frac{u^{2^p-1}(1-u)^2}{(1-u^{2^p})^2},$$

or, simplified

$$R_p(z) = \frac{u^{2^p-1}(1-u^2)}{(1-u^{2^p})(1+u^{2^p})} = \frac{1-u^2}{u}\frac{u^{2^p}}{1-u^{2^{p+1}}},$$

which is the formula that we needed to prove. Alternatively, this formula can also be proved by converting the sum into a telescoping sum, by extending the numerator and denominator by $(1-u^{2^j})$, and using a partial fraction decomposition. $\square$

Weighted unary-binary trees and Horton–Strahler numbers. The family of unary-binary trees $\mathscr{M}$ might be defined by the symbolic equation



The equation for the generating function is

$$M(z) = 1 + z(M(z) - 1) + zM(z)^2$$

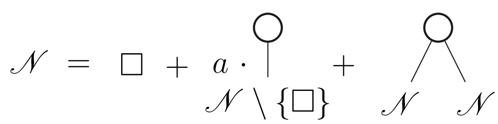
with the solution

$$M(z) = \frac{1 - z - \sqrt{1 - 6z + 5z^2}}{2z} = 1 + z + 3z^2 + 10z^3 + 36z^4 + \cdots;$$

the coefficients form again sequence A002212 in [52] and enumerate Schröder paths, among many other things. We will come to equivalent structures a bit later.

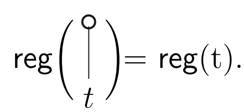
In the instance of unary-binary trees, we can also work with a substitution. Set $z = \frac{u}{1+3u+u^2}$, then $M(z) = 1 + u$. Unary-binary trees and the register function were investigated in [17], but the present favourable substitution was not used. Therefore, in this previous paper, asymptotic results were available but no explicit formulæ.

This works also with a weighted version, where we allow unary edges with a different colours. Then

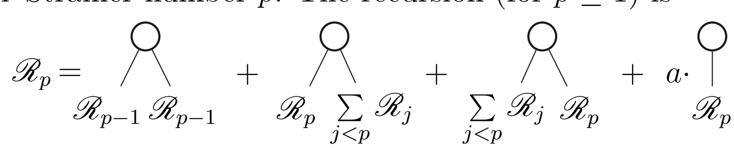


and with the substitution $z = \frac{u}{1+(a+2)u+u^2}$, the generating function is beautifully expressed as $N(z) = 1 + u$. For $a = 0$, this covers also binary trees.

We will consider the Horton–Strahler numbers of unary-binary trees in the sequel. The definition is naturally extended by



Now we can move again to $R_p(z)$, the generating function of (generalized) unary-binary trees with Horton–Strahler number p. The recursion (for $p \geq 1$) is



In terms of generating functions, these equations read as

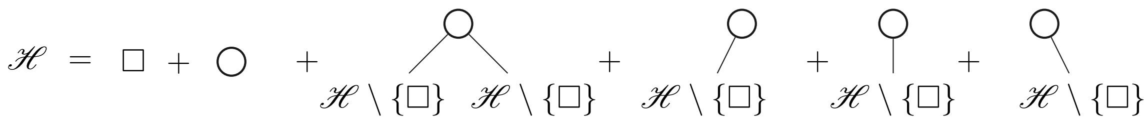
$$R_p(z) = zR_{p-1}^2(z) + 2zR_p(z)\sum_{j<p} R_j(z) + azR_p(z), \quad p \geq 1; \quad R_0(z) = 1.$$

Amazingly, with the substitution $z = \frac{u}{1+(a+2)u+u^2}$, formally we get the *same* solution as in the binary case:

$$R_p(z) = \frac{1-u^2}{u}\frac{u^{2^p}}{1-u^{2^{p+1}}}.$$

The proof by induction is as before. One sees another advantage of the substitution. On a formal level, many manipulations do not need to be repeated. Only when one switches back to the z-world, things become different.

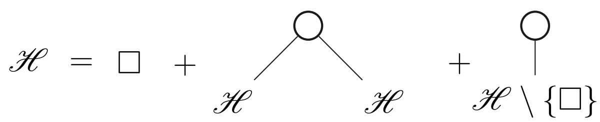
Hex-trees. Hex-trees either have two non-empty successors, or one of 3 types of unary successors (called left, middle, right). The author has seen this family first in [26], but one can find older literature following the references and the usual search engines [2, 23]. We start with a symbolic equation, as usual.



The generating function satisfies (by translation of the symbolic equation)

$$H(z) = 1 + z(H(z) - 1)^2 + z + 3z(H(z) - 1) = \frac{1 - z - \sqrt{(1-z)(1-5z)}}{2z}$$
$$= 1 + z + 3z^2 + 10z^3 + 36z^4 + 137z^5 + 543z^6 + 2219z^7 + 9285z^8 + 39587z^9 + \cdots .$$

The same generating function also appears in [24], and it is sequence `A002212` in the OEIS [52]. One can rewrite the symbolic equation as



and sees in this way that the hex-trees are just unary-binary trees (with parameter $a = 1$).

Continuing with enumerations. First, we will enumerate the number of (generalized) unary-binary trees with n (internal) nodes. For that we need the notion of generalized trinomial coefficients, viz.

$$\binom{n; 1, a, 1}{k} := [z^k](1 + az + z^2)^n.$$

Of course, for $a = 2$, this simplifies to a binomial coefficient $\binom{2n}{k}$. We will use contour integration to pull out coefficients, and the contour of integer, in whatever variable, is a small circle (or equivalent) around the origin. The desired number is (recall that $z = \frac{u}{1+(a+2)u+u^2}$)

$$\begin{aligned}
[z^n](1+u) &= \frac{1}{2\pi i}\oint \frac{dz}{z^{n+1}}(1+u) \\
&= \frac{1}{2\pi i}\oint \frac{du(1-u^2)(1+(a+2)u+u^2)^{n+1}}{(1+(a+2)u+u^2)^2 u^{n+1}}(1+u) \\
&= [u^{n+1}](1-u)(1+u)^2(1+(a+2)u+u^2)^{n-1} \\
&= \binom{n-1; 1, a+2, 1}{n+1} + \binom{n-1; 1, a+2, 1}{n} \\
&\qquad - \binom{n-1; 1, a+2, 1}{n-1} - \binom{n-1; 1, a+2, 1}{n-2}.
\end{aligned}$$

Then we introduce $S_p(z) = R_p(z) + R_{p+1}(z) + R_{p+2}(z) + \cdots$, the generating function of trees with Horton–Strahler number $\ge p$. Using the summation formula proved earlier, we get

$$S_p(z) = \frac{1-u^2}{u}\frac{u^{2^p}}{1-u^{2^p}} = \frac{1-u^2}{u}\sum_{k\ge1} u^{k2^p}.$$

Asymptotics. We start by deriving asymptotics for the number of (generalized) unary-binary trees with n (internal) nodes. This is a standard application of singularity analysis of generating functions, as described in [16] and [20].

We start from the generating function

$$N(z) = \frac{1 - az - \sqrt{1 - 2(a+2)z + a(a+4)z^2}}{2z}$$

and determine the singularity closest to the origin, which is the value making the square root disappear: $z = \frac{1}{a+4}$. After that, the local expansion of $N(z)$ around this singularity is determined:

$$N(z) \sim 2 - \sqrt{a+4}\sqrt{1-(a+4)z}.$$

The translation lemmas given in [16] and [20] provide the asymptotics:

$$\begin{aligned}[z^n]N(z) &\sim [z^n]\Big(2 - \sqrt{a+4}\sqrt{1-(a+4)z}\Big)\\ &= -\sqrt{a+4}\,(a+4)^n \frac{n^{-3/2}}{\Gamma(-\frac12)} = (a+4)^{n+1/2}\frac{1}{2\sqrt{\pi}n^{3/2}}.\end{aligned}$$

Just note that $a = 0$ is the well-known formula for binary trees with n nodes.

Now we move to the generating function for the average number of registers. Apart from normalization it is

$$\sum_{p\ge1} pR_p(z) = \sum_{p\ge1} S_p(z) = \frac{1-u^2}{u}\sum_{p\ge1}\sum_{k\ge1} u^{k2^p} = \frac{1-u^2}{u}\sum_{n\ge1} v_2(n)u^n,$$

where $v_2(n)$ is the highest exponent k such 2^k divides n.

This has to be studied around $u = 1$, which, upon setting $u = e^{-t}$, means around $t = 0$. Eventually, and that is the only thing that is different here, this is to be retranslated into a singular expansion of z around its singularity, which depends on the parameter a.

For the reader's convenience, we also repeat the steps that were known before. The first factor is elementary:

$$\frac{1-u^2}{u} \sim 2t + \frac13 t^3 + \cdots.$$

For

$$\sum_{p\ge1}\sum_{k\ge1} e^{-k2^pt},$$

one applies the Mellin transform, with the result

$$\frac{\Gamma(s)\zeta(s)}{2^s-1}.$$

Applying the inversion formula, one finds

$$\sum_{p\ge1}\sum_{k\ge1} e^{-k2^pt} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} t^{-s}\frac{\Gamma(s)\zeta(s)}{2^s-1}ds.$$

Shifting the line of integration to the left, the residues at the poles $s=1$, $s=0$, $s=\chi_k=\frac{2k\pi i}{\log 2}$, $k\neq 0$ provide enough terms for our asymptotic expansion.

$$\frac{1}{t}+\frac{\gamma}{2\log 2}-\frac{1}{4}-\frac{\log\pi}{2\log 2}+\frac{\log t}{2\log 2}+\frac{1}{\log 2}\sum_{k\neq 0}\Gamma(\chi_k)\zeta(\chi_k)t^{-\chi_k}.$$

Combined with the elementary factor, this leads to

$$2+\Big(\frac{\gamma}{\log 2}-\frac{1}{2}-\frac{\log\pi}{\log 2}+\frac{\log t}{\log 2}\Big)t+\frac{2t}{\log 2}\sum_{k\neq 0}\Gamma(\chi_k)\zeta(\chi_k)t^{-\chi_k}+O(t^2\log t).$$

Now we want to translate into the original z-world. Since $z=\frac{u}{1+(a+2)u+u^2}$, $u=1$ translates into the singularity $z=\frac{1}{a+4}$. Further,

$$t\sim\sqrt{a+4}\cdot\sqrt{1-z(a+4)},$$

let us abbreviate $A=a+4$, and we now want to get the asymptotic behaviour of the coefficients in the power series expansion of

$$\begin{aligned}&\frac{\sqrt{A}\cdot\sqrt{1-zA}\log(1-zA)}{2\log 2}\\&+\Big(\frac{\gamma}{\log 2}-\frac{1}{2}-\frac{\log\pi}{\log 2}+\frac{\log A}{2\log 2}\Big)\sqrt{A}\cdot\sqrt{1-zA}\\&+\frac{2}{\log 2}\sum_{k\neq 0}\Gamma(\chi_k)\zeta(\chi_k)A^{\frac{1-\chi_k}{2}}(1-zA)^{\frac{1-\chi_k}{2}}.\end{aligned}$$

By singularity analysis (see [16, 20]), one has

$$[z^n](1-z)^\alpha\sim\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}$$

and

$$[z^n]\log(1-z)\sqrt{1-z}\sim\frac{n^{-3/2}\log n}{2\sqrt{\pi}}+\frac{n^{-3/2}}{2\sqrt{\pi}}(-2+\gamma+2\log 2).$$

We start with the most complicated term:

$$\begin{aligned}\frac{[z^n]\frac{\sqrt{A}\cdot\sqrt{1-zA}\log(1-zA)}{2\log 2}}{[z^n]N(z)}&\sim\frac{\sqrt{A}}{2\log 2}\frac{A^n\Big(\frac{n^{-3/2}\log n}{2\sqrt{\pi}}+\frac{n^{-3/2}}{2\sqrt{\pi}}(-2+\gamma+2\log 2)\Big)}{A^{n+1/2}\frac{1}{2\sqrt{\pi}n^{3/2}}}\\&=\log_4 n+1+\frac{\gamma}{2\log 2}-\frac{1}{\log 2}.\end{aligned}$$

The next term we consider is

$$\begin{aligned}&\Big(\frac{\gamma}{\log 2}-\frac{1}{2}-\frac{\log\pi}{\log 2}+\frac{\log A}{2\log 2}\Big)\sqrt{A}\frac{[z^n]\sqrt{1-zA}}{[z^n]N(z)}\\&\qquad\sim\Big(\frac{\gamma}{\log 2}-\frac{1}{2}-\frac{\log\pi}{\log 2}+\frac{\log A}{2\log 2}\Big)\sqrt{A}\frac{[z^n]\sqrt{1-zA}}{-\sqrt{A}[z^n]\sqrt{1-zA}}\\&\qquad=-\frac{\gamma}{\log 2}+\frac{1}{2}+\frac{\log\pi}{\log 2}-\frac{\log A}{2\log 2}.\end{aligned}$$

The last term we consider is

$$\frac{2}{\log 2}\Gamma(\chi_k)\zeta(\chi_k)A^{\frac{1-\chi_k}{2}}\frac{[z^n](1-zA)^{\frac{1-\chi_k}{2}}}{-\sqrt{A}[z^n]\sqrt{1-zA}} \sim -\frac{4\sqrt{\pi}}{\log 2}\frac{\Gamma(\chi_k)\zeta(\chi_k)}{\Gamma\Big(\frac{\chi_k-1}{2}\Big)}A^{\frac{1-\chi_k}{2}}n^{\chi_k/2}.$$

Eventually we have evaluated the average value of the Horton–Strahler numbers:

Theorem 4.1. *The average Horton–Strahler number of weighted unary-binary trees with n nodes is given by the asymptotic formula*

$$\log_4 n - \frac{\gamma}{2\log 2} - \frac{1}{\log 2} + \frac{3}{2} + \frac{\log \pi}{\log 2} - \frac{\log A}{2\log 2} - \frac{4\sqrt{\pi A}}{\log 2}\sum_{k\neq 0}\frac{\Gamma(\chi_k)\zeta(\chi_k)}{\Gamma\Big(\frac{\chi_k-1}{2}\Big)}A^{\frac{-\chi_k}{2}}n^{\chi_k/2}$$
$$= \log_4 n - \frac{\gamma}{2\log 2} - \frac{1}{\log 2} + \frac{3}{2} + \frac{\log \pi}{\log 2} - \frac{\log A}{2\log 2} + \psi(\log_4 n),$$

with a tiny periodic function $\psi(x)$ of period 1.

These oscillations are usually bounded by 10^{-5}, say. See [19] for some explicit error bounds.

5. Marked ordered trees

In [12] we find the following variation of ordered trees. Each rightmost edge might be marked or not, if it does not lead to an endnode (leaf). We depict a marked edge by the red colour and draw all of them of size 4 (4 nodes):

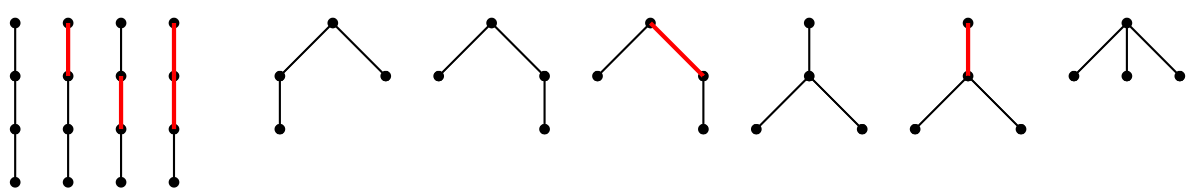
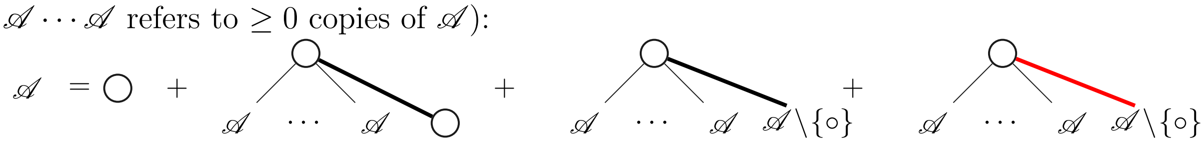


Figure 2. All 10 marked ordered trees with 4 nodes.

Accordingly, the marked ordered trees satisfy the following symbolic equation (where $\mathscr{A}\cdots\mathscr{A}$ refers to ≥ 0 copies of $\mathscr{A}$):



In terms of generating functions, this gives the functional equation

$$A(z) = z + \frac{z}{1-A(z)}z + \frac{z}{1-A(z)}2(A(z)-z),$$

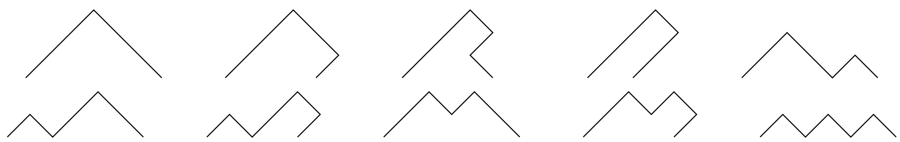
whose solution is

$$A(z) = \frac{1-z-\sqrt{1-6z+5z^2}}{2} = z + z^2 + z^3 + 3z^3 + 10z^4 + 36z^5 + \cdots.$$

In fact, as proved in [12], these trees are in bijection with an interesting family of lattice paths, called *skew Dyck paths*. The bijection performs a walk around the contour of the tree (that is, a depth-first search traversal) and translates it into a skew Dyck path as follows

- black or red edges on the way down become a $(+1,+1)$ step,
- black edges on the way up become a $(+1,-1)$ step,
- red edges on the way up become a $(-1,-1)$ step.

Thus, the 10 trees of Figure 2 translate as follows into skew Dyck paths of length 6:



We will analyze several parameters of skew Dyck paths in Section 7. But, as the present author believes that trees are more manageable (than these paths) when it comes to enumeration issues, let us now investigate these marked trees in more detail.

Parameters of marked ordered trees. There are many parameters, usually considered in the context of ordered trees, that can be considered for marked ordered trees. Of course, we cannot be encyclopedic about such parameters. We just consider a few parameters and leave further analysis to the future.

The number of leaves. To get this, it is most natural to use an additional variable u when translating the symbolic equation, so that $z^n u^k$ refers to trees with n nodes and k leaves. One obtains

$$F(z,u) = zu + \frac{z}{1-F(z,u)}\Big(zu + 2(F(z,u)-zu)\Big),$$

with the solution

$$\begin{aligned} F(z,u) &= -z + \frac{zu}{2} + \frac{1}{2} - \frac{1}{2}\sqrt{4z^2 - 4z + z^2u^2 - 2zu + 1} \\ &= zu + z^2u + \Big(2u + u^2\Big)\,z^3 + \Big(4u + 5u^2 + u^3\Big)\,z^4 + \cdots . \end{aligned}$$

The factor $4u + 5u^2 + u^3$ corresponds to distribution of leaves in the 10 trees of Figure 2.

Of interest is also the average number of leaves, when all marked ordered trees of size n are considered to be equally likely. For that, we differentiate $F(z,u)$ with respect to u, and set $u := 1$, with the result

$$\frac{z}{2} + \frac{z - z^2}{2\sqrt{1-6z+5z^2}} = \frac{z}{1-v}, \quad \text{with the parametrization} \quad z = \frac{v}{1+3v+v^2}. \tag{5.1}$$

Since $F(z,1) = z(1+v)$, it follows that the average is asymptotic to

$$\frac{[z^{n+1}]\frac{z}{1-v}}{[z^{n+1}]z(1+v)} = \frac{[z^n]\frac{1}{1-v}}{[z^n](1+v)} = \frac{[z^n]\frac{1}{\sqrt{5}}\frac{1}{\sqrt{1-5z}}}{5^{n+\frac{1}{2}}\frac{1}{2\sqrt{\pi}}n^{3/2}} = \frac{\frac{n^{-1/2}}{\Gamma(\frac{1}{2})}}{5^{n+\frac{1}{2}}\frac{1}{2\sqrt{\pi}}n^{3/2}} = \frac{n}{10}. \tag{5.2}$$

Note that the corresponding number for ordered trees (unmarked) is $\frac{n}{2}$, so we have significantly less leaves here.

The height. As in the seminal paper [9], we define the height in terms of the longest chain of nodes from the root to a leaf. Further, let $p_h = p_h(z)$ be the generating function of marked ordered trees of height at least h. From the symbolic equation, one has

$$p_{h+1} = z + \frac{z^2}{1-p_h} + \frac{2z(p_h - z)}{1-p_h} = -z + \frac{2z - z^2}{1-p_h} \quad \text{(for } h \ge 1\text{) and} \quad p_1 = z.$$

By some creative guessing, separating numerator and denominator, we find the solution (where we use the auxiliary algebraic function v, implicitly defined in (5.1)):

$$p_h = z(1+v)\frac{(1+2v)^{h-1} - v^h(v+2)^{h-1}}{(1+2v)^{h-1} - v^{h+1}(v+2)^{h-1}}.$$

This formula is in fact proved by induction (we start with $p_1 = z(1+v)\frac{1-v}{1-v^2} = z$ and, then, the induction step is best checked using a computer).

The limit of p_h for $h \to \infty$ is $z(1+v)$, the generating function of *all* marked ordered trees, as expected. Taking differences, we get the generating functions of trees of height at least h:

$$p_\infty - p_h = z(1-v^2)\frac{(v+2)^{h-1}v^h}{(1+2v)^{h-1} - v^{h+1}(v+2)^{h-1}}.$$

From this, the average height can be worked out, as in the model paper [24]. We sketch the essential steps. For the average height, one needs

$$\sum_{h\ge 0} z(1-v^2)\frac{(v+2)^{h-1}v^h}{(1+2v)^{h-1} - v^{h+1}(v+2)^{h-1}}$$

and its behaviour around $v = 1$, viz.

$$2z(1-v)\sum_{h\ge 0}\frac{3^{h-1}v^h}{3^{h-1} - v^{h+1}3^{h-1}} \sim 2z(1-v)\sum_{h\ge 1}\frac{v^h}{1-v^h}.$$

The behaviour of the series can be taken straight from [24]. We find there

$$\sum_{h\ge 1}\frac{v^h}{1-v^h} = -\frac{\log(1-v)}{1-v}$$

and

$$\sum_{h\ge 0} z(1-v^2)\frac{(v+2)^{h-1}v^h}{(1+2v)^{h-1} - v^{h+1}(v+2)^{h-1}} \sim -2z\log(1-v).$$

Thus, the coefficient of z^{n+1} is asymptotic to $-2[z^n]\log(1-v)$. Since $1 - v \sim \sqrt{5}\sqrt{1-5z}$,

$$-2z\log(1-v) \sim -2z\log\sqrt{1-5z} = -z\log(1-5z),$$

and the coefficient of z^{n+1} in it is asymptotic to $\frac{5^n}{n}$. This has to be divided (as derived inside Formula (5.2)) by

$$5^{n+\frac{1}{2}}\frac{1}{2\sqrt{\pi}n^{3/2}},$$

with the result

$$2\frac{5^n}{n}\frac{1}{5^{n+\frac{1}{2}}}\sqrt{\pi}n^{3/2} = \frac{2}{\sqrt{5}}\sqrt{\pi n}.$$

Note that the constant in front of $\sqrt{\pi n}$ for ordered trees is $\frac{2}{\sqrt{4}} = 1$, so the average height for marked ordered trees is indeed a bit smaller thanks to the extra markings.

6. A bijection between multi-edge trees and 3-coloured Motzkin paths

Multi-edge trees are like ordered (planar, plane, ...) trees, but instead of edges there are multiple edges. When drawing such a tree, instead of drawing, say 5 parallel edges, we just draw one edge and put the number 5 on it as a label. These trees were studied in [14, 24]. We also considered this model in our unpublished preprint arXiv:2105.03350; the bijection presented hereafter is new.

The generating function $F(z)$ (where one counts edges) satisfies

$$F(z) = \sum_{k\ge 0}\Big(\frac{z}{1-z}F(z)\Big)^k = \frac{1}{1-\frac{z}{1-z}F(z)},$$

whence

$$F(z) = \frac{1-z-\sqrt{1-6z+5z^2}}{2z} = 1 + z + 3z^2 + 10z^3 + 36z^4 + 137z^5 + 543z^6 + \cdots .$$

The coefficients form sequence `A002212` in the OEIS [52].

A Motzkin path consists of up-steps, down-steps, and horizontal steps; see sequence `A091965` in [52] and the references given there. As Dyck paths, they start at the origin and end, after n steps again at the x-axis, but are not allowed to go below the x-axis. A 3-coloured Motzkin path is built as a Motzkin path, but there are 3 different types of horizontal steps, which we call *red, green, blue*. The generating function $M(z)$ satisfies

$$M(z) = 1 + 3zM(z) + z^2M(z)^2 = \frac{1-3z-\sqrt{1-6z+5z^2}}{2z^2}, \quad \text{or} \quad F(z) = 1 + zM(z).$$

So multi-edge trees with N edges (counting the multiplicities) correspond to 3-coloured Motzkin paths of length $N-1$.

The purpose of this section is to describe a bijection. It transforms trees into paths, but all steps are reversible.

The details. As a first step, the multiplicities will be ignored, and the tree then has only n edges. The standard translation of such tree into the world of Dyck paths, which is in every book on combinatorics, leads to a Dyck path of length $2n$. Then the Dyck path will be transformed bijectively to a 2-coloured Motzkin path of length $n-1$ (the colours used are red and green). This transformation plays a prominent role in [13], but is most likely much older. I believe that people like Viennot know this for 40 years. I would be glad to get a proper historic account from the gentle readers.

The last step is then to use the third colour (blue) to deal with the multiplicities.

The first up-step and the last down-step of the Dyck path will be deleted. Then, the remaining $2n-2$ steps are coded pairwise into a 2-Motzkin path of length $n-1$:

The last step is to deal with the multiplicities. If an edge is labelled with the number a, we will insert $a-1$ horizontal blue steps in the following way. Since there are currently $n-1$ symbols in the path, we have n possible positions to enter something (in the beginning, in the end, between symbols). We go through the tree in pre-order, and enter the multiplicities one by one using the blue horizontal steps.

To illustrate this procedure, we give in Table 1 the list of 10 multi-edge trees with 3 edges, and the corresponding 3-Motzkin paths of length 2, with intermediate steps completely worked out.

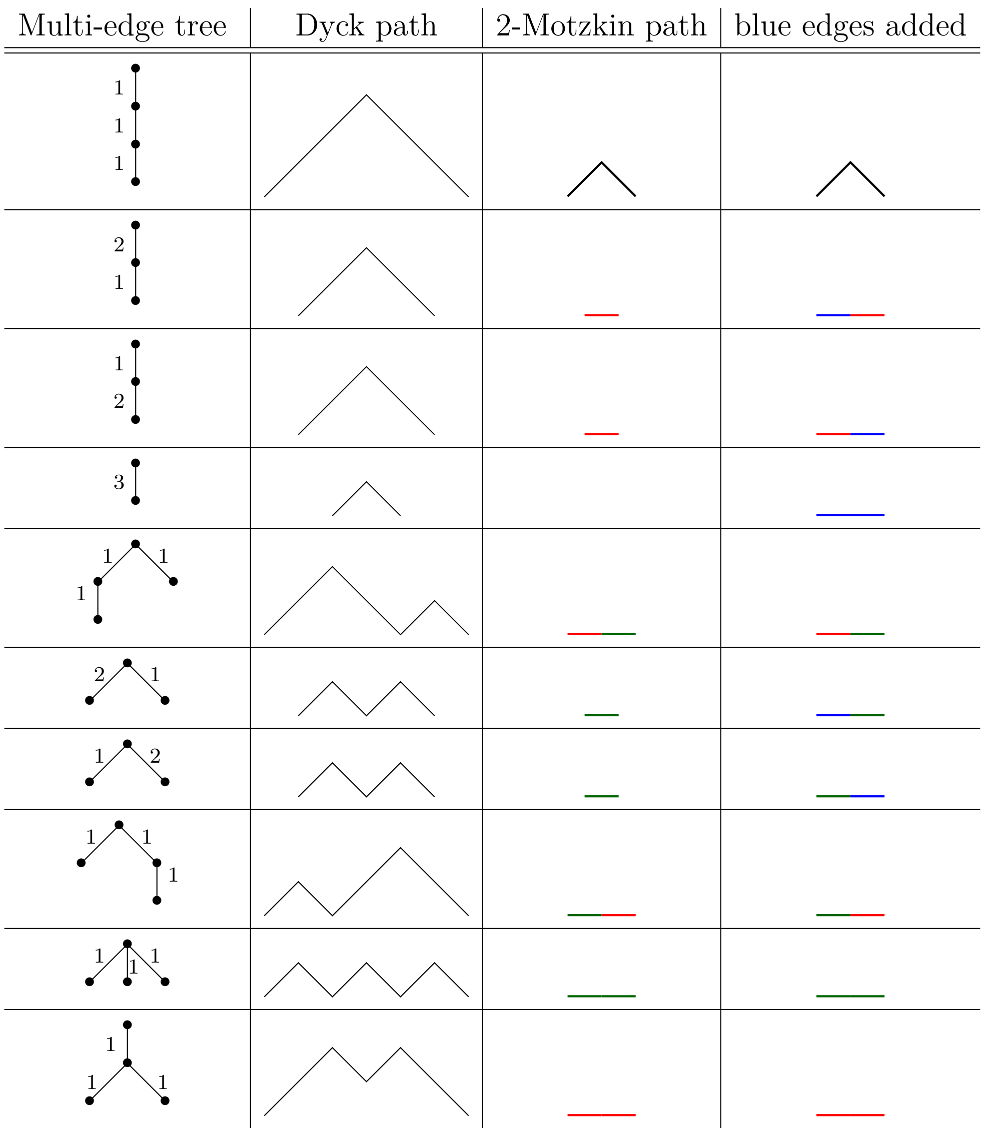


Table 1. First row is a multi-edge tree with 3 edges, second row is the standard Dyck path (multiplicities ignored), third row is cutting off first and last step, and then translated pairs of steps, fourth row is inserting blue horizontal edges, according to multiplicities.

Connecting unary-binary trees with multi-edge trees. This is not too difficult: We start from multi-edge trees, and ignore the multiplicities at the moment. Then we apply the classical rotation correspondence (also called: natural correspondence). Then we add vertical edges, if the multiplicity is higher than 1. To be precise, if there is a node, and an edge with multiplicity a leads to it from the top, we insert $a-1$ extra unary nodes in a chain on the top, and connect them with unary branches.

In Table 2 below, we illustrate this procedure on 10 objects.

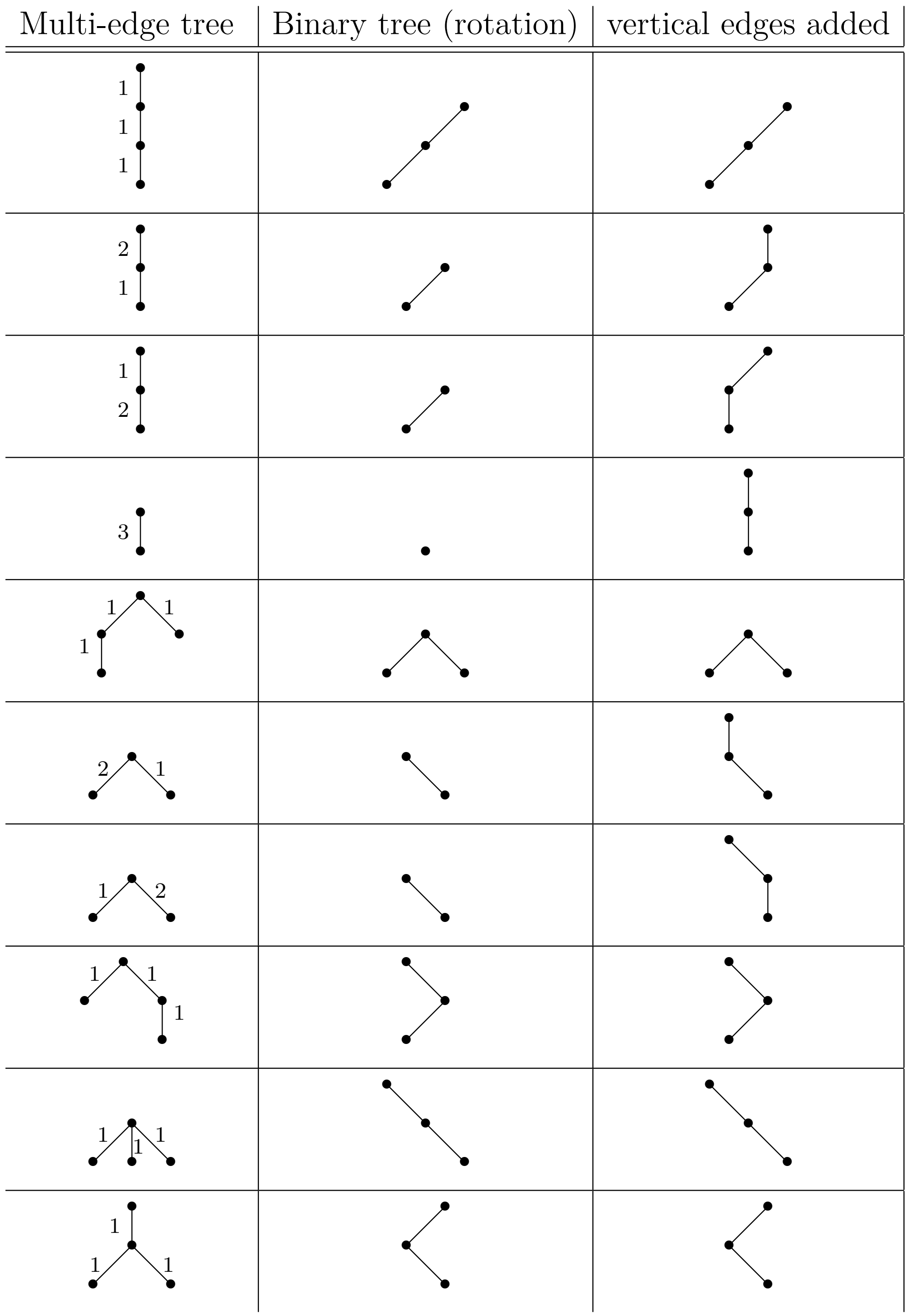


TABLE 2. First row is a multi-edge tree with 3 edges, second row the corresponding binary tree, according to the classical rotation correspondence, ignoring the unary branches. Third row is inserting extra horizontal edges when the multiplicities are higher than 1.

7. The combinatorics of skew Dyck paths

Let us come back to skew Dyck paths, which we introduced in Section 5 as objects in bijection with marked ordered trees. As we saw, skew Dyck paths are a variation of Dyck paths, where additionally to steps $(1,1)$ and $(1,-1)$ a south-west step $(-1,-1)$ is also allowed, provided that the path does not intersect itself. Also, like for Dyck paths, it must never go below the x-axis and end eventually (after $2n$ steps) on the x-axis. These paths were considered in [6, 12, 26, 44]. We extend here on our paper [46], giving here additional results on the prefixes of these paths. The enumerating sequence is $1, 1, 3, 10, 36, 137, 543, 2219, 9285, 39587, 171369, \dots$, which is `A002212` in the OEIS [52].

Let us now give a more thorough analysis of skew Dyck paths, using generating functions and the kernel method. Here is the list of the 10 skew Dyck paths consisting of 6 steps:

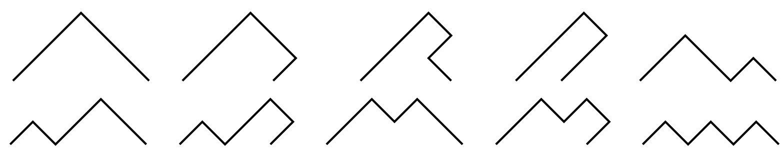


Figure 3. All 10 skew Dyck paths of length 6.

We prefer to work with the equivalent model (resembling more traditional Dyck paths) where we replace each step $(-1,-1)$ by $(1,-1)$ but label it red (see Figure 4, and compare with Figure 3):

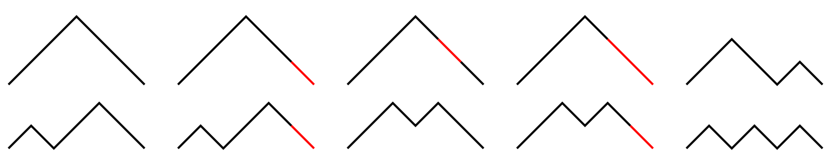
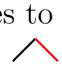


Figure 4. The 10 paths of length 6 redrawn, with red south-east edges instead of south-west edges.

The rules to generate such decorated Dyck paths are: each edge $(1,-1)$ may be black or red, but  and  are forbidden.

Our interest is in particular in *partial* decorated Dyck paths, ending at level j, for fixed $j \ge 0$; the instance $j = 0$ is the classical case. The analysis of partial skew Dyck paths was recently started by Baril et al. in [6] (using the notion 'prefix of a skew Dyck path') using Riordan arrays instead of our kernel method. The latter gives us *bivariate* generating functions, from which it is easier to draw conclusions. Two variables, z and u, are used, where z marks the length of the path and j marks the end-level. We briefly mention that one can, using a third variable w, also count the number of red edges.

Again, once all generating functions are explicitly known, many corollaries can be derived in a standard fashion. We only do this in a few instances. But we would like to emphasize that the substitution $z = \frac{v}{1+3v+v^2}$, which was used in [24, 44] allows to write *explicit enumerations*, using the notion of a (weighted) trinomial coefficient:

$$\binom{n;1,3,1}{k} := [t^k](1+3t+t^2)^n.$$

Generating functions and the kernel method. We catch the essence of a decorated Dyck path using a state-diagram:

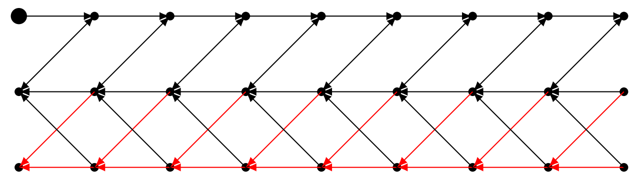


FIGURE 5. Three layers of states according to the type of steps leading to them (up, down-black, down-red).

It has three types of states, with j ranging from 0 to infinity; in the drawing, only $j = 0..8$ is shown. The first layer of states refers to an up-step leading to a state, the second layer refers to a black down-step leading to a state and the third layer refers to a red down-step leading to a state. We will work out generating functions describing all paths leading to a particular state. We will use the notations f_j, g_j, h_j for the three respective layers, from top to bottom. Note that the syntactic rules of forbidden patterns $\wedge$ and $\vee$ can be clearly seen from the picture. The functions depend on the variable z (marking the number of steps), but mostly we just write f_j instead of $f_j(z)$, etc.

The following recursions can be read off immediately from the diagram:

$$\begin{gathered} f_0 = 1, \quad f_{i+1} = zf_i + zg_i, \quad i \ge 0, \\ g_i = zf_{i+1} + zg_{i+1} + zh_{i+1}, \quad i \ge 0, \\ h_i = zg_{i+1} + zh_{i+1}, \quad i \ge 0. \end{gathered}$$

And now it is time to introduce the promised *bivariate* generating functions:

$$F(z,u) = \sum_{i\ge0} f_i(z)u^i, \quad G(z,u) = \sum_{i\ge0} g_i(z)u^i, \quad H(z,u) = \sum_{i\ge0} h_i(z)u^i.$$

Again, often we just write $F(u)$ instead of $F(z,u)$ and treat z as a 'silent' variable. Summing the recursions leads to

$$\begin{aligned} \sum_{i\ge0} u^i f_{i+1} &= \sum_{i\ge0} u^i z f_i + \sum_{i\ge0} u^i z g_i, \\ \sum_{i\ge0} u^i g_i &= \sum_{i\ge0} u^i z f_{i+1} + \sum_{i\ge0} u^i z g_{i+1} + \sum_{i\ge0} u^i z h_{i+1}, \\ \sum_{i\ge0} u^i h_i &= \sum_{i\ge0} u^i z h_{i+1} + \sum_{i\ge0} u^i z g_{i+1}. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} \frac{1}{u}(F(u)-1) &= zF(u) + zG(u), \\ G(u) &= \frac{z}{u}(F(u)-1) + \frac{z}{u}(G(u)-G(0)) + \frac{z}{u}(H(u)-H(0)), \\ H(u) &= \frac{z}{u}(G(u)-G(0)) + \frac{z}{u}(H(u)-H(0)). \end{aligned}$$

Such systems of equations having more unknowns than equations can be solved with the kernel method (see [37] for a gentle example-driven introduction to this method).

We begin by rewriting our system as

$$F(u) = \frac{z^2uG(0) + z^2uH(0) + z^2u - u - z^3 + 2z}{-z^3 - u + 2z + zu^2 - z^2u}, \tag{7.1}$$

$$G(u) = \frac{z(H(0) - uzH(0) + z^2 + G(0) - zuG(0) - zu)}{-z^3 - u + 2z + zu^2 - z^2u}, \tag{7.2}$$

$$H(u) = \frac{z(-uzH(0) - z^2 - zuG(0) + G(0) - z^2H(0) + H(0) - z^2G(0))}{-z^3 - u + 2z + zu^2 - z^2u}. \tag{7.3}$$

The denominator is the same for each equation and it factors as $z(u - r_1)(u - r_2)$, with

$$r_1 = \frac{1 + z^2 + \sqrt{1 - 6z^2 + 5z^4}}{2z}, \quad r_2 = \frac{1 + z^2 - \sqrt{1 - 6z^2 + 5z^4}}{2z}.$$

Consider Equation (7.1), since $F(u)$ is a power series in z, the factor $u - r_2$ in the denominator is "bad"[2], thus this factor must also be a factor of the *numerator* (seen as a polynomial of degree 1 in u). This implies

$$G(0) = -H(0) - 1 + \frac{1}{z^2} + \frac{z - 2/z}{r_2}.$$

Applying the same principle to either (7.2) or (7.3), we get after simplification

$$H(0) = \frac{1 - 4z^2 + z^4 + (z^2 - 1)\sqrt{1 - 6z^2 + 5z^4}}{2 - z^2}.$$

Thus, with $W = \sqrt{1 - 6z^2 + 5z^4} = \sqrt{(1 - z^2)(1 - 5z^2)}$, one has

$$F(u) = \frac{-1 - z^2 - W}{2z(u - r_1)} = \frac{1 + z^2 + W}{2zr_1(1 - u/r_1)},$$

$$G(u) = \frac{-1 + z^2 + W}{2z(u - r_1)} = \frac{1 - z^2 - W}{2zr_1(1 - u/r_1)},$$

$$H(u) = \frac{-1 + 3z^2 + W}{2z(u - r_1)} = \frac{1 - 3z^2 - W}{2zr_1(1 - u/r_1)}.$$

The total generating function (summing the 3 cases that lead to the same level) is

$$S(u) = F(u) + G(u) + H(u) = \frac{3 - 3z^2 - W}{2zr_1(1 - u/r_1)}.$$

The coefficient of $u^j z^n$ in $S(u)$ counts the partial paths of length n, ending at level j. We will write $s_j = [u^j]S(u)$. At this stage, we are only interested in

$$s_j = f_j + g_j + h_j = [u^j]\frac{3 - 3z^2 - W}{2zr_1(1 - u/r_1)} = \frac{3 - 3z^2 - W}{2zr_1^{j+1}},$$

which is the generating function of all (partial) paths ending at level j. Parity considerations give us that only coefficients $[z^n]s_j$ are non-zero if $n \equiv j \bmod 2$. To make this more transparent, we set

$$P(z) = zr_1 = \frac{1 + z^2 + \sqrt{1 - 6z^2 + 5z^4}}{2}.$$

[2] In an identity having the form $A(z,u) = B(z,u)/(u - r(z))$, we say that the factor $(u - r(z))$ is *bad* if $A(u,z)$ is a power series in z while $1/(u - r(z))$ is not.

We thus get

$$s_j = f_j + g_j + h_j = z^j \frac{3 - 3z^2 - W}{2P^{j+1}}.$$

Now we read off coefficients. We do this using residues and contour integration. The path of integration, in both variables x resp. v is a small circle or an equivalent contour. The following computation is abbreviated.

$$\begin{aligned}[z^{2m+j}]s_j &= \frac{1}{2\pi i} \oint \frac{dx}{x^{m+1}} \frac{(1+v)(1+2v)}{v^{j+1}(v+2)^{j+1}} (1 + 3v + v^2)^j \\ &= [v^{m+j+1}] \frac{(1+v)^2(1+2v)(1-v)}{(v+2)^{j+1}} (1 + 3v + v^2)^{m-1+j}.\end{aligned}$$

Note that

$$(1+v)^2(1+2v)(1-v) = -9 + 27(v+2) - 29(v+2)^2 + 13(v+2)^3 - 2(v+2)^4;$$

consequently

$$\begin{aligned}[v^k]\frac{(1+v)^2(1+2v)(1-v)}{(v+2)^{j+1}} &= -9\frac{1}{2^{j+1+k}}\binom{-j-1}{k} + 27\frac{1}{2^{j+k}}\binom{-j}{k} - 29\frac{1}{2^{j-1+k}}\binom{-j+1}{k} \\ &+ 13\frac{1}{2^{j-2+k}}\binom{-j+2}{k} - 2\frac{1}{2^{j-3+k}}\binom{-j+3}{k} =: \lambda_{j;k}.\end{aligned}$$

With this abbreviation $\lambda_{j;k}$ we find

$$[v^{m+j+1}]\frac{(1+v)^2(1+2v)(1-v)}{(v+2)^{j+1}}(1+3v+v^2)^{m-1+j} = \sum_{k=0}^{m+j+1} \lambda_{j;k} \binom{m-1+j; 1, 3, 1}{m+j+1-k}.$$

This is not extremely pretty but it is *explicit* and as good as it gets. Here are the first few generating functions:

- $s_0 = 1 + z^2 + 3z^4 + 10z^6 + 36z^8 + 137z^{10} + 543z^{12} + \cdots,$
- $s_1 = z + 2z^3 + 6z^5 + 21z^7 + 79z^9 + 311z^{11} + 1265z^{13} + \cdots.$

We could also give such lists for the functions f_j, g_j, h_j, if desired. We summarize the essential findings of this section in the following theorem.

Theorem 7.1. *The generating function of decorated (partial) Dyck paths, consisting of n steps, ending on level j, is given by*

$$S(z,u) = \frac{3 - 3z^2 - \sqrt{1 - 6z^2 + 5z^4}}{2zr_1(1 - u/r_1)},$$

with

$$r_1 = \frac{1 + z^2 + \sqrt{1 - 6z^2 + 5z^4}}{2z}.$$

Furthermore

$$[u^j]S(z,u) = \frac{3 - 3z^2 - \sqrt{1 - 6z^2 + 5z^4}}{2zr_1^{j+1}}.$$

Open ended paths. If we do not specify the end of the paths, in other words we sum over all $j \ge 0$, then at the level of generating functions this is very easy, since we only have to set $u := 1$. We find

$$\begin{aligned} S(1) &= -\frac{(z+1)(z^2+3z-2)+(z+2)\sqrt{1-6z^2+5z^4}}{2z(z^2+2z-1)} \\ &= 1+z+2z^2+3z^3+7z^4+11z^5+26z^6+43z^7+102z^8+175z^9+416z^{10}+\cdots. \end{aligned}$$

Counting red edges. We can use an extra variable, w, to count additionally the red edges that occur in a path. We use the same letters for generating functions. Eventually, the coefficient $[z^n u^j w^k]S$ is the number of (partial) paths consisting of n steps, leading to level j, and having passed k red edges. The endpoint of the original skew path has then coordinates $(n-2k, j)$. The computations are very similar, and we only sketch the key steps.

$$f_0 = 1, \quad f_{i+1} = zf_i + zg_i, \quad i \ge 0,$$
$$g_i = zf_{i+1} + zg_{i+1} + zh_{i+1}, \quad i \ge 0,$$
$$h_i = wzg_{i+1} + wzh_{i+1}, \quad i \ge 0;$$

$$\begin{aligned} \frac{1}{u}(F(u)-1) &= zF(u) + zG(u), \\ G(u) &= \frac{z}{u}(F(u)-1) + \frac{z}{u}(G(u)-G(0)) + \frac{z}{u}(H(u)-H(0)), \\ H(u) &= \frac{wz}{u}(G(u)-G(0)) + \frac{wz}{u}(H(u)-G(0)); \end{aligned}$$

$$\begin{aligned} F(u) &= \frac{z^2uG(0)+z^2uH(0)+z^2u-u-wz^3+z+wz}{-wz^3-u+z+wz+zu^2-wz^2u}, \\ G(u) &= \frac{z(H(0)-uzH(0)+wz^2+G(0)-zuG(0)-zu)}{-wz^3-u+z+wz+zu^2-wz^2u}, \\ H(u) &= \frac{wz(-uzH(0)-z^2-zuG(0)+G(0)-z^2H(0)+H(0)-z^2G(0))}{-wz^3-u+z+wz+zu^2-wz^2u}. \end{aligned}$$

The denominator factors as $z(u-r_1)(u-r_2)$, with

$$\begin{aligned} r_1 &= \frac{1+wz^2+\sqrt{1-(4+2w)z^2+(4w+w^2)z^4}}{2z}, \\ r_2 &= \frac{1+wz^2-\sqrt{1-(4+2w)z^2+(4w+w^2)z^4}}{2z}. \end{aligned}$$

Note the factorization $1-(4+2w)z^2+(4w+w^2)z^4 = (1-z^2w)(1-(4+w)z^2)$. Since the factor $u-r_2$ in the denominator is "bad," it must also cancel in the numerators. From this we eventually find, with the abbreviation $W = \sqrt{1-(4+2w)z^2+(4w+w^2)z^4}$:

$$F(u) = \frac{-1-wz^2-W}{2z(u-r_1)}, \qquad G(u) = \frac{-1+wz^2+W}{2z(u-r_1)}, \qquad H(u) = \frac{-1+(2+w)z^2+W}{2z(u-r_1)}.$$

The total generating function is

$$S(u)=F(u)+G(u)+H(u)=\frac{-2-w+z^2(w+w^2)+wW}{2z(u-r_1)}.$$

The special case $u=0$ (return to the x-axis) is to be noted:

$$S(0)=\frac{-2-w+z^2(w+w^2)+wW}{-2zr_1}=\frac{1-wz^2-W}{2z^2}.$$

Since there are only even powers of z in this function, we replace $x=z^2$ and get

$$\begin{aligned}S(0)&=\frac{1-wx-\sqrt{1-(4+2w)x+(4w+w^2)x^2}}{2x}\\&=1+x+(w+2)x^2+(w^2+4w+5)x^3+(w^3+6w^2+15w+14)x^4+\cdots.\end{aligned}$$

Compare the factor (w^2+4w+5) with the earlier drawing of the 10 paths.

There is again a substitution that allows for better results:

$$z=\frac{v}{1+(2+w)v+v^2},\quad \text{then}\quad S(0)=1+v.$$

Reading off coefficients can now be done using modified trinomial coefficients:

$$\binom{n;1,2+w,1}{k}=[t^k]\Big(1+(2+w)t+t^2\Big)^n.$$

Again, we use contour integration to extract coefficients:

$$\begin{aligned}[x^n](1+v)&=\frac{1}{2\pi i}\oint\frac{dx}{x^{n+1}}(1+v)\\&=\frac{1}{2\pi i}\oint\frac{dx}{v^{n+1}}\frac{1-v^2}{(1+(2+w)v+v^2)^2}(1+(2+w)v+v^2)^{n+1}(1+v)\\&=[v^n](1-v)(1+v)^2(1+(2+w)v+v^2)^{n-1}\\&=\binom{n-1;1,2+w,1}{n}+\binom{n-1;1,2+w,1}{n-1}\\&\quad-\binom{n-1;1,2+w,1}{n-2}-\binom{n-1;1,2+w,1}{n-3}.\end{aligned}$$

Now we want to count the average number of red edges. For that, we differentiate $S(0)$ with respect to w, and set $w:=1$. This leads to

$$\frac{-1+6x-5x^2+(1+3x)\sqrt{1-6x+5x^2}}{2(1-x)(1-5x)}.$$

A simple application of singularity analysis leads to $\dfrac{\frac{1}{2\sqrt5}[x^n]\frac{1}{\sqrt{1-5x}}}{-\sqrt5[x^n]\sqrt{1-5x}}\sim\dfrac{n}{5}$.

So, a random path consisting of $2n$ steps has about $n/5$ red steps, on average. For readers who are not familiar with singularity analysis of generating functions [16, 20], we just mention that one determines the local expansion around the dominating singularity, which is at $z=\frac15$ in our instance. In the denominator, we just have the total number of skew Dyck paths, according to the sequence A002212 in the OEIS [52]. In the example of Figure 2, the exact average is 6/10, which curiously is exactly the same as 3/5.

We finish the discussion by considering fixed powers of w in $S(0)$, counting skew Dyck paths consisting of zero, one, two, three, ... red edges. We find

$$[w^0]S(0) = \frac{1-\sqrt{1-4x}}{2x}, \quad [w^1]S(0) = \frac{1-2x-\sqrt{1-4x}}{2\sqrt{1-4x}},$$
$$[w^2]S(0) = \frac{x^3}{(1-4x)^{3/2}}, \quad [w^3]S(0) = \frac{x^4(1-2x)}{(1-4x)^{5/2}}, \quad \text{\&c.}$$

The generating function $[w^0]S(0)$ is of course the generating function of Catalan numbers, since no red edges just means: ordinary Dyck paths. We can also conclude that the asymptotic behaviour is of the form $n^{k-3/2}4^n$, where the polynomial contribution gets higher, but the exponential growth stays the same: 4^n. This is compared to the scenario of an *arbitrary* number of red edges, when we get an exponential growth of the form 5^n.

Dual skew Dyck paths. The mirrored version of skew Dyck paths with two types of up-steps, $(1,1)$ and $(-1,1)$ are also cited among the objects in A002212 in the OEIS [52]. We call them dual skew paths and drop the 'dual' when it isn't necessary. When the paths come back to the x-axis, no new enumeration is necessary, but this is no longer true for paths ending at level j.

Here is a list of the 10 skew paths consisting of 6 steps:

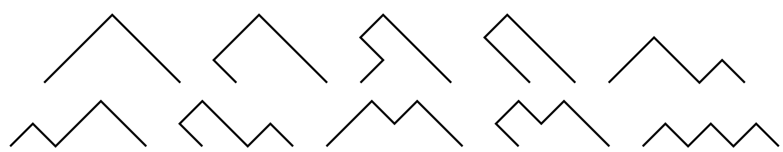


FIGURE 6. All 10 dual skew Dyck paths of length 6 (consisting of 6 steps).

We prefer to work with the equivalent model (resembling more traditional Dyck paths) where we replace each step $(-1,-1)$ by $(1,-1)$ but label it blue. Here is the list of the 10 paths again (Figure 2):

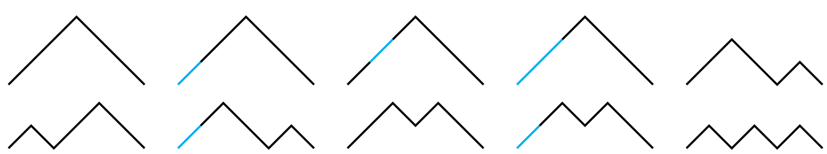
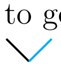
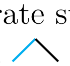


FIGURE 7. All 10 dual skew Dyck paths of length 6 (consisting of 6 steps).

The rules to generate such decorated Dyck paths are: Each edge $(1,-1)$ may be black or blue, but  and  are forbidden.

Our interest is in particular in *partial* decorated Dyck paths, ending at level j, for fixed $j \ge 0$; the instance $j = 0$ is the classical case.

As before, two variables, z and u, are used, where z marks the length of the path and j marks the end-level. We briefly mention that one can, using a third variable w, also count the number of blue edges. The substitution $x = \dfrac{v}{1+3v+v^2}$ is again the key to the success.

Generating functions and the kernel method. We catch the essence of a decorated (dual skew) Dyck path using a state-diagram:

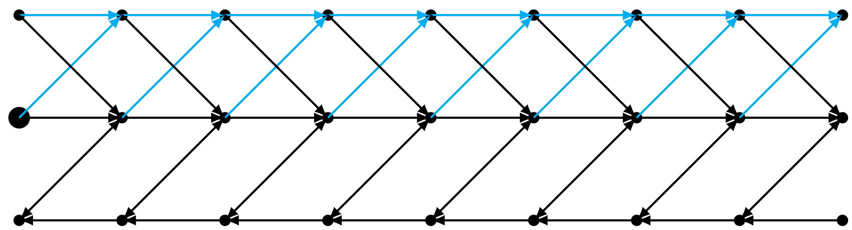


FIGURE 8. Three layers of states according to the type of steps leading to them (down, up-black, up-blue).

It has three types of states, with j ranging from 0 to infinity; in the drawing, only $j = 0..8$ is shown. The first layer of states refers to an up-step leading to a state, the second layer refers to a black down-step leading to a state and the third layer refers to a blue down-step leading to a state. We will work out generating functions describing all paths leading to a particular state. We will use the notations c_j, a_j, b_j for the three respective layers, from top to bottom. Note that the syntactic rules of forbidden patterns and can be clearly seen from the picture. The functions depend on the variable z (marking the number of steps), but mostly we just write a_j instead of $a_j(z)$, etc.

The following recursions can be read off immediately from the diagram:

$$a_0 = 1, \quad a_{i+1} = za_i + zb_i + zc_i, \quad i \ge 0,$$
$$b_i = za_{i+1} + zb_{i+1}, \quad i \ge 0,$$
$$c_{i+1} = za_i + zc_i, \quad i \ge 0.$$

And now it is time to introduce the *bivariate* generating functions:

$$A(z,u) = \sum_{i\ge 0} a_i(z)u^i, \quad B(z,u) = \sum_{i\ge 0} b_i(z)u^i, \quad C(z,u) = \sum_{i\ge 0} c_i(z)u^i.$$

Summing the recursions leads to

$$\sum_{i\ge 0} u^i a_i = 1 + u\sum_{i\ge 0} u^i(za_i + zb_i + zc_i) = 1 + uzA(u) + uzB(u) + uzC(u),$$
$$\sum_{i\ge 0} u^i b_i = \sum_{i\ge 0} u^i(za_{i+1} + zb_{i+1}) = \frac{z}{u}\sum_{i\ge 1} u^i a_i + \frac{z}{u}\sum_{i\ge 1} u^i b_i,$$
$$\sum_{i\ge 1} u^i c_i = uz\sum_{i\ge 0} u^i a_i + uz\sum_{i\ge 0} u^i c_i.$$

This can be rewritten as

$$A(u) = 1 + uzA(u) + uzB(u) + uzC(u),$$
$$B(u) = \frac{z}{u}(A(u) - a_0) + \frac{z}{u}(B(u) - b_0),$$
$$C(u) = c_0 + uzA(u) + uzC(u).$$

Note that $a_0 = 1$, $c_0 = 0$. Simplification leads to

$$C(u) = \frac{uzA(u)}{1-uz} \quad \text{and} \quad B(u) = \frac{z(A(u) - 1 - B(0))}{u - z}.$$

This leaves us with just one equation

$$A(u) = \frac{(z - u + uz^2 + uz^2 B(0))\,(uz-1)}{u^2z^3 + uz^2 - 2u^2z - z + u}.$$

This is again a typical application of the kernel method: One writes

$$u^2z^3 + uz^2 - 2u^2z - z + u = z(z^2-2)(u-s_1)(u-s_2).$$

The denominator thus factors as $2z(z^2-2)(u-s_1)(u-s_2)$, with

$$s_1 = \frac{1+z^2+\sqrt{1-6z^2+5z^4}}{2z(2-z^2)}, \quad s_2 = \frac{1+z^2-\sqrt{1-6z^2+5z^4}}{2z(2-z^2)}.$$

Note that $s_1 s_2 = \frac{1}{2-z^2}$. Since the factor $u-s_2$ in the denominator is "bad," it must also cancel in the numerators. We get $B(0) = \frac{zs_2}{1-2zs_2}$ and, again with the abbreviation $W = \sqrt{1-6z^2+5z^4}$,

$$A(u) = \frac{(1-uz)(1+z^2+W)}{2z(z^2-2)(u-s_1)}, \quad B(u) = \frac{1-2z^2-W}{z(2-z^2)(u-s_1)}, \quad C(u) = \frac{1+z^2+W}{2(z^2-2)}\frac{u}{u-s_1},$$

and for the function of main interest

$$G(u) = A(u) + B(u) + C(u) = \frac{3z^2-3+W}{2z(2-z^2)(u-s_1)}.$$

Since one has

$$\frac{1}{s_1} = \frac{1+z^2-\sqrt{1-6z^2+5z^4}}{2z} = zS \qquad \text{and} \qquad \frac{1}{s_2} = \frac{1+z^2+\sqrt{1-6z^2+5z^4}}{2z},$$

we then get

$$[u^j]G(u) = [u^j]\frac{3z^2-3+W}{2z(z^2-2)s_1(1-u/s_1)} = \frac{3z^2-3+W}{2z(z^2-2)s_1^{j+1}} = \frac{3z^2-3+W}{2(z^2-2)}z^jS^{j+1}.$$

So $[u^j]G(u)$ contains only powers of the form z^{j+2N}. Now we continue

$$\begin{aligned}
[z^{j+2N}u^j]G(u) &= [z^{2N}]\frac{3z^2-3+W}{2(z^2-2)}S^{j+1} \\
&= [x^N]\frac{3x-3+\sqrt{1-6x+5x^2}}{2(x-2)}\left(\frac{1+x-\sqrt{1-6x+5x^2}}{2x}\right)^{j+1} \\
&= [x^N](v+1)(v+2)^j
\end{aligned}$$

which is the generating function of all (partial) paths ending at level j.

Now we read off coefficients as before:

$$\begin{aligned}
[z^{j+2N}u^j]G(u) &= [x^N](v+1)(v+2)^j \\
&= \frac{1}{2\pi i}\oint \frac{dx}{x^{N+1}}(v+1)(v+2)^j \\
&= \frac{1}{2\pi i}\oint \frac{dv}{v^{N+1}}(1+3v+v^2)^{N+1}\frac{(1-v^2)}{(1+3v+v^2)^2}(v+1)(v+2)^j \\
&= [v^N](1+3v+v^2)^{N-1}(1-v)(1+v)^2(v+2)^j.
\end{aligned}$$

Note that

$$(1-v)(1+v)^2 = 3-7(v+2)+5(v+2)^2-(v+2)^3;$$

consequently

$$[z^{j+2N}u^j]G(u) = [v^N](1+3v+v^2)^{N-1}\Big[3-7(v+2)+5(v+2)^2-(v+2)^3\Big](v+2)^j.$$

We abbreviate:

$$\begin{aligned}\mu_{j;k} &= [v^k]\Big[3(v+2)^j-7(v+2)^{j+1}+5(v+2)^{j+2}-(v+2)^{j+3}\Big]\\ &= 3\binom{j}{k}2^{j-k}-7\binom{j+1}{k}2^{j+1-k}+5\binom{j+2}{k}2^{j+2-k}-\binom{j+3}{k}2^{j+3-k}.\end{aligned}$$

With this notation we get

$$[z^{j+2N}u^j]G(u) = \sum_{0\le k\le N-1}\mu_{j;k}\binom{N-1;1,3,1}{N-k}.$$

Here are the first few generating functions:

- $G_0 = 1+z^2+3z^4+10z^6+36z^8+137z^{10}+543z^{12}+2219z^{14}+\cdots$,
- $G_1 = 2z+3z^3+10z^5+36z^7+137z^9+543z^{11}+2219z^{13}+9285z^{15}+\cdots$,
- $G_2 = 4z^2+8z^4+29z^6+111z^8+442z^{10}+1813z^{12}+7609z^{14}+32521z^{16}+\cdots$,
- $G_3 = 8z^3+20z^5+78z^7+315z^9+1306z^{11}+5527z^{13}+23779z^{15}+103699z^{17}+\cdots$.

We could also give such lists for the functions a_j, b_j, c_j, if desired. We summarize the essential findings of the rest of this section:

Theorem 7.2. *The generating function of decorated (partial) dual skew Dyck paths, consisting of n steps, ending on level j, is given by*

$$G(z,u) = \frac{3z^2-3+\sqrt{1-6z^2+5z^4}}{2z(2-z^2)(u-s_1)},$$

with

$$s_1 = \frac{2z}{1+z^2-\sqrt{1-6z^2+5z^4}}.$$

Furthermore

$$[u^j]G(z,u) = \frac{3z^2-3+\sqrt{1-6z^2+5z^4}}{2(z^2-2)}z^jS^{j+1},$$

with

$$S = \frac{1+z^2-\sqrt{1-6z^2+5z^4}}{2z^2}.$$

Open ended paths. If we do not specify the end of the paths, in other words we sum over all $j\ge 0$, then at the level of generating functions this is very easy, since we only have to set $u := 1$. We find

$$\begin{aligned}G(1) &= \frac{(1+z)(1-3z)}{2z(z^2+2z-1)-\sqrt{1-6z^2+5z^4}}\\ &= 1+2z+5z^2+11z^3+27z^4+62z^5+151z^6+354z^7+859z^8+2036z^9+\cdots.\end{aligned}$$

Counting blue edges. We use an extra variable w as before to count additionally the blue edges that occur in a path. Eventually, the coefficient $[z^n u^j w^k]S$ is the number of (partial) paths consisting of n steps, leading to level j, and having passed k blue edges. The endpoint of the original skew path has then coordinates $(n-2k, j)$. The computations are very similar, and we only sketch the key steps.

$$\begin{aligned}
a_0 = 1, \quad a_{i+1} &= za_i + zb_i + zc_i, \quad i \ge 0,\\
b_i &= za_{i+1} + zb_{i+1}, \quad i \ge 0,\\
c_{i+1} &= wza_i + wzc_i, \quad i \ge 0.
\end{aligned}$$

This leads to

$$\begin{aligned}
A(u) &= 1 + uzA(u) + uzB(u) + uzC(u),\\
B(u) &= \frac{z}{u}(A(u) - a_0) + \frac{z}{u}(B(u) - b_0),\\
C(u) &= c_0 + wuzA(u) + wuzC(u).
\end{aligned}$$

Solving,

$$S(u) = A(u) + B(u) + C(u) = \frac{u - wuz^2 - zA(0) - zB(0) + uwz^2A(0) + uwz^2B(0)}{u^2z^3w + u - wu^2z - u^2z - z + wuz^2}.$$

The denominator factors as $-z(1+w-z^2w)(u-s_1)(u-s_2)$, with

$$\begin{aligned}
s_1 &= \frac{1 + z^2w + \sqrt{1 - 2\,z^2w + z^4w^2 - 4\,z^2 + 4z^4w}}{2z\,(1 + w - z^2w)},\\
s_2 &= \frac{1 + z^2w - \sqrt{1 - 2\,z^2w + z^4w^2 - 4\,z^2 + 4z^4w}}{2z\,(1 + w - z^2w)}.
\end{aligned}$$

Note the factorization $1-(4+2w)z^2+(4w+w^2)z^4 = (1-z^2w)(1-(4+w)z^2)$. Since the factor $u - r_2$ in the denominator is “bad,” it must also cancel in the numerators. From this we eventually find, with the abbreviation $W = \sqrt{1-(4+2w)z^2+(4w+w^2)z^4}\,)$

$$G(0) = \frac{1 - z^2w - W}{2z^2},$$

and further

$$G(u) = \frac{w - z^2w^2 - wW + 2 - 2z^2w}{2z\,(-w - 1 + z^2w)\,(u - s_1)}.$$

The special case $u = 0$ (return to the x-axis) is to be noted:

$$G(0) = 1 + z^2 + (w+2)\,z^4 + \left(w^2 + 4w + 5\right)z^6 + (w+2)\left(w^2 + 4w + 7\right)z^8 + \cdots.$$

Compare the factor $(w^2 + 4w + 5)$ with the earlier drawing of the 10 paths. There is again a substitution that allows for better results:

$$z = \frac{v}{1 + (2+w)v + v^2}, \quad \text{then} \quad G(0) = 1 + v.$$

Since $S(u) = G(u)$ with $S(u)$ from the first part of the paper, as it means the same objects, read from left to right resp. from right to left, no new analysis is required.

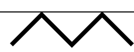
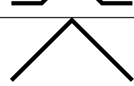
8. More about Motzkin paths

Several other interesting problems related to Motzkin paths were considered in the literature. To wet the readers appetite, we briefly mention below the Retakh paths, the amplitude of paths, and then analyze in more detail skew Motzkin paths.

Retakh's Motzkin paths. V. Retakh [15] introduced a restricted class of Dyck paths: Peaks are only allowed on level 1 and on even-numbered levels. For the analysis of this class using generating functions, including also the average height and the number of leaves; see [43].

The amplitude of Motzkin paths. Another interesting parameter is the amplitude of Motzkin paths that was recently analyzed in [45]. Here, we want to give a few introductory remarks. The average height of a random Motzkin path of length n was considered in an early paper [35], it is asymptotically given by $\sqrt{\frac{\pi n}{3}}$.

In the recent paper [8] an interesting new concept was introduced: the *amplitude*. It is basically twice the height, but with a twist. If there exists a horizontal step on level h, which is the height, the amplitude is $2h+1$, otherwise it is $2h$. To clarify the concept, we created a list of all 9 Motzkin paths of length 4 with height and amplitude given.

Motzkin path	horizontal on maximal level	height	amplitude
	Yes	0	1
	No	1	2
	No	1	2
	Yes	1	3
	Yes	1	3
	Yes	1	3
	No	1	2
	No	1	2
	No	2	4

The goal of an extended analysis is to investigate this new parameter, using trinomial coefficients $\binom{n,3}{k} = [t^k](1+t+t^2)^n$ (notation following Comtet's book [7]). The intuitive result that the average amplitude is about twice the average height, can be confirmed.

Skew Motzkin paths. This section was written to provide a more complete analysis related to [30]. The methods are of course by now familiar to the readers.

As seen before, Motzkin paths allow additional flat (horizontal) steps of unit length. A skew path allows 'left' step $(-1,-1)$ as well, but the path is not allowed to intersect itself. We prefer 'red' steps $(1,-1)$; see our analysis in [46]. For Motzkin paths, some analysis was provided in [30]. Here, we provide further analysis that allows to consider partial paths as well, so we do not need to land at the x-axis. It uses the kernel method [37].

Apart from being not below the x-axis, the restrictions are that a left (red) step cannot follow or precede an up-step. The situation is best described by a graph (state-diagram); see Figure 9.

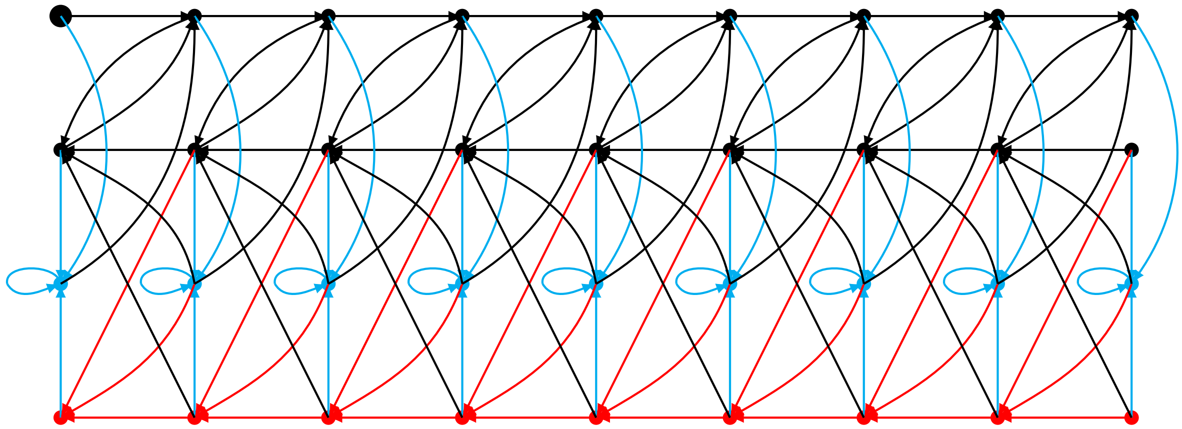


FIGURE 9. Four layers of states according to the type of steps leading to them. Traditional up-steps and down-steps are black, level-steps are blue, and left steps are red.

In further sections, the asymptotic equivalent for the number of skew Motzkin paths of given size is derived, as well as the *height*, meaning that the generating function of paths with a bounded height (bounded by H) is given, as well as the average height, which is approximately $\mathsf{const}\cdot\sqrt{n}$, which is typical for families of paths.

Generating functions for skew Motzkin paths. We translate the state diagram accordingly; f_j, g_j, h_j, k_j are generating functions in the variable z (marking the length of the path), ending at level j. The four families are related to the four layers of states.

$$\begin{aligned} f_{j+1} &= zf_j + zg_j + zh_j,\ \ f_0 = 1,\\ g_j &= zf_{j+1} + zg_{j+1} + zh_{j+1} + zk_{j+1},\\ h_j &= zf_j + zg_j + zh_j + zk_j,\\ k_j &= zg_{j+1} + zh_{j+1} + zk_{j+1}. \end{aligned}$$

Now we introduce bivariate generating functions, namely

$$F(u) := \sum_{j\ge0} f_j u^j, \quad G(u) := \sum_{j\ge0} g_j u^j, \quad H(u) := \sum_{j\ge0} h_j u^j, \quad K(u) := \sum_{j\ge0} k_j u^j.$$

The recursions then take this form:

$$\begin{aligned} F(u) &= 1 + zuF(u) + zuG(u) + zuH(u),\\ uG(u) &= zF(u) + zG(u) + zH(u) + zK(u) - z - zg_0 - zh_0 - zk_0,\\ H(u) &= zF(u) + zG(u) + zH(u) + zK(u),\\ uK(u) &= zG(u) + zH(u) + K(u) - zg_0 + zh_0 + zk_0. \end{aligned}$$

Solving the system,

$$F(u)=\frac{\mathscr{F}}{2z-u+zu-z^2u+zu^2-z^3-z^3u},$$
$$G(u)=\frac{\mathscr{G}}{2z-u+zu-z^2u+zu^2-z^3-z^3u},$$
$$H(u)=\frac{\mathscr{H}}{2z-u+zu-z^2u+zu^2-z^3-z^3u},$$
$$K(u)=\frac{\mathscr{K}}{2z-u+zu-z^2u+zu^2-z^3-z^3u},$$

with

$$\mathscr{F}=-z^3+2z-u+zu+z^2u+z^2ug_0+z^2uh_0+z^2uk_0+z^3ug_0+z^3uh_0+z^3uk_0,$$
$$\mathscr{G}=-z^2h_0+z^4+z^4k_0-z^2ug_0-z^2u-z^2k_0-z^2-z^2uh_0-z^2uk_0+z^4h_0+z^3-z^2g_0+zh_0 \\ +zg_0+zk_0+z^4g_0,$$
$$\mathscr{H}=-z^4+2z^2g_0+2z^2h_0+2z^2k_0+2z^2-zu-z^4g_0-z^4h_0-z^4k_0-z^3ug_0-z^3uh_0-z^3uk_0,$$
$$\mathscr{K}=zg_0-z^3-z^2g_0-z^2h_0+zk_0-z^2k_0-z^3g_0-z^3h_0-z^3k_0.$$

One cannot immediately insert $u=0$ to identify the constants, but one can use the kernel method. For that, one factorizes the denominator:

$$2z-u+zu-z^2u+zu^2-z^3-z^3u=z(u-u_1)(u-u_2)$$

with

$$u_1=\frac{1-z+z^2+z^3+(1+z)W}{2z},\quad u_2=\frac{1-z+z^2+z^3-(1+z)W}{2z}$$

and

$$W=\sqrt{(1-z)(1-3z-z^2-z^3)}=\sqrt{1-4z+2z^2+z^4}.$$

Since $u_2\sim 2z$ for small z, $u-u_2$ is a 'bad' factor and must cancel from both, numerator and denominator. This yields

$$F(u)=\frac{-1+z+z^2+z^2g_0+z^2h_0+z^2k_0+z^3g_0+z^3h_0+z^3k_0}{z(u-u_1)},$$
$$G(u)=\frac{-z^2-z^2g_0-z^2h_0-z^2k_0}{z(u-u_1)},$$
$$H(u)=\frac{-z-z^3g_0-z^3h_0-z^3k_0}{z(u-u_1)},$$
$$K(u)=\frac{-z^2g_0-z^2h_0-z^2k_0}{z(u-u_1)}.$$

Now we can plug in $u=0$ and identify the constants:

$$g_0=\frac{-z^5+z^3W-z^2-zW+3z-1+W}{2z^2\left(-2+z^2\right)},$$
$$h_0=-\frac{-z^2+2z-1+W}{2z},$$
$$k_0=-\frac{-z^4+z^3+z^2W+zW-3z+1-W}{2z^2\left(-2+z^2\right)}.$$

Adding these quantities yields

$$1 + g_0 + h_0 + k_0 = -\frac{-z^2 + 2z - 1 + W}{2z^2},$$

which is the generating function of the number of skew Motzkin paths (returning to the x-axis); the series expansion is

$$1 + z + 2z^2 + 5z^3 + 13z^4 + 35z^5 + 97z^6 + 275z^7 + 794z^8 + 2327z^9 + 6905z^{10} + 20705z^{11} + \cdots ,$$

as already given in [30], the coefficients are the sequence A082582 in the OEIS [52].

We further get

$$\begin{aligned}
F(u) &= \frac{-1 + z - z^2 - z^3 + u_2 z}{z(u - u_1)},\\
G(u) &= \frac{(z - u_2)}{(u - u_1)(1 + z)},\\
H(u) &= \frac{-1 - z + 2z^2 + z^3 - zu_2}{(u - u_1)(1 + z)},\\
K(u) &= \frac{z^2 + 2z - u_2}{(u - u_1)(1 + z)}.
\end{aligned}$$

Altogether,

$$F(z) + G(z) + H(z) + K(z) = \frac{-1 - z + 2z^2 + z^3 - zu_2}{z(u - u_1)(1 + z)}$$

and

$$[u^j]\Big(F(z) + G(z) + H(z) + K(z)\Big) = \frac{1 + z - 2z^2 - z^3 + zu_2}{z(1 + z)u_1^{j+1}},$$

which is the generating function of partial skew Motzkin paths, landing on level j. Here are the examples for $j = 1, 2, 3, 4$ (leading terms only):

$$\begin{aligned}
&z + 2z^2 + 5z^3 + 13z^4 + 36z^5 + 102z^6 + 295z^7 + 866z^8 + 2574z^9 + 7730z^{10} + 23419z^{11},\\
&z^2 + 3z^3 + 9z^4 + 26z^5 + 77z^6 + 230z^7 + 694z^8 + 2110z^9 + 6459z^{10} + 19890z^{11} + 61577z^{12},\\
&z^3 + 4z^4 + 14z^5 + 45z^6 + 143z^7 + 451z^8 + 1421z^9 + 4478z^{10} + 14129z^{11} + 44654z^{12},\\
&z^4 + 5z^5 + 20z^6 + 71z^7 + 242z^8 + 806z^9 + 2653z^{10} + 8670z^{11} + 28213z^{12}.
\end{aligned}$$

One can also substitute $u = 1$, which means that *all* partial skew Motzkin paths are counted with respect to length, regardless on which level they end:

$$\frac{2 - 3z - 7z^2 - z^3 + z^4 - (2 + z)(1 + z)W}{2z(1 + z)(2z^2 + 3z - 1)}.$$

The series expansion is

$$1 + 2z + 5z^2 + 14z^3 + 40z^4 + 117z^5 + 348z^6 + 1049z^7 + 3196z^8 + 9823z^9 + 30413z^{10} + \cdots$$

Counting flat and left (red) steps. Using two extra variables t and w, we can count the number of flat resp. left steps in a skew Motzkin path. The recursions are self-explanatory.

$$\begin{aligned} f_{j+1} &= zf_j + zg_j + zh_j,\ \ f_0 = 1,\\ g_j &= zf_{j+1} + zg_{j+1} + zh_{j+1} + zk_{j+1},\\ h_j &= ztf_j + ztg_j + zth_j + ztk_j,\\ k_j &= zwg_{j+1} + zwh_{j+1} + zwk_{j+1}. \end{aligned}$$

Again, here is the system for the multi-variate generating functions;

$$\begin{aligned} F(u) &= 1 + zuF(u) + zuG(u) + zuH(u),\\ uG(u) &= zF(u) + zG(u) + zH(u) + zK(u) - z - zg_0 - zh_0 - zk_0,\\ H(u) &= ztF(u) + ztG(u) + ztH(u) + ztK(u),\\ uK(u) &= zwG(u) + zwH(u) + zwK(u) - zwg_0 + zwh_0 + zwk_0. \end{aligned}$$

And following a similar procedure as before we get

$$\begin{aligned} 1 + g_0 + h_0 + k_0 &= \frac{-zw + u_2}{z(1+wt)}\\ &= 1 + tz + (t^2+1)z^2 + (tw + 3t + t^3)z^3 + (2 + 6t^2 + w + 3wt^2 + t^4)z^4 + \cdots ; \end{aligned}$$

the quantity u_2 is now

$$u_2 = \frac{1 - tz + wz^2 + twz^3 - \sqrt{(1-z^2w)(1 - 2\,tz + (t^2 - 4 - w)\,z^2 - 2twz^3 - wt^2z^4)}}{2z}.$$

Quantities like $F(u)$, $G(u)$, $H(u)$, $K(u)$ can also be computed easily, following the approach from the previous section.

Asymptotics for the number of skew Motzkin paths. We must analyze the generating function

$$\mathscr{S}\mathscr{M} = \frac{(1-z)^2 - \sqrt{(1-z)(1-3z-z^2-z^3)}}{2z^2}$$

which is of the sqrt-type [16, 20] around the singularity ρ closest to the origin, which we call ρ. It is a solution of $1 - 3z - z^2 - z^3 = 0$ and can be expressed as

$$\rho = \frac{\sqrt[3]{26 + 6\sqrt{33}}}{3} - \frac{8}{3\sqrt[3]{26 + 6\sqrt{33}}} - \frac{1}{3} \approx 0.295597.$$

Expanding the generating function around $z = \rho$, we get

$$\mathscr{S}\mathscr{M} \sim \frac{(1-\rho)^2 + 2\sqrt{(1-\rho-\rho^3)(z-\rho)}}{2\rho^2}.$$

Singularity analysis of generating function [16, 20] gives the estimate

$$[z^n]\mathscr{S}\mathscr{M} \sim \frac{\sqrt{1-\rho-\rho^3}}{2\sqrt{\pi\rho^3}}\rho^{-n}n^{-3/2}. \tag{8.1}$$

The error at $n = 100$ is about 3%. This is to be expected by this type of approximation.

Skew Motzkin paths of bounded height. Now we introduce a parameter h and do not allow the path to reach any level higher than h. We can still work with the system

$$\begin{aligned} f_{j+1} &= zf_j + zg_j + zh_j,\ 0 \le j \le h-1,\ f_0 = 1,\\ g_j &= zf_{j+1} + zg_{j+1} + zh_{j+1} + zk_{j+1},\ 0 \le j < h,\\ h_j &= zf_j + zg_j + zh_j + zk_j,\ 0 \le j \le h,\\ k_j &= zg_{j+1} + zh_{j+1} + zk_{j+1},\ 0 \le j < h. \end{aligned}$$

This is now a finite linear system, and we are only interested in paths that return to the x-axis. For a given h, we write $s[h] = f_0 + g_0 + h_0 + k_0$ for the generating function of path of height $\le h$. It can be proved that both the numerator and the denominator of $s[h]$ satisfy the recursion

$$X_{h+2} + (-1 + z - z^2 - z^3)X_{h+1} + (2z^2 - z^4)X_h = 0.$$

Thus, adjusting this to the initial conditions, we get

$$s[h] = \frac{A_o(1 + z^3 + z^2 - z + \omega)^h + B_o(1 + z^3 + z^2 - z - \omega)^h}{A_u(1 + z^3 + z^2 - z + \omega)^h + B_u(1 + z^3 + z^2 - z - \omega)^h}$$

with

$$\begin{aligned} \omega &= \sqrt{z^6 + 2z^5 + 3z^4 - 5z^2 - 2z + 1},\\ A_o &= (z^3 + z^2 + 3z - 1)(z + 1) + (z - 1)\omega,\\ B_o &= (z^3 + z^2 + 3z - 1)(z + 1) - (z - 1)\omega,\\ A_u &= (1 - z^2)(z^3 + z^2 + 3z - 1) + \frac{z^3 - z^2 + 3z - 1}{1 - z}\omega,\\ B_u &= (1 - z^2)(z^3 + z^2 + 3z - 1) - \frac{z^3 - z^2 + 3z - 1}{1 - z}\omega. \end{aligned}$$

When h goes to infinity, the second terms go away, and we are left with

$$s[\infty] = \frac{A_o}{A_u} = \frac{(1 - z)^2 - \sqrt{(1 - z)(1 - 3z - z^2 - z^3)}}{2z^2} = \mathscr{S}\mathscr{M},$$

as expected. Now we consider $s[> h]$, the generating function of skew Motzkin paths of height $> h$. Taking differences, we find

$$\begin{aligned} s[> h] = s[\infty] - s[h] &= \frac{A_oB_u - A_uB_o}{A_u} \frac{(1 + z^3 + z^2 - z - \omega)^h}{A_u(1 + z^3 + z^2 - z + \omega)^h + B_u(1 + z^3 + z^2 - z - \omega)^h}\\ &\sim \frac{A_oB_u - A_uB_o}{A_u^2} \frac{\left(\dfrac{1 + z^3 + z^2 - z - \omega}{1 + z^3 + z^2 - z + \omega}\right)^h}{1 - \left(\dfrac{1 + z^3 + z^2 - z - \omega}{1 + z^3 + z^2 - z + \omega}\right)^h}. \end{aligned}$$

A computer computation leads to (always in the neighbourhood of $z = \rho$)

$$\frac{A_oB_u - A_uB_o}{A_u^2} \sim 18.854986275200314363\sqrt{\rho - z}.$$

Now we approximate and write for convenience:

$$\frac{1+z^3+z^2-z-\omega}{1+z^3+z^2-z+\omega} \sim 1-5.2213516788791457598\sqrt{\rho-z}$$
$$\sim \exp\Bigl(-5.2213516788791457598\sqrt{\rho-z}\Bigr)=e^{-t}.$$

For the average height, we need apart from the leading factor,

$$\sum_{h\ge0} s[>h] \sim \sum_{h\ge0}\frac{e^{-th}}{1-e^{-th}}.$$

Since we only compute the leading term of the asymptotics of the average height, we might start the sum at $h\ge1$, and expand the geometric series:

$$\sum_{h\ge1} s[>h] \sim \sum_{h,k\ge1} e^{-thk} = \sum_{k\ge1} d(k)e^{-kt} \sim -\frac{\log t}{t},$$

with $d(k)$ being the number of divisors of k. This type of analysis, although having been done often before, has been described in much detail in [24]. Together with the factor in front, we are at

$$-18.854986275200314363\sqrt{\rho-z}\,\frac{\log\sqrt{\rho-z}}{5.2213516788791457598\sqrt{\rho-z}}$$
$$\sim -1.8055656307800996608\log(1-z/\rho).$$

Singularity analysis [16] gives the following estimate for the coefficient of z^n:

$$1.8055656307800996608\frac{\rho^{-n}}{n}.$$

For the average height we need to normalize, that is, we divide by the total number of skew Motzkin numbers of size n given by (8.1):

$$\frac{1.8055656307800996608\frac{\rho^{-n}}{n}}{5.1256244361431546460\frac{1}{2\sqrt{\pi}}\rho^{-n}n^{-3/2}} = 0.70452513767814089508\sqrt{\pi n}.$$

9. Oscillations in Dyck paths revisited

This section in honour of Rainer Kemp was written for this personal survey.

Rainer Kemp's paper [25] was unfortunately largely overlooked. An extension was published quickly [28], and then it fell into oblivion. We want to come back to this gem, with modern methods, in particular, the kernel method and singularity analysis. Kemp was interested in peaks and valleys of Dyck paths, which he called MAX-turns and MIN-turns, probably motivated by computer science applications. The peaks/valleys are enumerated from left to right, and the height of the j-th one is analyzed. In the corresponding ordered (plane) tree, the peaks correspond to the leaves.

Very precise information is available for leaves of binary trees [22,27,33,34] but the situation is a bit different for Dyck paths since the number of peaks/valleys isn't directly related to the length of the Dyck path. (Narayana numbers enumerate them.) Kemp's results in a nutshell are: The average height of the m-th peak/valley is $\sim 4\sqrt{2m/\pi}$ (it is asymptotically independent of the length n of the path), and the difference between the height of the peak and the next valley is about 2, with more terms being available in principle.

A trivariate generating function for heights of valleys. The goal is to derive an expression for $\Phi(u) = \Phi(u; z, w)$, where z is used for the length of the path, w for the enumeration of the *valleys* (w^m corresponds to the m-th valley), and u is used to record the height of the m-th (and last) valley of a partial Dyck path (the path does not need to return to the x-axis). We could think about it continued in any possible fashion, as in the following figure. We will figure out the generating function of partial Dyck paths with m valleys, and the generating function of the 'rest', which (if it is not empty) is a partial Dyck path starting with an up-step and ends on the x-axis, where the number of valleys is immaterial. The enumeration of the rest is easy, when one thinks about it from right to left, since then it is just a Dyck path ending on a prescribed level j with a down-step. This can be obtained from the first part by setting $w = 1$, i.e., not counting the valleys.

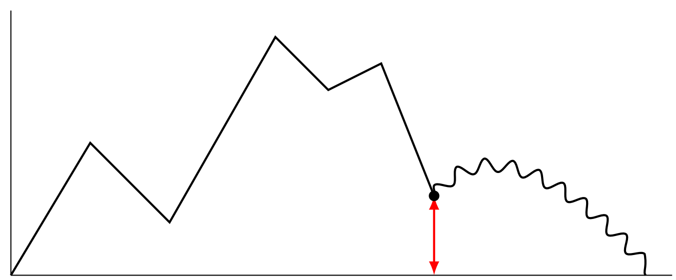


FIGURE 10. The third valley at level j.

Our goal is, as often, to use the adding-a-new-slice technique, namely adding another 'mountain' (a maximal sequence of up-steps, followed by a maximal sequence of down-steps), or going from the m-th valley to the $(m+1)$-st valley. We investigate what can happen to u^j:

$$\sum_{l\ge1}\sum_{i=1}^{j+l} z^l u^{j+l} z^i u^{-i}.$$

Working this out, the following substitution is essential for our problem:

$$u^j \longrightarrow \frac{z^2u^k}{(u-z)(1-zu^k)}u^j - \frac{z^{k+2}}{(u-z)(1-z^{k+1})}z^j.$$

Working this into a generating function of the type

$$\Phi(u) = \sum_{m\ge0} w^m \varphi_m(u),$$

where the variable w keeps track of the number of mountains, we find from the substitution

$$\Phi(u) = 1 + \frac{wz^2u}{(u-z)(1-zu)}\Phi(u) - \frac{wz^3}{(u-z)(1-z^2)}\Phi(z),$$

where 1 stands for the empty path having no mountains. Rearranging,

$$\Phi(u)\frac{z(u-s_1)(u-s_2)}{(u-z)(zu-1)} = 1 - \frac{wz^3}{(u-z)(1-z^2)}\Phi(z),$$

and

$$\Phi(u) = \frac{(zu-1)}{z(u-s_1)(u-s_2)}\left[u - z - \frac{wz^3}{(1-z^2)}\Phi(z)\right].$$

Here,

$$s_2 = \frac{z^2 + 1 - wz^2 - \sqrt{z^4 - 2z^2 - 2z^4w + 1 - 2wz^2 + w^2z^4}}{2z} \quad \text{and} \quad s_1 = \frac{1}{s_2}.$$

In the spirit of the kernel method, the factor $u - s_2$ is 'bad' and must cancel out. That leads first to

$$\Phi(z) = \frac{(1-z^2)(s_2 - z)}{wz^3}$$

and further to

$$\begin{aligned}\Phi(u) &= \frac{(zu-1)}{z(u-s_1)} = \frac{s_2(1-zu)}{z(1-us_2)} \\ &= 1 + wz^2 + wuz^3 + (w^2 + w + wu^2)z^4 + (2w^2u + wu + wu^3)z^5 + \cdots .\end{aligned}$$

From this it is easy to read off coefficients in general:

$$[u^j]\Phi(u) = [u^j]\frac{s_2(1-zu)}{z(1-us_2)} = \frac{1}{z}s_2^{j+1} - s_2^j.$$

Note that setting $w = 1$ ignores the number of mountains, and the generating function would then be enumerating partial Dyck paths ending on level j with a down-step. The answer could then be derived by combinatorial means as well.

For Kemp's problem, we need

$$S = \sum_{j\ge 0} j\Big(\frac{1}{z}s_2^{j+1} - s_2^j\Big)\cdot\Big(\frac{1}{z}s_2^{j+1} - s_2^j\Big)\Big|_{w=1}.$$

Recall that the two parts of the Dyck path, according to our decomposition, are glued together, which just means multiplication of generating functions. The factor j comes in because of the average value of the height of the valley, the first factor is what we just worked out, and the third factor is the rest, which, when read from right to left, is just what we discussed, since the number of valleys or mountains in the rest is irrelevant. Thanks to computer algebra (not available when Kemp worked on the oscillations), we get

$$S = 4\frac{(-3z + W_1z - W_1 + 1)\,(-W_2 + wzW_2 + 1 + z^2w^2 - wz^2 - 2wz - z)}{z\,(-3z - W_1z + 1 - W_1 - wz + wzW_1 - W_2 + W_2W_1)^2}$$

with

$$W_1 = \sqrt{1-4z} \quad \text{and} \quad W_2 = \sqrt{z^2 - 2z - 2z^2w + 1 - 2wz + w^2z^2}.$$

Note carefully that z^2 was replaced by z, since Dyck paths (returning to the x-axis) must have an even number of steps. Their enumeration is classical:

$$D(z) = \frac{1-\sqrt{1-4z}}{2z} \sim 2 - 2\sqrt{1-4z},$$

for z close to the (dominant) singularity $z = \frac{1}{4}$. We are in the regime of the subcritical case; see [20, Section IX-3].

The function S has a similar local expansion:

$$S \sim C_1(w) - C_2(w)\sqrt{1-4z},$$

and the function $\frac{C_2(w)}{2}$ is the resulting generating function. Working out the details,

$$S \sim \frac{w + \sqrt{(1-w)(9-w)} - 3}{-1+w} - \sqrt{1-4z}\left(\frac{w^2+2w-3+(1+w)\sqrt{(1-w)(9-w)}}{(1-w)^2}\right) + \cdots .$$

Eventually we are led to

$$\mathsf{Valley}(w) := \frac{w^2+2w-3+(1+w)\sqrt{(1-w)(9-w)}}{2(1-w)^2}.$$

To say it again, the coefficient of w^m in this is the average value of the m-th valley in a 'very long' Dyck path. To say more about it, we can use singularity analysis again and expand (this time around $w=1$, which is dominant):

$$\mathsf{Valley}(w) \sim \frac{2\sqrt{2}}{(1-w)^{3/2}} - \frac{2}{1-w} - \frac{7}{8}\frac{\sqrt{2}}{\sqrt{1-w}}.$$

The traditional translation theorems [16, 20] lead to the average value of the height of the m-th valley:

$$4\sqrt{2}\sqrt{\frac{m}{\pi}} - 2 + \frac{5\sqrt{2}}{8\sqrt{\pi m}} + \cdots .$$

From valleys to peaks. We do not need too many new computations, as we can modify the previous results. If one adds an arbitrary non-empty number of up-steps after the m-th valley, one has reached the $(m+1)$-st peak! This is basically a substitution!

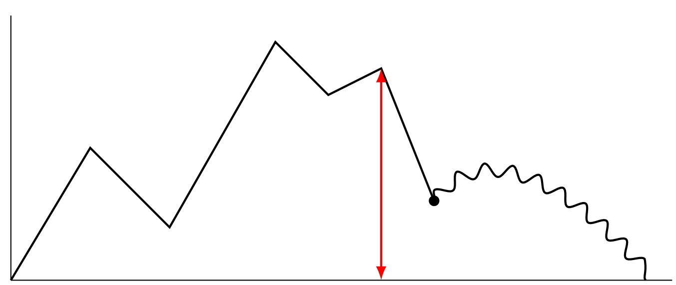


FIGURE 11. The third peak at level j.

Start from

$$\Phi(u) = \frac{s_2(1-zu)}{z(1-us_2)}$$

and attach a sequence of up-steps: $u^j \to \frac{zu}{1-zu}u^j$. A factor w is also important, since the m-th valley corresponds to the $(m+1)$-st peak. Now

$$\frac{zuw}{1-zu}\frac{s_2(1-zu)}{z(1-us_2)} = \frac{us_2w}{1-us_2} = w\sum_{j\ge1} u^j s_2^j.$$

The computation

$$w\sum_{j\ge 0} js_2^j\cdot s_2^j\Big|_{w=1}$$

was basically done before, and the local expansion leads to

$$\frac{2w}{1-w}-\frac{2w\sqrt{(1-w)(9-w)}}{(1-w)^2}\sqrt{1-4z},$$

and the generating function of the average values of the m-th peak is

$$\mathsf{Peak}(w)=\frac{w\sqrt{(1-w)(9-w)}}{(1-w)^2}.$$

A local expansion of this results in

$$\mathsf{Peak}(w)\sim\frac{2\sqrt{2}}{(1-w)^{3/2}}-\frac{15}{8}\frac{\sqrt{2}}{\sqrt{1-w}}.$$

Taking differences:

$$\mathsf{Peak}(w)-\mathsf{Valley}(w)\sim\frac{2}{1-w}-\frac{\sqrt{2}}{\sqrt{1-w}},$$

and translating into asymptotics:

$$2-\frac{\sqrt{2}}{\sqrt{\pi m}}.$$

The formula $2+O(m^{-1/2})$ was already known to Kemp [25]. As Kemp stated in [25], which was confirmed in [28], the generating functions $\mathsf{Peak}(w)$ and $\mathsf{Valley}(w)$ can be expressed by Legendre polynomials at special values. This is a bit artificial and not too useful in itself.

10. Deutsch-paths in a strip

Emeric Deutsch [11] had the idea to consider a variation of ordinary Dyck paths, by augmenting the usual up-steps and down-steps by one unit each, by down-steps of size $3,5,7,\dots$; the set of down-steps is $\{(1,-1),(1,-3),(1,-5),\dots\}$. This leads to ternary equations, as can be seen for instance in [41].

The present author started to investigate a related but simpler model of down-steps $1,2,3,4,\dots$ and investigated it (named Deutsch paths in honour of Emeric Deutsch) in a series of papers [39, 40, 42]. This simpler model can also be seen in the context of Łukasiewicz paths, except that horizontal steps are not allowed; see [5]. Another relevant paper that deals with infinite step sets is [3].

This section is a further member of this series (and extends our unpublished preprint arXiv:2108.12797): The condition that (as with Dyck paths) the paths cannot enter negative territory is relaxed by introducing a negative boundary $-t$. Here are two recent publications about such a negative boundary: [51] and [49].

Instead of allowing negative altitudes, we think about the whole system shifted up by t units, and start at the point $(0,t)$ instead. This is much better for the generating functions that we are going to investigate. Eventually, the results can be re-interpreted as results about enumerations with respect to a negative boundary.

The setting with flexible initial level t and final level j allows us to consider the Deutsch paths also from right to left (they are not symmetric!), without any new computations.

The next sections achieves this, using the celebrated kernel method. An additional upper bound is introduced, so that the Deutsch paths live now in a strip. The way to attack this is linear algebra. Once everything has been computed, one can relax the conditions and let lower/upper boundary go to $\mp\infty$.

Generating functions and the kernel method. As discussed, we consider Deutsch paths starting at $(0,t)$ and ending at (n,j), for $n,t,j\ge 0$. First we consider univariate generating functions $f_j(z)$, where z^n stays for n steps done, and j is the final destination. The recursion is immediate:

$$f_j(z) = [\![t=j]\!] + zf_{j-1}(z) + z\sum_{k>j} f_k(z),$$

where $f_{-1}(z)=0$. Next, we consider $F(z,u) := \sum_{j\ge 0} f_j(z)u^j$, and get

$$\begin{aligned}
F(z,u) &= u^t + zuF(z,u) + z\sum_{j\ge 0} u^j \sum_{k>j} f_k(z) = u^t + zuF(z,u) + z\sum_{k>0} f_k(z) \sum_{0\le j<k} u^j\\
&= u^t + zuF(z,u) + z\sum_{k\ge 0} f_k(z)\frac{1-u^k}{1-u} = u^t + zuF(z,u) + \frac{z}{1-u}[F(z,1)-F(z,u)]\\
&= \frac{u^t(1-u)+zF(z,1)}{z-zu+zu^2+1-u}.
\end{aligned}$$

Since the critical value is around $u=1$, we write the denominator as

$$z(u-1)^2 + (u-1)(z-1) + z = z(u-1-r_1)(u-1-r_2),$$

with

$$r_1 = \frac{1-z+\sqrt{1-2z-3z^2}}{2z}, \quad r_2 = \frac{1-z-\sqrt{1-2z-3z^2}}{2z}.$$

The factor $(u-1-r_2)$ is bad, so the numerator must vanish for $[u^t(1-u)+zF(z,1)]\big|_{u=1+r_2}$, therefore $zF(z,1) = (1+r_2)^t r_2$. Furthermore

$$F(z,u) = \frac{\frac{u^t(1-u)+zF(z,1)}{u-r_2}}{z(u-r_1)}.$$

The expressions become prettier using the substitution $z = \frac{v}{1+v+v^2}$; then $r_1 = \frac{1}{v}$, $r_2 = v$. It can be proved by induction (or computer algebra) that

$$\frac{u^t(1-u)+v(1+v)^t}{u-1-v} = -v\sum_{k=0}^{t-1}(1+v)^{t-1-k} - u^t.$$

Furthermore

$$\frac{1}{z(u-1-r_1)} = -\frac{1}{z(1+r_1)(1-\frac{u}{1+r_1})},$$

and so

$$f_j(z) = [u^j]F(z,u) = [u^j]\left[v\sum_{k=0}^{t-1}(1+v)^{t-1-k}u^k + u^t\right]\sum_{\ell\ge 0}\frac{u^\ell}{z(1+r_1)^{\ell+1}}.$$

Les us mention two interesting special cases: the case $t = 0$ (which was also studied before [39])

$$f_j = \frac{(1+v+v^2)v^j}{(1+v)^{j+1}},$$

and the case $j = 0$ for general t, as it may be interpreted as Deutsch paths read from right to left, starting at level 0 and ending at level $t \geq 1$ (for $t = 0$, the previous formula applies). It gives

$$\begin{aligned} f_0(z) &= [u^0]\Bigg[v\sum_{k=0}^{t-1}(1+v)^{t-1-k}u^k + u^t\Bigg]\sum_{\ell\geq 0}\frac{u^\ell}{z(1+r_1)^{\ell+1}} \\ &= v(1+v)^{t-1}\frac{1}{z(1+r_1)} = v(1+v+v^2)(1+v)^{t-2}. \end{aligned}$$

The next section will present a simplification of the expression for $f_j(z)$, which could be obtained directly by distinguishing cases and summing some geometric series.

Refined analysis: lower and upper boundary. Now we consider Deutsch paths bounded from below by zero and bounded from above by $m-1$; they start at level t and end at level j after n steps. For that, we use generating functions $\varphi_j(z)$ (the quantity t is a silent parameter here). The recursions that are straight-forwarded are best organized in a matrix, as the following example shows.

$$\begin{pmatrix} 1 & -z & -z & -z & -z & -z & -z & -z \\ -z & 1 & -z & -z & -z & -z & -z & -z \\ 0 & -z & 1 & -z & -z & -z & -z & -z \\ 0 & 0 & -z & 1 & -z & -z & -z & -z \\ 0 & 0 & 0 & -z & 1 & -z & -z & -z \\ 0 & 0 & 0 & 0 & -z & 1 & -z & -z \\ 0 & 0 & 0 & 0 & 0 & -z & 1 & -z \\ 0 & 0 & 0 & 0 & 0 & 0 & -z & 1 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{array}{l} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} t \\ \vphantom{\begin{matrix}0\\0\\0\\0\\0\end{matrix}} \end{array}$$

The goal is now to solve this system. For that the substitution $z = \frac{v}{1+v+v^2}$ is used throughout. The method is to use Cramer's rule, which means that the right-hand side has to replace various columns of the matrix, and determinants have to be computed. At the end, one has to divide by the determinant of the system.

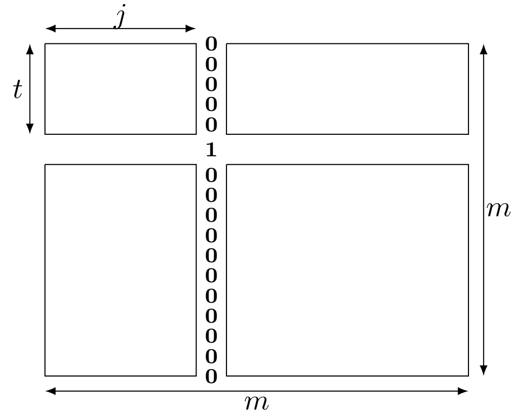
Let D_m be the determinant of the matrix with m rows and columns. The recursion

$$(1+v+v^2)^2 m_{n+2} - (1+v+v^2)(1+v)^2 D_{m+1} + v(1+v)^2 D_m = 0$$

appeared already in [39] and is not difficult to derive and to solve:

$$D_m = \frac{(1+v)^{m-1}}{(1+v+v^2)^m}\frac{1-v^{m+2}}{1-v}.$$

To solve the system with Cramer's rule, we must compute a determinant of shape



where the various rows are replaced by the right-hand side. While it is not impossible to solve this recursion by hand, it is very easy to make mistakes, so it is best to employ a computer. Let $D(m;t,j)$ the determinant according to the drawing.

It is not unexpected that the results are different for $j<t$ resp. $j\ge t$. Here is what we found:

$$D(m;t,j)=\frac{(1+v)^{t-j-3+m}(1-v^{j+1})v(1-v^{m-t})}{(1-v)^2(1+v+v^2)^{m-1}},\quad \text{for } j<t,$$
$$D(m;t,j)=\frac{v^{j-t}(1-v^{t+2})(1-v^{1-j+m})}{(1-v)^2(1+v+v^2)^{m-1}(1+v)^{j-t+3-m}},\quad \text{for } j\ge t.$$

To solve the system, we have to divide by the determinant D_m, with the result

$$\varphi_j=\frac{D(m;t,j)}{D_m}=\frac{(1+v)^{t-j-2}(1-v^{j+1})v(1-v^{m-t})(1+v+v^2)}{(1-v)(1-v^{m+2})},\quad \text{for } j<t,$$
$$\varphi_j=\frac{D(m;t,j)}{D_m}=\frac{v^{j-t}(1-v^{t+2})(1-v^{1-j+m})(1+v+v^2)}{(1-v)(1+v)^{j-t+2}(1-v^{m+2})},\quad \text{for } j\ge t.$$

We found all this using computer algebra. Some critical minds may argue that this is only experimental. One way of rectifying this would be to show that indeed the functions φ_j solve the system, which consists of summing various geometric series; again, a computer could be helpful for such an enterprise.

Of interest are also the limits for $m\to\infty$, i.e., no upper boundary:

$$\varphi_j=\lim_{m\to\infty}\frac{D(m;t,j)}{D_m}=\frac{(1+v)^{t-j-2}(1-v^{j+1})v(1+v+v^2)}{(1-v)},\quad \text{for } j<t,$$
$$\varphi_j=\frac{v^{j-t}(1-v^{t+2})(1+v+v^2)}{(1-v)(1+v)^{j-t+2}},\quad \text{for } j\ge t.$$

The special case $t=0$ appeared already in the previous section:

$$\varphi_j=\frac{v^j(1+v+v^2)}{(1+v)^{j+1}}.$$

Likewise, for $t\ge 1$,

$$\varphi_0=v(1+v+v^2)(1+v)^{t-2}.$$

In particular, the formulæ show that the expression from the previous section can be simplified in general, which could have been seen directly, of course.

Theorem 10.1. *The generating function of Deutsch path with lower boundary 0 and upper boundary $m-1$, starting at $(0,t)$ and ending at (n,j) is given by*

$$\frac{(1+v)^{t-j-2}(1-v^{j+1})v(1-v^{m-t})(1+v+v^2)}{(1-v)(1-v^{m+2})}, \quad \textit{for } j<t,$$
$$\frac{v^{j-t}(1-v^{t+2})(1-v^{1-j+m})(1+v+v^2)}{(1-v)(1+v)^{j-t+2}(1-v^{m+2})}, \quad \textit{for } j\ge t,$$

with the substitution $z=\dfrac{v}{1+v+v^2}$.

By shifting everything down, we can interpret the results as Deutsch walks between boundaries $-t$ and $m-1-t$, starting at the origin $(0,0)$ and ending at $(n,j-t)$.

Theorem 10.2. *The generating function of Deutsch path with lower boundary $-t$ and upper boundary h, starting at $(0,0)$ and ending at (n,i) with $-t\le i\le h$ is given by*

$$\frac{(1+v)^{i-2}(1-v^{i+t+1})v(1-v^{h+1})(1+v+v^2)}{(1-v)(1-v^{h+t+3})}, \quad \textit{for } i<0,$$
$$\frac{v^{i}(1-v^{t+2})(1-v^{2-i+h})(1+v+v^2)}{(1-v)(1+v)^{i+2}(1-v^{h+t+3})}, \quad \textit{for } i\ge 0.$$

It is possible to consider the limits $t\to\infty$ and/or $h\to\infty$ resulting in simplified formulæ.

Theorem 10.3. *The generating function of Deutsch path with lower boundary $-t$ and upper boundary ∞, starting at $(0,0)$ and ending at (n,i) with $-t\le i$ is given by*

$$\frac{(1+v)^{i-2}(1-v^{i+t+1})v(1+v+v^2)}{(1-v)}, \quad \textit{for } i<0,$$
$$\frac{v^{i}(1-v^{t+2})(1+v+v^2)}{(1-v)(1+v)^{i+2}}, \quad \textit{for } i\ge 0.$$

Theorem 10.4. *The generating function of Deutsch path with lower boundary $-\infty$ and upper boundary h, starting at $(0,0)$ and ending at (n,i) with $\le i\le h$ is given by*

$$\frac{(1+v)^{i-2}v(1-v^{h+1})(1+v+v^2)}{(1-v)}, \ \textit{for } i<0, \qquad \frac{v^{i}(1-v^{2-i+h})(1+v+v^2)}{(1-v)(1+v)^{i+2}}, \ \textit{for } i\ge 0.$$

Theorem 10.5. *The generating function of unbounded Deutsch path starting at $(0,0)$ and ending at (n,i) is given by*

$$\frac{(1+v)^{i-2}v(1+v+v^2)}{(1-v)}, \ \textit{for } i<0, \qquad \frac{v^{i}(1+v+v^2)}{(1-v)(1+v)^{i+2}}, \ \textit{for } i\ge 0.$$

11. Conclusion

After this personal survey was completed (it will never be complete!) a few more papers about skew Dyck path papers were written; see, e.g., Baril, Kirgizov, Maréchal, Vajnovszki [4] and [47].

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♩♪♫ Article dedicated to Neil Peart (1952–2020) ♫♪♩

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A NUMERICAL STUDY OF L-CONVEX POLYOMINOES AND 201-AVOIDING ASCENT SEQUENCES

ANTHONY J. GUTTMANN[1] AND VÁCLAV KOTĚŠOVEC[2]

[1]School of Mathematics and Statistics, The University of Melbourne, Vic. 3010, Australia; https://blogs.unimelb.edu.au/tony-guttmann

[2]Prague, Czech Republic; http://www.kotesovec.cz/math.htm

Abstract. For L-convex polyominoes we give the conjectured asymptotics of the generating function coefficients, obtained by analysis of the coefficients derived from the functional equation given by Castiglione et al.

For 201-avoiding ascent sequences, we conjecture the solution, obtained from the first twenty-three coefficients of the generating function. This solution is D-finite, indeed algebraic. The conjectured solution then correctly generates all subsequent coefficients. We also obtain the asymptotics, both from direct analysis of the coefficients, and from the conjectured solution.

As well as presenting these new results, our purpose is to illustrate the methods used, so that they may be more widely applied.

Keywords: L-convex polyominoes, ascent sequences.

1. Introduction

In [4], Castiglione et al. gave a functional equation for the number of L-convex polyominoes. These are defined as polyominoes with the property that any two cells may be joined by an L-shaped path, that is to say, a path with at most one right-angle bend. An example is shown in Figure 1. It can be seen that such polygons can be described as a stack polyomino placed atop an upside-down stack polyomino. A stack polyomino is just a row-convex bargraph polyomino. The perimeter generating function of L-convex polyominoes has a simple, rational expression,

$$P(x) = \frac{(1-x)^2}{2(1-x)^2 - 1} = 1 + 2x + 7x^2 + 24x^3 + \cdots ,$$

and is the sequence A003480 in the On-line Encyclopaedia of Integer Sequences (OEIS), [9]. Accordingly, one has

$$[x^n]P(x) = \frac{(2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1}}{4\sqrt{2}} \sim \frac{1+\sqrt{2}}{4}(2+\sqrt{2})^n.$$

The area generating function is given by Castiglione et al. [4]

$$A(q) = 1 + \sum_{k\geq 0} \frac{q^{k+1} f_k(q)}{(1-q)^2(1-q^2)^2 \cdots (1-q^k)^2(1-q^{k+1})} = 1 + q + 2q^2 + 6q^3 + 15q^4 + \cdots , \tag{1.1}$$

where

$$f_k(q) = 2f_{k-1}(q) - (1-q^k)^2 f_{k-2},$$

with initial conditions $f_0(q) = 1$, and $f_1(q) = 1 + 2q - q^2$. We used this expression to generate 2000 terms of the sequence, and these are given in the OEIS as sequence A126764. Analysis of this sequence allowed us to derive the conjectured asymptotics as

$$[q^n]A(q) \sim \frac{13\sqrt{2}}{768 \cdot n^{3/2}} \exp(\pi\sqrt{13n/6}).$$

In the next section we will describe how this estimate was obtained.

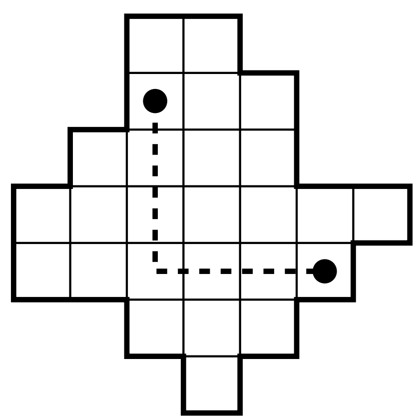

FIGURE 1. An L-convex polyomino.

The second problem we are considering is that of 201-avoiding ascent sequences, defined below. Given a sequence of non-negative integers, $n_1 n_2 n_3 \dots n_k$, the number of *ascents* in this sequence is

$$asc(n_1 n_2 n_3 \dots n_k) = |\{1 \le j < i : n_j \le n_{j+1}\}|.$$

The given sequence is an *ascent sequence* of length k if it satisfies $n_1 = 0$ and

$$n_i \in [0, 1 + asc(n_1 n_2 n_3 \dots n_{k-1})] \text{ for all } 2 \le i \le k.$$

For example, $(0,1,0,2,3,1,0,2)$ is an ascent sequence, but $(0,1,2,1,4,3)$ is not, as $4 > asc(0121) + 1 = 3$.

Ascent sequences came to prominence when Bousquet-Mélou et al. [2] related them to $(2+2)$-free posets. They have subsequently been linked to other combinatorial structures. See [10] for a number of examples. Later, Duncan and Steingrímsson [6] studied *pattern-avoiding ascent sequences.*

A pattern is simply a word on non-negative integers (repetitions allowed). Given an ascent sequence $(n_1 n_2 n_3 \ldots n_k)$, a pattern p is a subsequence $n_{i_1} n_{i_2} \ldots n_{i_j}$, where j is just the length of p, and where the letters appear in the same relative order of size as those in p. For example, the ascent sequence $(0, 1, 0, 2, 3, 1)$ has three occurrences of the sequence 001, namely 002, 003 and 001. If an ascent sequence does not contain a given pattern, it is said to be *pattern avoiding*.

The connection between pattern-avoiding ascent sequences and other combinatorial objects, such as set partitions, is the subject of [6], while the connection between pattern-avoiding ascent sequences and a number of stack sorting problems is explored in [5].

Considering patterns of length three, the number of ascent sequences of length n avoiding the patterns 001, 010, 011, and 012 is 2^{n-1} (the sequence A000079 in the OEIS). For the pattern 102 the number is $(3^n + 1)/2$ (OEIS A007051), while for 101 and 021 the number is just given by the n^{th} Catalan number (OEIS A000108).

For the pattern 201, the first twenty-eight terms of the generating function are given in the OEIS as sequence A202062, and it is this sequence that we have used in our investigation. First, we found, experimentally, that the coefficients given in the OEIS satisfied a recurrence relation, given in Section 3 below. This recurrence can be converted to a second-order inhomogeneous ODE, or, as we prefer, a third-order homogeneous ODE. The smallest root of the polynomial multiplying the third derivative in the ODE is $x = 0.1370633395\ldots$ and is the radius of convergence of the generating function, and of course the reciprocal of the growth constant $\mu = 7.295896946\ldots$

This ODE, readily converted into differential operator form, can be factored into the direct sum of two differential operators, one of first order and one of second order. The solution of the first order ODE is a rational function while the solution of the second turns out to satisfy a cubic algebraic equation. This can be solved by one's favourite computer algebra package (we give the solution below), and expanding this solution, and adding it to the expansion of the solution of the first-order ODE, gives the required expansion.

This analysis required only the first 24 terms given in OEIS, so the correct prediction of the next four terms gives us confidence that this is indeed the exact solution. Expanding this solution and analysing the coefficients, as described in Section 3, leads to us conjecturing the following asymptotic behaviour for these coefficients:

$$u(n) \sim C \frac{\mu^n}{n^{9/2}},$$

where

$$\mu = \frac{14}{3} \cos\left(\frac{\arccos(\frac{13}{14})}{3}\right) + \frac{8}{3}$$

and

$$C = \frac{35}{16}\left(\frac{4107}{\pi} - \frac{84}{\pi}\sqrt{9289}\cos\left(\frac{\pi}{3} + \frac{1}{3}\arccos\left[\frac{255709\sqrt{9289}}{24653006}\right]\right)\right)^{1/2}.$$

In the next two sections we give the derivation of the results given above.

2. L-convex polyominoes

As mentioned above, a typical L-convex polyomino can be considered as a stack polyomino placed atop an upside-down stack polyomino. Stack polyominoes counted by area have generating function

$$S(q) = \sum s_n q^n = \sum_{n \geq 1} \frac{q^n}{(q)_{n-1}(q)_n},$$

where $(q)_n := \prod_{k=1}^{n}(1-q^k)$, and, as first shown by Auluck [1], one has

$$s_n \sim \frac{\exp(2\pi\sqrt{n/3})}{8 \cdot 3^{3/4} \cdot n^{5/4}}.$$

Thus putting two such objects together, one would expect a similar expression for the asymptotic form of the coefficients of the generating function (1.1), that is to say, an expression of the form

$$l_n \sim \frac{\exp(a\pi n^{\beta})}{cn^{\delta}}, \tag{2.1}$$

where we write $L(x) = \sum l_n x^n$ for the ordinary generating function of L-convex polyominoes. We expect both exponents β and δ to be simple rationals, as for stack polyominoes, and the constants a and c to be products of integers and small fractional powers.

The analysis of series with asymptotics of this type is described in detail in [8] and we will not repeat that discussion here, but simply apply the methods described there.

First, we consider the ratios of successive coefficients, $r_n = l_n/l_{n-1}$. For a power-law singularity, one expects the sequence of ratios to approach the growth constant linearly when plotted against $1/n$. In our case the growth constant is 1. That is to say, there is no exponential growth. From the asymptotic behaviour (2.1), which is called of *stretched exponential* type, it follows that the ratio of coefficients behaves as

$$r_n = \frac{l_n}{l_{n-1}} = 1 + \frac{a\beta\pi}{n^{1-\beta}} + O\left(\frac{1}{n}\right), \tag{2.2}$$

so we expect the ratios to approach a limit of 1 linearly when plotted against $1/n^{1-\beta}$, and to display curvature when plotted against $1/n$. We show the ratios plotted against $1/n$ and $1/\sqrt{n}$ in Figure 2.

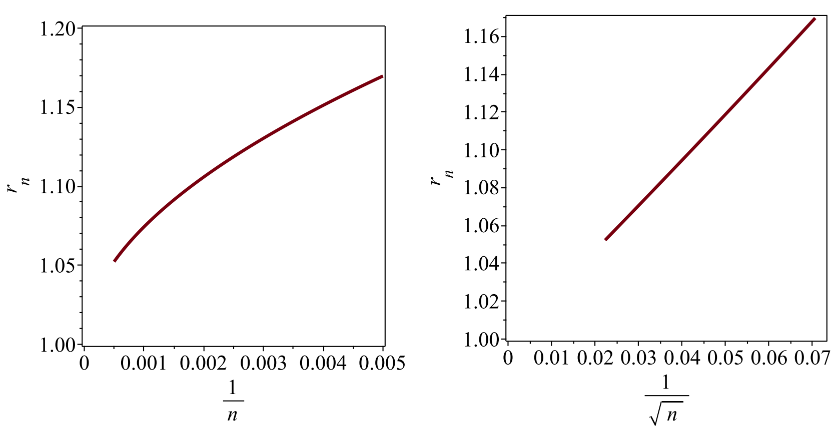

FIGURE 2. L-convex ratios r_n plotted against $1/n$ (left) and against $1/\sqrt{n}$ (right).

These plots are behaving as expected, with the plot against $1/n$ displaying considerable curvature, while the plot against $1/\sqrt{n}$ is visually linear. This is strong evidence that $\beta = 1/2$, just as is the case for stack polyominoes.

In fact we can easily refine this estimate. From Equation (2.2), one sees that

$$r_n - 1 = a\beta\pi \cdot n^{\beta-1} + O\left(\frac{1}{n}\right).$$

Accordingly, a plot of $\log(r_n - 1)$ versus $\log n$ should be linear, with gradient $\beta - 1$. We would expect an estimate of β close to that which linearised the ratio plot. In Figure 3 we show the log-log plot, and in Figure 4 we show the local gradient plotted against $1/\sqrt{n}$. The linearity of the first plot is obvious, while the second is convincingly going to a limit of -0.5 as $n \to \infty$.

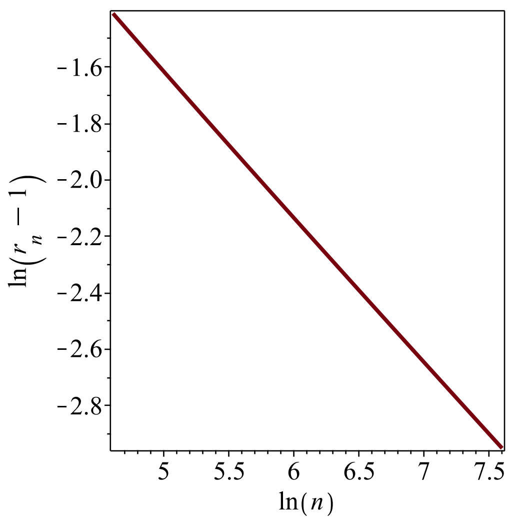

FIGURE 3. Log-log plot of $r_n - 1$ against n.

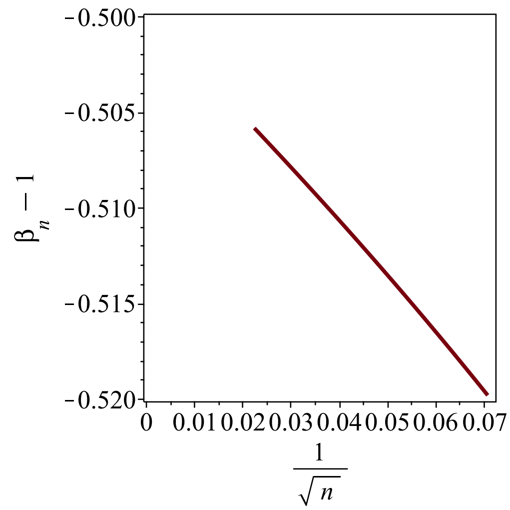

FIGURE 4. Gradient of log-log plot.

Having convincingly established that $\beta = 1/2$, just as for stack polyominoes, it remains to determine the other parameters. There are several ways one might proceed, but here is one that works quite well. From the conjectured asymptotic form, we write

$$\lambda_n := \frac{\log(l_n)}{\pi\sqrt{n}} \sim a - \frac{\delta \log n}{\pi\sqrt{n}} - \frac{\log c}{\pi\sqrt{n}},$$

so one can readily fit successive triple of coefficients $\lambda_{k-1}, \lambda_k, \lambda_{k+1}$, to the linear equation

$$\lambda_n = e_1 + e_2 \frac{\log n}{\pi\sqrt{n}} + e_3 \frac{1}{\pi\sqrt{n}},$$

with k increasing until one runs out of known coefficients. Then e_1 should give an estimator of a, e_2 should give an estimator of $-\delta$ and e_3 should give an estimator of $-\log(c)$. The result of doing this is shown for e_1 and e_2 in Figures 5 and 6 respectively.

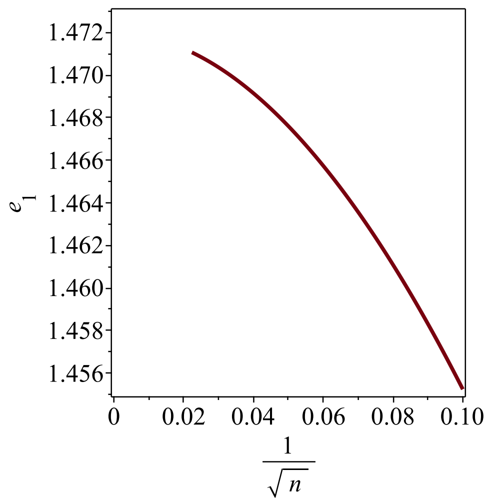

FIGURE 5. Plot of e_1 against $1/\sqrt{n}$.

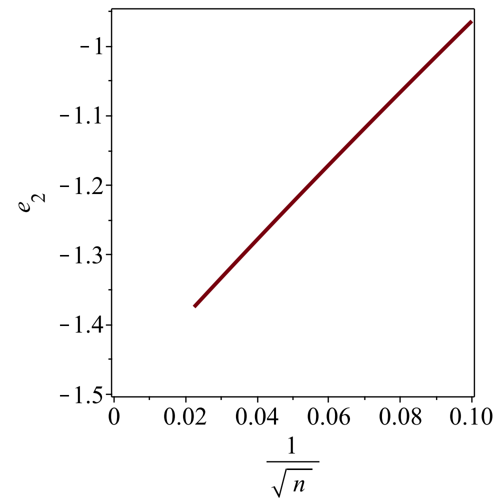

FIGURE 6. Plot of e_2 against $1/\sqrt{n}$.

We estimate the limits as $n \to \infty$ of e_1 as approximately 1.472, and e_2 as -1.5. From the asymptotic expression for s_n, we expect a to likely involve a square root. So we look at $e_1^2 = 2.16678$, which we conjecture to be $13/6$. The exponent δ is expected to be a simple rational, and $3/2$ is indeed a simple rational! We don't show the plot for e_3, as it does not give a precise enough estimate to conjecture the value of $\log(c)$ with any precision.

So at this stage we can reasonably conjecture that

$$l_n \sim \frac{\exp(\pi\sqrt{13n/6})}{c \cdot n^{3/2}}. \tag{2.3}$$

We reached this stage based on only 100 terms in the expansion. In order to both gain more confidence in the conjectured form, and to calculate the constant, we needed more terms, and eventually generated 2000 terms from Equation (1.1).

With hindsight, an arguably more elegant way to analyse this series is to consider only the coefficients l_{n^2}. Denote $\ell_n := l_{n^2}$. The conjectured form (2.3) then becomes $\ell_n \sim \frac{\exp(n\pi\sqrt{13/6})}{c \cdot n^3}$. We have 44 coefficients of the series ℓ_n available, and these grow in the usual power-law manner, that is, $\ell_n \sim D \cdot \mu^n \cdot n^g$.

We now analyse this sequence assuming its asymptotic form to be

$$\ell_n \sim \frac{\exp(n\pi\sqrt{a})}{c \cdot n^b}, \tag{2.4}$$

with a, b, and c to be determined. Then we form the ratios,

$$r_n^{(sq)} = \ell_n/\ell_{n-1} = \mu(1 - b/n + o(1/n)),$$

where $\mu = \exp(\pi\sqrt{a})$ and where the superscript (sq) is a mnemonic recalling that we here consider sequences derived from the subsequence $\ell_n = l_{n^2}$ of square indices. Plotting the ratios $r_n^{(sq)}$ against $1/n$ should give a linear plot with gradient $-b\mu$ and ordinate μ. For a pure power-law the term $o(1/n)$ is $O(1/n^2)$, and the estimate of μ can thus be refined by plotting the linear intercepts $\ell_n^{(sq)} = n \cdot r_n - (n-1) \cdot r_{n-1}$ against $1/n^2$. The results of doing this are shown in Figures 7 and 8 for the ratios and linear intercepts respectively.

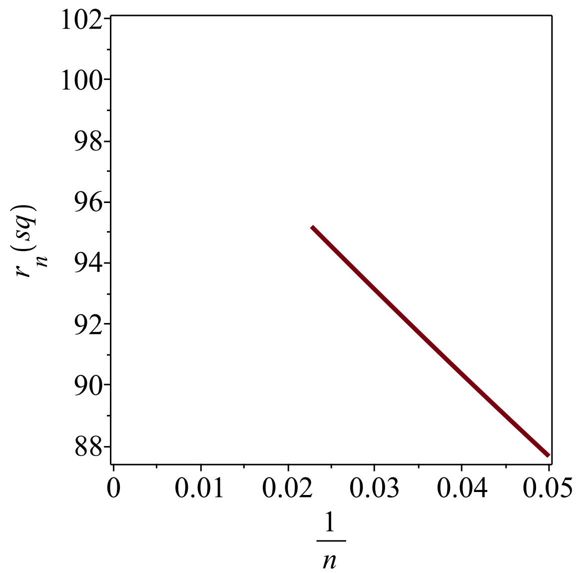

FIGURE 7. Plot of ratios $r_n^{(sq)}$ against $1/n$.

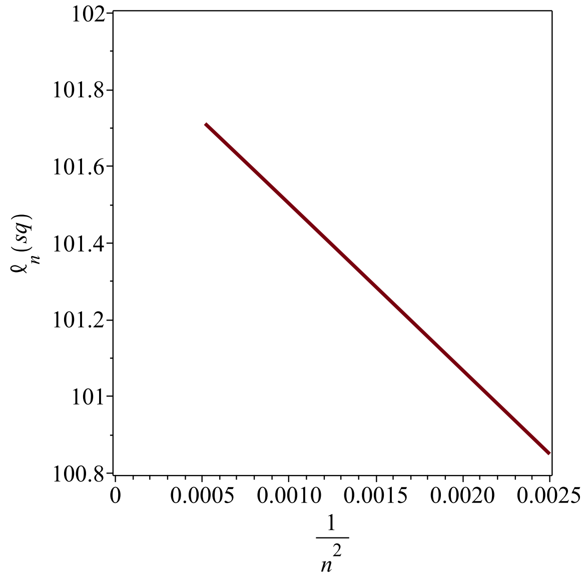

FIGURE 8. Plot of linear intercepts $\ell_n^{(sq)}$ against $1/n^2$.

It can be seen that the linear intercepts have a faster convergence. We can go further and eliminate the $O(1/n^2)$ term by forming the sequence $t_n = (n^2 \cdot \ell_n^{(sq)} - (n-1)^2 \cdot \ell_{n-1}^{(sq)})/(2n-1)$, and these are shown in Figure 9. From this we estimate that the intercept of the plot with the ordinate is about 101.931. This is the growth constant $\mu = \exp(\pi\sqrt{a})$, from which we find $a \approx 2.16666$, which strongly suggests that $a = 13/6$ exactly.

To estimate the exponent b in the asymptotic form (2.4), we introduce

$$g_n = (r_n^{(sq)}/\mu - 1) \cdot n,$$

noting that $\lim_n g_n = g = -b$. Then, using the estimate of μ just given, we obtain the plot shown in Figure 10. This is rather convincingly approaching $g = -3$.

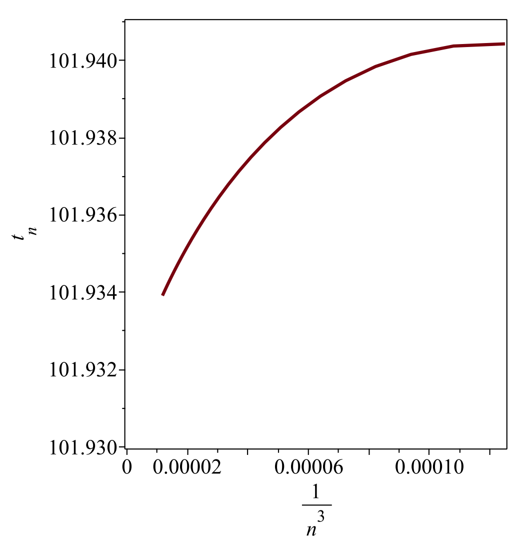

FIGURE 9. Plot of sequence t_n against $1/n^3$.

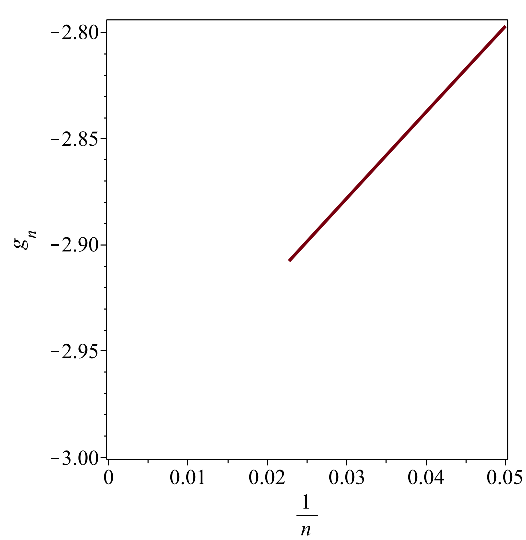

FIGURE 10. Plot of exponent estimates g_n against $1/n$.

We can do better by calculating the linear intercepts $g2_n := n \cdot g_n - (n-1) \cdot g_{n-1}$. A plot of $g2_n$ against $1/n^2$ is shown in Figure 11. The result $g = -3$ is totally convincing.

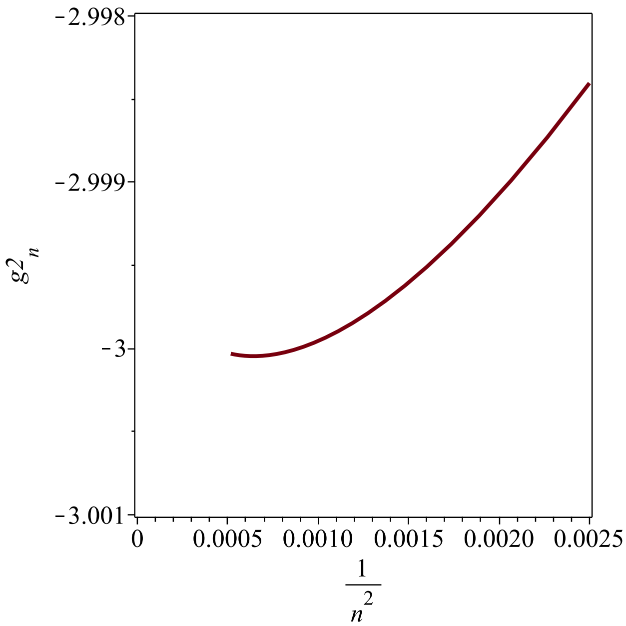

FIGURE 11. Plot of sequence $g2_n$ against $1/n^3$.

In order to calculate the constant c in the asymptotic form (2.4), we introduce the sequence

$$c_n := \frac{\exp(\pi\sqrt{13n/6})}{l_n \cdot n^{3/2}},$$

and extrapolate the sequence c_n using any of a variety of standard methods.

For this extrapolation, we used the Bulirsch–Stoer method (see [12, Chapter 3.5] or [3] for more details), applied to the coefficient sequence $\{\ell_n\}$, with parameter $1/2$, and 44 terms in the sequence (corresponding to $44^2 = 1936$ terms in the original series). This gave the estimate $c \approx 0.023938510821419$. This unknown number is likely to involve a square root, cube root or fourth root of a small integer, just as did s_n.

We investigated this by dividing by various powers of small integers, and tried to identify the result. Fortuitously, dividing the approximate value by $\sqrt{2}$ gave a result that the *Maple* command *identify* reported as $13/768$. This implies $c = 13\sqrt{2}/768 = 0.023938510821419577\ldots$, agreeing to all quoted digits with the approximate value. The occurrence of 13 in this fraction, as well as in the exponent square-root, is a reassuring feature, as is the factorisation of 768 as $3 \cdot 2^8$.

Thus we conclude with the confident conjecture that the asymptotic form of the coefficients of L-convex polyominoes is

$$l_n \sim \frac{3 \cdot 2^8 \cdot \exp(\pi\sqrt{13n/6})}{13\sqrt{2} \cdot n^{3/2}}.$$

3. 201-avoiding ascent sequence

From the coefficients $u(n)$ for $n = 0, \ldots, 27$ (this is the sequence A202062 in the OEIS), we used the *gfun* package of *Maple* [11] and immediately found that the coefficients satisfy the recurrence relation

$$\begin{aligned}
&(2n^2+n)u(n) + (6n^2+45n+60)u(n+1) + (-34n^2-263n-480)u(n+2) \\
&+ (44n^2+421n+984)u(n+3) + (-20n^2-235n-684)u(n+4) \\
&+ (2n^2+31n+120)u(n+5) = 0, \\
&\text{with } \ u(0)=1, u(1)=1, u(2)=2, u(3)=5, u(4)=15.
\end{aligned}$$

This recurrence can be converted to a second-order inhomogeneous ODE, or, as we prefer, a third-order homogeneous ODE, using the gfun command *diffeqtohomdiffeq*, giving

$$P_3(x)f^{'''}(x) + P_2(x)f^{''}(x) + P_1(x)f^{'}(x) + P_0(x)f(x) = 0,$$

where

$$\begin{aligned}
P_3(x) &= -2x^2(x^3+5x^2-8x+1)(4x^4-30x^3+48x^2-36x+15)(x-1)^2, \\
P_2(x) &= -3x(x-1)(12x^8-30x^7-652x^6+2734x^5-4767x^4+4758x^3-2843x^2+870x-85), \\
P_1(x) &= -24x^9+30x^8+2754x^7-13278x^6+28884x^5-38106x^4+32436x^3-16620x^2+4350x-420, \\
P_0(x) &= \ 30(3x-2)(3x^5-10x^4+19x^3-28x^2+24x-7).
\end{aligned}$$

The smallest root of the cubic factor in $P_3(x)$ is $x = 0.1370633395\ldots$ and is the radius of convergence of the solution. Accordingly, the growth constant μ thus satisfies

$$\mu = \frac{1}{x} = \frac{14}{3}\cos\left(\frac{\arccos(\frac{13}{14})}{3}\right) + \frac{8}{3} = 7.295896946\ldots$$

This ODE can then be studied using the *Maple* package *DEtools*. We first convert the ODE to differential operator form through the command *de2diffop*, then factor this into the direct sum of two differential operators by the command *DFactorLCLM*. One of these operators is first order and one is second order.

The solution of the first order ODE is immediately given by the *dsolve* command, and is the rational function

$$y_1(x) = \frac{x^4+26x^3-45x^2+18x+1}{12(x-1)x^3}.$$

To solve the second-order ODE, we obtain a series solution, the first term of which is $\mathrm{O}(x^{-3})$. We multiply the solution by x^3 to obtain a regular power series, then use the *gfun* command *seriestoalgeq* to discover the cubic equation,

$$\begin{aligned}
&4(x-1)^3 y_2(x)^3 \\
&\quad - 3(x-1)(x^2-x+1)(x^6-235x^5+1430x^4-1695x^3+270x^2+229x+1)y_2(x) \\
&\quad + x^{12}+510x^{11}-14631x^{10}+80090x^9-218058x^8+316290x^7-253239x^6 \\
&\quad + 131562x^5-70998x^4+37950x^3-8955x^2-522x+1 = 0.
\end{aligned} \tag{3.1}$$

This can be solved by *Maple*'s, *solve* command, giving three solutions. Inspection of their expansion reveals the appropriate one, and simplifying this gives the following rather cumbersome solution: Let

$$\begin{aligned}P_1 &= x^{12} + 510x^{11} - 14631x^{10} + 80090x^9 - 218058x^8 + 316290x^7 - 253239x^6 + 131562x^5 \\ &\quad - 70998x^4 + 37950x^3 - 8955x^2 - 522x + 1 - 24\sqrt{3x(x-1)(x^3+5x^2-8x+1)^7},\\ P_2 &= (x^2-x+1)(x-1)^4(x^6 - 235x^5 + 1430x^4 - 1695x^3 + 270x^2 + 229x + 1),\end{aligned}$$

$P_3 = (3^{5/6}i + 3^{1/3}),$ and $P_4 = (-3^{5/6}i + 3^{1/3})$. Then, one has

$$y_2(x) = \frac{-3^{2/3}\left(P_4\left(-P_1\cdot(x-1)^6\right)^{2/3} + P_2\cdot P_3\right)}{12\,\left(-P_1\cdot(x-1)^6\right)^{1/3}(x-1)^3}.$$

The solution to the original ODE is then

$$y(x) = \frac{y_2(x)}{12x^3} - y_1(x) = 1 + x + 2x^2 + 5x^3 + 15x^4 + \cdots$$

This analysis required only the first 24 terms given in the OEIS, so the correct prediction of the next four terms gives us confidence that this is indeed the exact solution.

We next obtained the first 5000 terms in only a few minutes of computer time by expanding this solution. We used these terms to calculate the amplitude. That is to say, we now know that the coefficients behave asymptotically as $u(n) \sim C\mu^n n^{-9/2}$. Equivalently, the generating function behaves as

$$U(x) = \sum u(n)x^n = A(1-\mu\cdot x)^{7/2},$$

where $C = A/\Gamma(-7/2) = 105A/(16\sqrt{\pi})$. We estimate C by assuming a pure power law, so that

$$\frac{u(n)\cdot n^{9/2}}{\mu^n} = C(1 + \sum_{k\ge 1} a_k/n^k).$$

We calculated the first twenty coefficients of this expansion, which allowed us to estimate $C = 13.4299960869\ldots$ with 74-digit accuracy (as checked later). Unless one is very fortunate (for example, when the *Maple* command *identify* determines an expression for this constant, which it doesn't in our case), to identify this constant requires some experience-based guesswork.

Such constants in favourable cases are a product of rational numbers and square roots of small integers, sometimes with integer or half-integer powers of π. These powers of π usually arise from the conversion factor in going from the generating function amplitude A to the coefficient amplitude C. That is to say, we might expect the amplitude A to be simpler than C. And, to eliminate square-roots, we will try and identify A^2 rather than A.

We do this by seeking the minimal polynomial with root A^2, using the command *MinimalPolynomial* in either *Maple* or *Mathematica*. In fact, one only requires 20 digit accuracy in the estimate of A^2 to establish the minimal polynomial, $A^6 - 1369A^4 + 17839A^2 + 1$, which can be solved to give

$$C = \frac{35}{16}\left(\frac{4107}{\pi} - \frac{84}{\pi}\sqrt{9289}\cos\left(\frac{\pi}{3} + \frac{1}{3}\arccos\left[\frac{255709\sqrt{9289}}{24653006}\right]\right)\right)^{1/2}.$$

This derivation includes a degree of hindsight. In fact we searched for the minimal polynomial for the amplitude C, by including various powers of π, and then choose the polynomial of minimal degree. This required a much greater degree of precision in our estimate of C to ensure we found the correct minimal polynomial.

It has been pointed out to us by Jean-Marie Maillard that the amplitude A can be obtained directly from the solution of the cubic equation (3.1), by extracting the coefficient of $(1-\mu\cdot x)^{7/2}$, as explained in [7, Chapter VII.7.1]. This gives the minimal polynomial that we obtained by numerical experimentation. This alternative way to derive asymptotic expansions is a more elegant method, as it is automatic, but it only works for algebraic functions. There are thus many sequences for which it is not applicable, as in the case of L-convex polyominoes (for which the generating function is not algebraic, as its radius of convergence is not algebraic), while our numerical approach can still yield conjecturally exact results.

4. Conclusion

We have shown how experimental mathematics can be used to conjecture exact asymptotics, in the case of L-convex polyominoes, and to conjecture an exact solution, in the case of 201-avoiding ascent sequences. We hope that the results will be of interest, and that the methods will be more widely applied, as there are many outstanding combinatorial problems that lend themselves to such an approach.

We recognise that these results are conjectural. We leave proofs to those more capable, and in the hope that the maxim of the late lamented J. M. Hammersley to the effect that “it is much easier to prove something when you know that it is true” will aid that endeavour.

Acknowledgements. AJG wishes to acknowledge helpful discussions with Jean-Marie Maillard, and with Paolo Massazza on the topic of L-convex polyominoes, and to thank the ARC Centre of Excellence for Mathematical and Statistical Frontiers (ACEMS) for support.

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INTRODUCING DASEP: THE DOUBLY ASYMMETRIC SIMPLE EXCLUSION PROCESS

DAVID W. ASH[1]

[1]OptIn Inc., San Francisco Bay Area, CA, USA; https://orcid.org/0000-0001-8970-5784

Abstract. Research in combinatorics has often explored the asymmetric simple exclusion process (ASEP). The ASEP, inspired by examples from statistical mechanics, involves particles of various species moving around a lattice. With the traditional ASEP particles of a given species can move but do not change species. In this paper a new combinatorial formalism, the DASEP (doubly asymmetric simple exclusion process), is explored. The DASEP is inspired by biological processes where, unlike the ASEP, the particles can change from one species to another. The combinatorics of the DASEP on a one dimensional lattice are explored, including the associated generating function. The stationary probabilities of the DASEP are explored, and results are proven relating these stationary probabilities to those of the simpler ASEP.

Keywords: ASEP, DASEP, lattice, algebraic combinatorics, steady state probabilities, species, lattice paths.

1. Introduction

The ASEP (asymmetric simple exclusion process) is a structure that has frequently been referred to in the combinatorics literature. In its simplest form, the ASEP consists of a one dimensional infinite lattice, with each point on the lattice being populated with either a particle or a hole. At random intervals, each particle attempts to move either to the left or the right with different but fixed probabilities (hence the term 'asymmetric'). The ASEP can be thought of as a form of Markov process as noted in [4] by Corteel et al. Multiline queues [5] were introduced by Ferrari et al. as a combinatorial approach to the analysis of the ASEP. Originally the ASEP particles were thought of as all belonging to a single species. More recent work by Cantini et al. [3] generalized the concept to multiple species and uncovered a link with Macdonald polynomials. Although we focus on the homogeneous ASEP (transition probabilities do not depend on position in the lattice), several researchers (Lam et al. [7], Ayyer et al. [1], Cantini [2], Mandelshtam [9], and Kim et al. [6]) have explored the inhomogeneous ASEP in which transition probabilities do depend on lattice position.

2. Definitions

Following [4], a partition can be defined as follows:

Definition 2.1. A partition λ is a nonincreasing sequence of n nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0)$.

We will start by working through a simple example of the ASEP before introducing the new concept of the DASEP. We will ordinarily write a partition as defined above as an n-tuple: $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$.

Definition 2.2. We write $S_n(\lambda)$ to mean the set of all permutations of λ.

Example 2.3. So, for $\lambda = (2, 2, 1)$, $S_3(\lambda) = \{(2, 2, 1), (2, 1, 2), (1, 2, 2)\}$.

The multispecies asymmetric simple exclusion process ASEP(λ) is then defined to be a Markov process on $S_n(\lambda)$ with certain specific transition probabilities:

Definition 2.4. For all partitions λ as defined in Definition 2.1, ASEP(λ) is a Markov process on $S_n(\lambda)$. We let t be a constant with $0 \leq t \leq 1$. The transition probability $P_{\mu,\nu}$ between two permutations $\mu \in S_n(\lambda)$ and $\nu \in S_n(\lambda)$ is given by:

- If $\mu = (\mu_1, \ldots, \mu_k, i, j, \mu_{k+2}, \ldots, \mu_n)$ and $\nu = (\mu_1, \ldots, \mu_k, j, i, \mu_{k+2}, \ldots, \mu_n)$, with $i \neq j$, then $P_{\mu,\nu} = \frac{t}{n}$ if $i > j$ and $P_{\mu,\nu} = \frac{1}{n}$ if $j > i$.
- If $\mu = (i, \mu_2, \mu_3, \ldots, \mu_{n-1}, j)$ and $\nu = (j, \mu_2, \mu_3, \ldots, \mu_{n-1}, i)$ with $i \neq j$, then $P_{\mu,\nu} = \frac{t}{n}$ if $j > i$ and $P_{\mu,\nu} = \frac{1}{n}$ if $i > j$.
- If neither of the above conditions apply but $\nu \neq \mu$ then $P_{\mu,\nu} = 0$. If $\nu = \mu$ then $P_{\mu,\mu} = 1 - \sum_{\nu \neq \mu} P_{\mu,\nu}$.

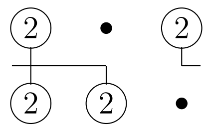
It is possible to compute steady state probabilities for ASEP(λ). For the purposes of the example that we will develop as we introduce DASEP, we are primarily interested in ASEP(λ) for $\lambda = (2, 2, 0)$, $\lambda = (2, 1, 0)$, and $\lambda = (1, 1, 0)$, so we will focus mostly on these three processes as we work through the computation of the steady state probabilities. Continuing to follow [4] as we develop this example, to compute these probabilities we need to define the concept of a multiline queue.

Definition 2.5. A ball system B is an $L \times n$ matrix each element of which is either 0 or 1. Moreover for all i the number of 1's in row $i + 1$ is less than or equal to the number of 1's in row i.

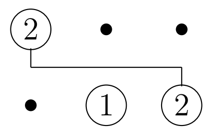
Definition 2.6. Given a ball system B a multiline queue Q is obtained by augmenting B with a labeling and matching system. Each cell in B will be labelled with a number from 0 to L inclusive, and each cell with a 1 element in row $i + 1$, for $i \geq 1$, will be matched to a cell with a 1 element in row i. Such a matching must be obtained through an application of the following algorithm:

- Step 1: Find the highest numbered row with unlabelled 1 elements. Label each of those elements with the number of the row. If this is row 1, or there are no remaining unlabelled 1 elements in the matrix, exit.
- Step 2: Find the row with labelled but unmatched elements. If this is row 1, go back to step 1. If it is row $i + 1$, for $i \geq 1$, first match each labelled but unmatched element that can be matched to an unlabelled element directly below it to that element. This is considered a *trivial* match. Then proceed from right to left (highest to lowest numbered columns) matching each remaining labelled but unmatched element to an unlabelled element in the row below–these are the *nontrivial* matches. Give all newly matched elements in row i the same label as the element it has just been matched to. Repeat step 2.

A multiline queue is often visualized as a ball system with an element with a 1 value being shown as a ball and a 0 value by the absence of a ball. Matches between elements (balls) are drawn by lines between the matched balls. The following shows a multiline queue associated with ASEP(λ) where $\lambda = (2, 2, 0)$. Note that the line matching the ball at upper right to the one at the lower middle wraps around to the right.



The labels in the bottom row determine the partition of the associated ASEP. The above multiline queue has $\lambda = (2, 2, 0)$ since the bottom row includes two 2's and a 0–by convention an element without a ball is assumed to be labeled with a 0. Likewise the following would be a multiline queue with $\lambda = (2, 1, 0)$:



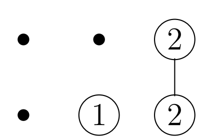
Each multiline queue is also associated with a permutation $\alpha \in S_n(\lambda)$ corresponding to the labels of its bottom row in *unsorted* order. For example, for the above multiline queue, $\lambda = (2, 1, 0)$ but $\alpha = (0, 1, 2)$. We will write $\lambda(Q) = \lambda$ and $\alpha(Q) = \alpha$.

3. Steady state probabilities with example

To determine steady state probabilities–and continue with the example started in the introduction–we next assign to each nontrivial matching p in Q two values $f(p)$ and $s(p)$. $f(p)$ is the number of choices that were available for the match when the match was made. $s(p)$ is the number of legal matches that were skipped, if we imagine ourselves considering possible matches from left to right and wrapping around the end if needed, before the actual choice was made. We can then define a weight on p as $\mathrm{wt}(p) = \frac{(1-t)t^{s(p)}}{1-t^{f(p)}}$. Here we are proceeding from [4] but with the simplifying assumption that $q = 1$, since in the sequel we will rely on a theorem that requires $q = 1$. Next we can define a weight on the entire multiline queue $\mathrm{wt}(Q) = \prod_{p\in Q} \mathrm{wt}(p)$ where the product is taken over all *nontrivial* matches p in Q. A theorem due to Martin [10] then gives the required steady state probabilities:

$$\Pr(\alpha) = \frac{\sum_{\alpha(Q)=\alpha} \mathrm{wt}(Q)}{\sum_{\lambda(Q)=\lambda} \mathrm{wt}(Q)}.$$

Before moving on to the DASEP, we need to evaluate the steady state probabilities for the examples that we will ultimately use to develop the DASEP. For the above multiline queue, there is exactly one nontrivial pair p. When this pair is matched, there are two available options so $f(p) = 2$. As we picked the second available option, $s(p) = 1$. So $\mathrm{wt}(Q) = \frac{(1-t)t}{1-t^2}$. As noted above, $\alpha = (0, 1, 2)$ and the only other multiline queue with $\alpha = (0, 1, 2)$ is as follows:



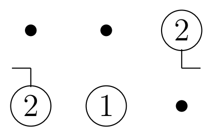
Here there is no nontrivial matching pair, so $\mathrm{wt}(Q) = 1$. Hence:

$$\sum_{\alpha(Q)=(0,1,2)} \mathrm{wt}(Q) = 1 + \frac{(1-t)t}{1-t^2} = \frac{1+2t}{1+t}.$$

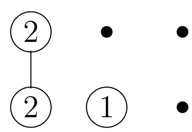
For reasons of symmetry:

$$\sum_{\alpha(Q)=(0,1,2)} \mathrm{wt}(Q) = \sum_{\alpha(Q)=(1,2,0)} \mathrm{wt}(Q) = \sum_{\alpha(Q)=(2,0,1)} \mathrm{wt}(Q) = \frac{1+2t}{1+t}.$$

Next we look at $\alpha = (0,1,2)$, for which there are also two multiline queues. The first of these is as follows:



Here there is one nontrivial matching pair p. When this pair is matched, there are two available options so $f(p) = 2$. As we picked the first available option, $s(p) = 0$. So $\mathrm{wt}(Q) = \frac{1-t}{1-t^2}$. The other multiline queue with $\alpha = (2,1,0)$ is as follows:



Again there is no nontrivial matching pair, so $\mathrm{wt}(Q) = 1$. Hence:

$$\sum_{\alpha(Q)=(2,1,0)} \mathrm{wt}(Q) = 1 + \frac{1-t}{1-t^2} = \frac{2+t}{1+t}.$$

For reasons of symmetry, one has

$$\sum_{\alpha(Q)=(2,1,0)} \mathrm{wt}(Q) = \sum_{\alpha(Q)=(1,0,2)} \mathrm{wt}(Q) = \sum_{\alpha(Q)=(0,2,1)} \mathrm{wt}(Q) = \frac{2+t}{1+t}.$$

So we get

$$\sum_{\lambda(Q)=(2,1,0)} \mathrm{wt}(Q) = 3(\frac{1+2t}{1+t}) + 3(\frac{2+t}{1+t}) = 9.$$

We are now ready to give the steady state probabilities

$$\Pr(0,1,2) = \Pr(1,2,0) = \Pr(2,0,1) = \frac{1+2t}{9(1+t)}$$

and

$$\Pr(2,1,0) = \Pr(1,0,2) = \Pr(0,2,1) = \frac{2+t}{9(1+t)}.$$

Trivial computations also give

$$\Pr(0,1,1) = \Pr(1,1,0) = \Pr(1,0,1) = \frac{1}{3}$$

and

$$\Pr(0,2,2) = \Pr(2,2,0) = \Pr(2,0,2) = \frac{1}{3}.$$

This concludes our computation for the steady state probabilities of this model; in the next section we introduce the DASEP model.

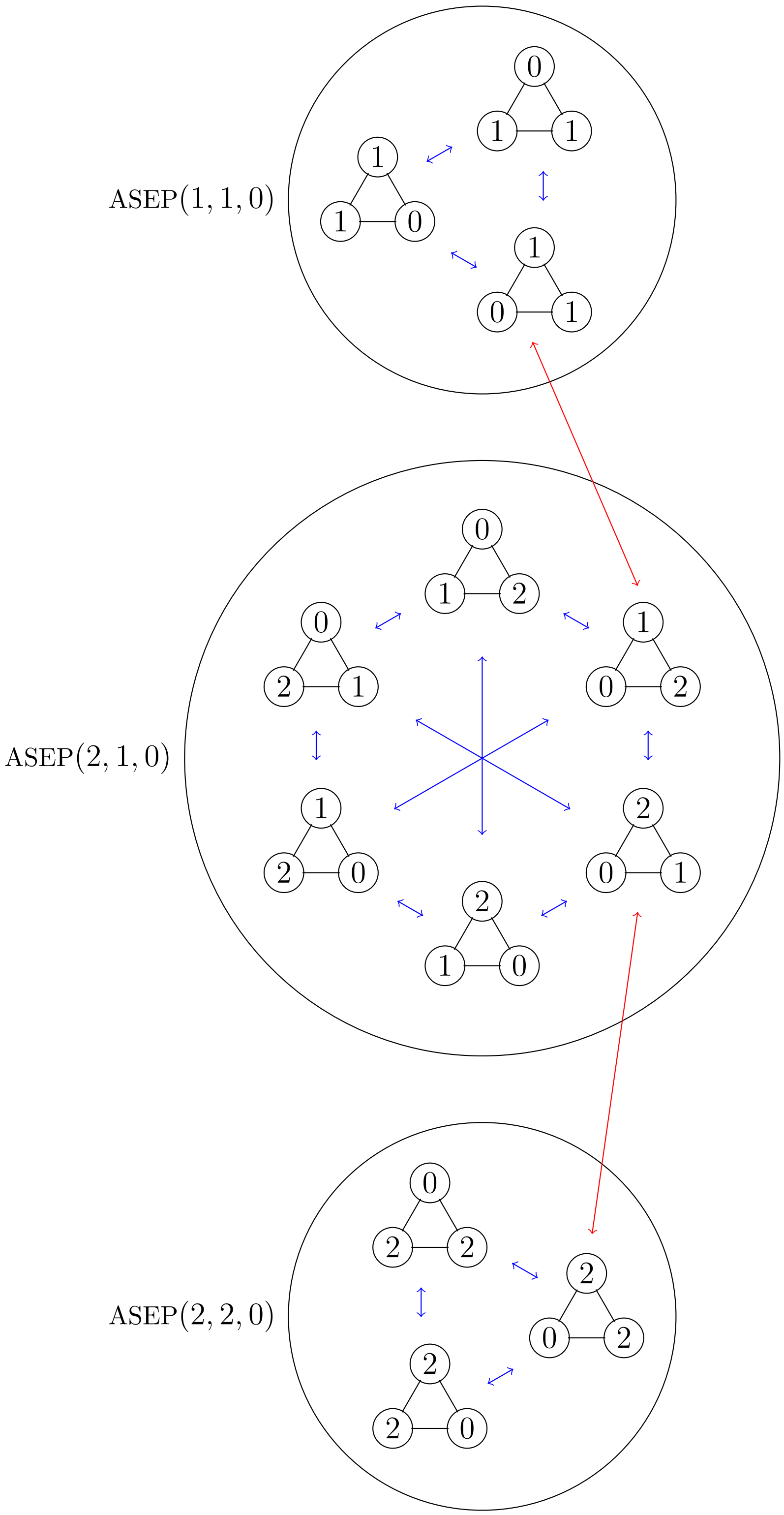


FIGURE 1. An example of DASEP(n,p,q): DASEP(3,2,2).

Each of the 12 small triangles represents a state of the DASEP in the circular lattice with $n = 3$ sites with $q = 2$ balls (i.e., the nonzero labels), each nonzero label is $\leq p = 2$ (i.e., one has $p = 2$ species). Each state corresponds to a permutation of the partition $(2,2,0)$, $(2,1,0)$, or $(1,1,0)$. The transitions are explained in Definition 4.1 hereafter.

4. Doubly asymmetric simple exclusion process

We are now ready to introduce the DASEP (doubly asymmetric simple exclusion process). While the ASEP is inspired by statistical mechanics where particles do not change species, the DASEP, by contrast, is inspired by biological processes where particles can change species, which we denote by DASEP(n,p,q) where n is the number of positions on the lattice, p is the number of types of species, and q is the number of particles.

Definition 4.1. For all positive integers n, p, and q with $n > q$, DASEP(n,p,q) is a Markov process on the set $\bigcup_{\lambda_1 \le p,\ \lambda'_1 = q} S_n(\lambda)$, where one uses the notation of Definition 2.2, and where $\lambda'_1 = q$ refers to the dual partition [8] of λ, namely λ', and uses the fact that λ'_1 gives the number of nonzero terms in the original partition λ. The transition probability $P_{\mu,\nu}$ on two permutations μ and ν is as follows:

- If $\mu = (\mu_1, \ldots, \mu_k, i, j, \mu_{k+2}, \ldots, \mu_n)$ and $\nu = (\mu_1, \mu_2, ..., \mu_k, j, i, \mu_{k+2}, \ldots, \mu_n)$ with $i \neq j$, then $P_{\mu,\nu} = \frac{t}{3n}$ if $i > j$ and $P_{\mu,\nu} = \frac{1}{3n}$ if $j > i$.
- If $\mu = (i, \mu_2, \ldots, \mu_{n-1}, j)$ and $\nu = (j, \mu_2, \mu_3, ..., \mu_{n-1}, i)$ with $i \neq j$, then $P_{\mu,\nu} = \frac{t}{3n}$ if $j > i$ and $P_{\mu,\nu} = \frac{1}{3n}$ if $i > j$.
- If $\mu = (\mu_1, \ldots, \mu_k, i, \mu_{k+2}, \ldots, \mu_n)$ and $\nu = (\mu_1, \ldots, \mu_k, i+1, \mu_{k+2}, \ldots, \mu_n)$ with $i \geq 1$, then $P_{\mu,\nu} = \frac{u}{3n}$.
- If $\mu = (\mu_1, \ldots, \mu_k, i+1, \mu_{k+2}, \ldots, \mu_n)$ and $\nu = (\mu_1, \ldots, \mu_k, i, \mu_{k+2}, \ldots, \mu_n)$ with $i \geq 1$, then $P_{\mu,\nu} = \frac{1}{3n}$.
- If none of the above conditions apply but $\nu \neq \mu$ then $P_{\mu,\nu} = 0$. If $\nu = \mu$ then $P_{\mu,\mu} = 1 - \sum_{\nu \neq \mu} P_{\mu,\nu}$.

Figure 1 shows the simple example of the DASEP that we are working through. All possible transitions within a single ASEP (the first and second bullet points in the definition above) are shown with blue arrows on this diagram. To keep the diagram relatively clean in appearance, only selected transitions between different ASEPs (the third and fourth bullet points) are shown (with red arrows). Other ASEPs such as ASEP$(1,0,0)$ or ASEP$(2,0,0)$ are not shown since these are not part of DASEP$(3,2,2)$. This is because, per Definition 4.1, for DASEP$(3,2,2)$ we always have $\lambda'_1 = 2$, whereas for ASEP$(1,0,0)$ and ASEP$(2,0,0)$, we would have $\lambda'_1 = 1$.

Similar to with the ASEP, with the DASEP we wish to compute steady state probabilities for permutations α which we will call $\mathrm{Pd}(\alpha)$. We will focus on continuing to develop the example we have been working on which turns out to be DASEP$(3,2,2)$. Here $n = 3$ means that the particles move on the circular lattice with 3 sites, $p = 2$ means that each particle is allowed to take on the value 0, 1, or 2, and $q = 2$ means that each permutation α has exactly 2 nonzero values. We therefore find ourselves interested in the following 12 steady state probabilities:

$$\mathrm{Pd}(0,1,1), \mathrm{Pd}(0,1,2), \mathrm{Pd}(0,2,1), \mathrm{Pd}(0,2,2), \mathrm{Pd}(1,0,1), \mathrm{Pd}(1,0,2),$$
$$\mathrm{Pd}(1,1,0), \mathrm{Pd}(1,2,0), \mathrm{Pd}(2,0,1), \mathrm{Pd}(2,0,2), \mathrm{Pd}(2,1,0), \mathrm{Pd}(2,2,0).$$

Note here that particles in the DASEP are allowed to switch back and forth between species 1 and 2, but not back and forth from 0 to anything else. That is because a value of 0 is understood to not so much be a species but the absence of a species. Due to symmetries we can now focus on solving for the following four probabilities:

$$w = \mathrm{Pd}(0,1,1), \quad x = \mathrm{Pd}(0,1,2), \quad y = \mathrm{Pd}(0,2,1), \quad z = \mathrm{Pd}(0,2,2).$$

From the above transition probabilities, this reduces to solving the system

$$\begin{cases} 2uw = x + y \\ (2+t)x + x + ux = (1+2t)y + z + uw \\ (1+2t)y + y + uy = (2+t)x + uw + z \\ 2z = u(x+y) \end{cases}$$

which in turn implies the relation

$$(5 + 2t + u)x = (3 + 4t + u)y.$$

We can then ask ourselves the question of when the proportions of steady state probabilities for the DASEP are the same as for the previous ASEP. Noting that $\mathrm{Pr}(0,1,2) = \frac{1+2t}{9(1+t)}$ and $\mathrm{Pr}(2,1,0) = \frac{2+t}{9(1+t)}$ such equality will happen if $(5+2t+u)(1+2t) = (3+4t+u)(2+t)$, or $5 + 2t + u + 10t + 4t^2 + 2tu = 6 + 8t + 2u + 3t + 4t^2 + tu$, or $t(1+u) = 1+u$. So this will happen if and only if $t = 1$. We have therefore proven the following proposition.

Proposition 4.2. *If $D = \mathrm{DASEP}(3,2,2)$ is parameterized as described above by t and u, then the following two statements are equivalent:*

- *$t = 1$.*
- *For all partitions λ with $S_n(\lambda) \subseteq D$ and all permutations μ, $\nu \in S_n(\lambda)$, the following equality holds: $\frac{\mathrm{Pr}(\mu)}{\mathrm{Pr}(\nu)} = \frac{\mathrm{Pd}(\mu)}{\mathrm{Pd}(\nu)}$. That is, the ratio between steady state probabilities does not change in moving from the ASEP to the DASEP.*

In fact, we conjecture the following more general statement.

Conjecture 4.3. *If $D = \mathrm{DASEP}(n,p,q)$ is parameterized as described above by t and u, then the following two statements are equivalent:*

- *$t = 1$.*
- *For all partitions λ with $S_n(\lambda) \subseteq D$ and all permutations μ, $\nu \in S_n(\lambda)$, the following equality holds: $\frac{\mathrm{Pr}(\mu)}{\mathrm{Pr}(\nu)} = \frac{\mathrm{Pd}(\mu)}{\mathrm{Pd}(\nu)}$. That is, the ratio between steady state probabilities does not change in moving from the ASEP to the DASEP.*

Partial proof. We will prove this only in the $\Longrightarrow$ direction. If $t = 1$ we can replace λ with a similar partition but with species of the same type being replaced by similar distinct species. For example, if $\lambda = (3,3,3,2,1,0,...)$ we would map this to $\hat{\lambda} = (3_1, 3_2, 3_3, 2, 1, 0, ...)$ and allow adjacent species originally of the same type to be exchanged with the same transition probability. This will create a completely symmetric situation, so all steady state probabilities are equal. As an equal number of $\hat{\lambda}$'s can be derived from each λ this means all original steady state probabilities are equal as well, so $\frac{\mathrm{Pr}(\mu)}{\mathrm{Pr}(\nu)} = \frac{\mathrm{Pd}(\mu)}{\mathrm{Pd}(\nu)} = 1$. This completes the proof in the $\Longrightarrow$ direction. ☐

Let us motivate Conjecture 4.3 by showing that it holds on one example. Following are the nine values (of the nine steady state probabilities) we must solve for to prove this conjecture for DASEP$(3,3,2)$:

$$\begin{aligned} a_1 &= \mathrm{Pd}(0,1,1), & a_2 &= \mathrm{Pd}(0,2,2), & a_3 &= \mathrm{Pd}(0,3,3),\\ b_1 &= \mathrm{Pd}(0,2,3), & b_2 &= \mathrm{Pd}(0,1,3), & b_3 &= \mathrm{Pd}(0,1,2),\\ c_1 &= \mathrm{Pd}(0,3,2), & c_2 &= \mathrm{Pd}(0,3,1), & c_3 &= \mathrm{Pd}(0,2,1). \end{aligned}$$

These values can be obtained by solving the following set of nine equations:

$$\begin{cases} 2ua_1 = b_3 + c_3\\ (2+t)b_3 + ub_3 + ub_3 + b_3 = (1+2t)c_3 + b_2 + a_2 + ua_1\\ (1+2t)c_3 + uc_3 + uc_3 + c_3 = (2+t)b_3 + c_2 + a_2 + ua_1\\ a_2 + a_2 + ua_2 + ua_2 = b_1 + c_1 + ub_3 + uc_3\\ (2+t)b_2 + ub_2 + b_2 = (1+2t)c_2 + b_1 + ub_3\\ (1+2t)c_2 + uc_2 + c_2 = (2+t)b_2 + c_1 + uc_3\\ 2a_3 = ub_1 + uc_1\\ (2+t)b_1 + ub_1 + b_1 + b_1 = (1+2t)c_1 + a_3 + ub_2 + ua_2\\ (1+2t)c_1 + uc_1 + c_1 + c_1 = (2+t)b_1 + a_3 + ua_2 + uc_2. \end{cases}$$

Without working through all the details, this can be solved to give

$$\begin{aligned} &(4u^3 + 36u^2t + 90ut^2 + 72t^3 + 32u^2 + 206ut + 270t^2 + 108u + 322t + 120)c_3\\ &= (4u^3 + 24u^2t + 54ut^2 + 36t^3 + 44u^2 + 190ut + 198t^2 + 160u + 350t + 200)b_3. \end{aligned}$$

As previously discussed, $\Pr(0,1,2) = \frac{1+2t}{9(1+t)}$ and $\Pr(2,1,0) = \frac{2+t}{9(1+t)}$, so for $b_3 = \mathrm{Pd}(0,1,2)$ and $c_3 = \mathrm{Pd}(2,1,0)$ to be in the same ratio we would require $b_3 = k(1+2t)$ and $c_3 = k(2+t)$ for some k. It follows, after also dividing through by 2, that

$$\begin{aligned} &(2u^3 + 18u^2t + 45ut^2 + 36t^3 + 16u^2 + 103ut + 135t^2 + 54u + 161t + 60)(t+2)\\ &= (2u^3 + 12u^2t + 27ut^2 + 18t^3 + 22u^2 + 95ut + 99t^2 + 80u + 175t + 100)(2t+1). \end{aligned}$$

This can be expanded to

$$\begin{aligned} &2u^3t + 18u^2t^2 + 45ut^3 + 36t^4 + 4u^3 + 52u^2t + 193ut^2\\ &\quad + 207t^3 + 32u^2 + 260ut + 431t^2 + 108u + 382t + 120\\ &= 4u^3t + 24u^2t^2 + 54ut^3 + 36t^4 + 2u^3 + 56u^2t +\\ &\quad 217ut^2 + 216t^3 + 22u^2 + 255ut + 449t^2 + 80u + 375t + 100. \end{aligned}$$

This can be reduced to

$$2u^3t + 6u^2t^2 + 9ut^3 - 2u^3 + 4u^2t + 24ut^2 + 9t^3 - 10u^2 - 5ut + 18t^2 - 28u - 7t - 20 = 0.$$

This can be factored as

$$(t-1)(2u^3 + 6u^2t + 9ut^2 + 10u^2 + 33ut + 9t^2 + 28u + 27t + 20) = 0.$$

Since $u \geq 0$ and $t \geq 0$, it follows that $t = 1$. This completes the proof in the $\Longleftarrow$ direction for the DASEP$(3,3,2)$ case.

5. Proof of the conjecture for DASEP$(3, p, 2)$

It would be an endless game to prove the conjecture "case by case", with more and more cumbersome computations, so let us now prove it for an infinite family of models. More precisely, we now prove Conjecture 4.3 for DASEP$(3, p, 2)$ (our previous examples covered the cases $p = 2$ and $p = 3$). To solve this case we essentially need to solve for each of p^2 prior probabilities $p_{i,j} = \mathrm{Pd}(0, i, j)$ for $1 \le i, j \le p$. The steady state probabilities can be obtained by solving a set of p^2 linear equations each of which essentially demands equilibrium for each of the possible states of the process. The generic form of such an equation, for $i < j$, is given by

$$(4 + t + 2u)p_{i,j} = (1 + 2t)p_{j,i} + p_{i+1,j} + p_{i,j+1} + up_{i-1,j} + up_{i,j-1}. \tag{5.1}$$

For $i > j$ the equation is

$$(3 + 2t + 2u)p_{i,j} = (2 + t)p_{j,i} + p_{i+1,j} + p_{i,j+1} + up_{i-1,j} + up_{i,j-1}$$

For $i = j$ the equation simplifies to

$$(2 + 2u)p_{i,i} = p_{i+1,i} + p_{i,i+1} + up_{i-1,i} + up_{i,i-1}.$$

The equation may be similarly simplified for other edge cases such as $i = 1 < j$, $i < j = p$, $i = 1 < j = p$, $i > j = 1$, $i = p > j$, $i = p > j = 1$, $i = j = 1$, and $i = j = p$. For the sake of brevity we do not list all such cases in detail.

From the first above equation we can define a polynomial $A_{i,j}$ by gathering all terms on the left:

$$A_{i,j} := (4 + t + 2u)p_{i,j} - (1 + 2t)p_{j,i} - p_{i+1,j} - p_{i,j+1} - up_{i-1,j} - up_{i,j-1}.$$

We can similarly define $A_{i,j}$ under the conditions stated for the various edge cases. We next define a $p^2 \times p^2$ matrix B as follows:

$$B_{p(i_1-1)+j_1,p(i_2-1)+j_2} = [p_{i_1,j_1}]A_{i_2,j_2}.$$

The next step is to prove that the rank of B is $p^2 - 1$. To see this, we first observe that the sum of all rows of B is identically zero, meaning that the rank cannot be p^2. For the rank to then be $p^2 - 1$, we would then need to show that no nontrivial linear combination of a proper subset of the rows can be zero. If we let row i, j be $R_{i,j}$ and for some coefficients $c_{i,j}$ we have $\sum_{i,j} c_{i,j}R_{i,j} = 0$, then we need to show that if any $c_{i,j} = 0$, then all $c_{i,j} = 0$. The only rows with a t term in column i, j will be $R_{i,j}$ and $R_{j,i}$. Hence if $c_{i,j} = 0$, it follows that $c_{j,i}$=0.

We next show that if $c_{i,j} = 0$ it follows that $c_{i-1,j-1} = 0$. We can do this by first showing that $c_{i-1,j}$ and $c_{i,j-1}$ must be negations of one another. The only rows with a u term in column i, j will be $R_{i,j}$, $R_{i-1,j}$, and $R_{i,j-1}$, with the latter two having the same coefficient. Hence the following two statements are equivalent: $c_{i,j} = 0$ and $c_{i-1,j} + c_{i,j-1} = 0$. We can similarly show that $c_{i,j} = 0$ and $c_{i+1,j} + c_{i,j+1} = 0$ are equivalent. So from $c_{i,j} = 0$ we can derive $c_{i-1,j-1} = 0$. By repeated application of the same argument we will get $c_{k,1} = 0$ or $c_{1,k} = 0$ for some k.

Likewise, using the equations for the edge cases $i = 1 < j$ and $i > j = 1$, the only rows with a u term in column $1, k$ will be $R_{1,k}$ and $R_{1,k-1}$ and the only rows with a u term in column $k, 1$ will be $R_{k,1}$ and $R_{k-1,1}$. So from $c_{k,1} = 0$ we can derive $c_{k-1,1} = 0$ and from $c_{1,k} = 0$ we can derive $c_{1,k-1} = 0$. By repeated application of this we will get to $c_{1,1} = 0$. By reversing the above arguments it follows that $c_{i,j} = 0$ for any i, j and we have proven:

Lemma 5.1. *The rank of the matrix B as defined above is $p^2 - 1$.*

We next prove a result about the values of the $p_{i,j}$.

Proposition 5.2. *One has*

$$p_{i,j} + p_{j,i} = \frac{2u^{i+j-2}}{\left(\sum_{k=0}^{n-1} u^k\right)^2} \qquad \textit{and} \qquad p_{i,i} = \frac{u^{2i-2}}{\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

Proof. This can be proven by eliminating the variable t from the set of linear equations above. For example, if we add the equations for $i < j$ and $j < i$ we get the following:

$$\begin{aligned}
&(4 + t + 2u)p_{i,j} + (3 + 2t + 2u)p_{j,i} \\
&= (2 + t)p_{i,j} + (1 + 2t)p_{j,i} + p_{i+1,j} + p_{j,i+1} + p_{i,j+1} + p_{j+1,i} \\
&\quad + up_{i-1,j} + up_{j,i-1} + up_{i,j-1} + up_{j-1,i}.
\end{aligned}$$

If we let $q_{i,j} = p_{i,j} + p_{j,i}$ the above can be simplified to

$$(2 + 2u)q_{i,j} = q_{i+1,j} + q_{i,j+1} + uq_{i-1,j} + uq_{i,j-1}.$$

If we substitute in the values for $q_{i,j}$ from the theorem we are attempting to prove to the above equation, we see that it does satisfy the above equation. Therefore the values of $q_{i,j}$ given in the theorem represent one possible feasible solution to the set of equations. Moreover, via Lemma 5.1 about the rank of B, the solution must be unique. This completes the proof. □

To continue with the proof of Conjecture 4.3 in the $\Longleftarrow$ direction, we note that from $\frac{\Pr(\mu)}{\Pr(\nu)} = \frac{\mathrm{Pd}(\mu)}{\mathrm{Pd}(\nu)}$ it follows that $\frac{\Pr(0,2,1)}{\Pr(0,1,2)} = \frac{\mathrm{Pd}(0,2,1)}{\mathrm{Pd}(0,1,2)}$ or $\frac{2+t}{1+2t} = \frac{p_{2,1}}{p_{1,2}}$. This expands as $(2 + t)p_{1,2} = (1 + 2t)p_{2,1}$. From the above theorem we know that

$$p_{1,2} + p_{2,1} = \frac{2u}{\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

We can then solve for $p_{1,2}$ giving

$$p_{1,2} = \frac{2(1 + 2t)u}{3(1 + t)\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

From the equation (5.1) for $i = 1 < j$ we get

$$(3 + t + 2u)p_{1,2} = (1 + 2t)p_{2,1} + p_{2,2} + p_{1,3} + up_{1,1}.$$

Substitute in to get

$$\frac{2(3 + t + 2u)(1 + 2t)u}{3(1 + t)\left(\sum_{k=0}^{n-1} u^k\right)^2} = \frac{2(2 + t)(1 + 2t)u}{3(1 + t)\left(\sum_{k=0}^{n-1} u^k\right)^2} + \frac{3(1 + t)(1 + u)u}{3(1 + t)\left(\sum_{k=0}^{n-1} u^k\right)^2} + p_{1,3}.$$

This simplifies to

$$p_{1,3} = \frac{(5ut + u + t - 1)u}{3(1+t)\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

A similar argument to that used to produce the above equation for $p_{1,2}$ will give us

$$p_{1,3} = \frac{2(1+2t)u^2}{3(1+t)\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

Equating the last two equations and solving gives us $t = 1$. This completes the proof of

Theorem 5.3. *Conjecture 4.3 holds for* $D = \text{DASEP}(3, p, 2)$*.*

6. Future work

Three main potential directions for future work are indicated. One is that further results should be obtained with a view to eventually proving Conjecture 4.3. We proved it for DASEP$(3, p, 2)$ and the suggestion would be to prove it for DASEP$(n, 2, 2)$ and DASEP$(n, 2, q)$ before eventually proceeding to DASEP(n, p, q). Similarly considering the case where 0 represents a ball with species 0 rather than the absence of a species is a variant that should be explored. The other, and more ambitious, possible goal for future research would be to come up with a complete combinatorial characterization of the steady state probabilities for the DASEP. For the ASEP, this has been done in [4] and [10] leading to a deep relationship being discovered between the ASEP and Macdonald polynomials.

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SIGNED AREA ENUMERATION FOR LATTICE WALKS

STÉPHANE OUVRY[1] AND ALEXIOS P. POLYCHRONAKOS[2]

[1]LPTMS, CNRS, Université Paris-Saclay, France;
http://lptms.u-psud.fr/fr/blog/stphane-ouvry

[2]Physics Department, City College of New York, USA;
Graduate Center, City University of New York, USA;
https://www.ccny.cuny.edu/profiles/alexios-polychronakos

Abstract. We give a summary of recent progress on the signed area enumeration of closed walks on planar lattices. Several connections are made with quantum mechanics and statistical mechanics. Explicit combinatorial formulae are proposed which rely on sums labelled by the multicompositions of the length of the walks.

Keywords: lattice walks, signed area enumeration, Hofstadter model, exclusion statistics.

1. Introduction

The seminal problem of the signed area enumeration of walks on planar lattices of various kinds has been around for a long time. It is well known that this purely combinatorial problem can be equivalently reformulated in the realm of Hofstadter-like quantum mechanics models (note that, in physics, the "signed area" is often called the "algebraic area"). Recently, in [16], this problem has been given a boost in the form of an explicit enumeration formula which in turn could be reinterpreted [5, 14, 15] in terms of statistical mechanics models with exclusion statistics, again a purely quantum concept. It is a striking fact that an enumeration quest regarding classical random walks should be in the end so intimately connected to quantum physics.

In this note we give a summary of this recent progress starting with the original signed area enumeration problem for closed walks on a square lattice and then enlarging the perspective to other kinds of lattices and walks via the statistical mechanics reinterpretation. The first question we address is: Among the $\binom{N}{N/2}^2$ closed N-step walks that one can draw on a square lattice starting from and returning to a given point (note that N is then necessarily even), how many of them enclose a given signed area A?

The signed area enclosed by a directed walk is weighted by its *winding number*: If the walk moves around a region in a counterclockwise direction, its area counts as positive, otherwise it counts as negative; if the walk winds around more than once, the area is counted with multiplicity. These regions inside the walk are called winding sectors.

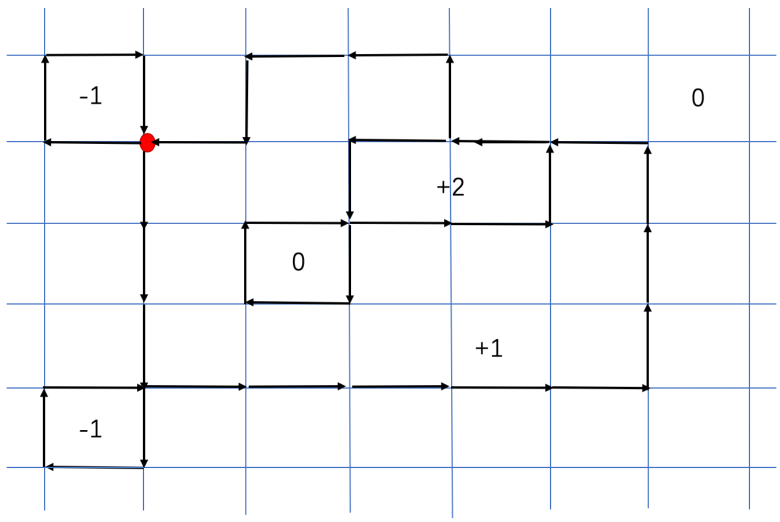


FIGURE 1. A closed walk of length $N = 36$ starting from and returning to the same bullet red point with its various winding sectors $m = +2, +1, 0, -1, -1$ (containing respectively $2, 14, 1, 1, 1$ unit lattice cells). Note the double arrow on the horizontal link (above the $+2$ sector) which indicates that the walk has moved twice on this link, here in the same left direction.

In Figure 1, the 0-winding sector inside the walk arises from a superposition of a $+1$ and a -1 winding. Summing the areas of each sector, with the corresponding multiplicative weight, gives the signed area $A = (-1) \times 2 + (+0) \times 1 + (+1) \times 14 + (+2) \times 2 = 16$.

More formally, if $\gamma : [0, 1] \to \mathbb{R}^2$ is a closed path that begins and ends at the origin, the signed area of this path is

$$A = \int_{\mathbb{R}^2} \eta(\gamma, \mathbf{x}) d\mathbf{x} = \sum_{m=-\infty}^{\infty} m S_m,$$

where $\eta(\gamma, \mathbf{x})$ is the winding number of γ around the point $\mathbf{x} \in \mathbb{R}^2$, and where S_m denotes the classical area of the m-winding sectors inside the path (i.e. the number of unit lattice cells it encloses with winding number m, where m can be positive or negative).

Winding sectors for continuous Brownian curves as well as for discrete lattice walks have been the subject of study for a long time. In this respect, we note in the last few years some advances in [2] where[1] an explicit formula for the expected area $\langle S_m \rangle$ of the m-winding sectors inside square-lattice walks is proposed, to the exception of the 0-winding sector, for the simple reason that the latter is difficult to distinguish from the outside (i.e. 0-winding again) sector, which is of infinite size. Taking the continuous limit allows us to recover the results previously obtained in [4] for Brownian curves. One notes that for Brownian curves the expected area $\langle S_0 \rangle$ of the 0-winding sectors is also known by other means thanks to the SLE machinery [7]. However, it remains an open problem for discrete lattice walks.

[1]Note that Timothy Budd gave a talk on his article [2] at the conference Lattice Paths, Combinatorics and Interactions, at CIRM, in 2021. A video of his talk is available on the website of this conference.

Counting the number of closed walks of length N on the square lattice enclosing a signed area A can be achieved in a most straightforward way by introducing two lattice hopping operators (symbols) u and v respectively in the right and up directions, as well as operators u^{-1} and v^{-1} corresponding to hops in the left and down direction. A directed walk on the square lattice starting at the origin is then represented by the ordered product of the hopping operators corresponding to its individual steps. By convention we order the operators *from right to left* as we trace the steps of the walk: vu corresponds to a up step u, followed by a right step v. Clearly the set of all walks of length N on the lattice is reproduced by the 4^N terms in the expansion of

$$\Big(u + u^{-1} + v + v^{-1}\Big)^N \tag{1.1}$$

into monomials of products of symbols, each with N factors. The operator $u+u^{-1}+v+v^{-1}$ can be considered as the generator of walks.

We are interested in closed walks, and in counting their multiplicity according to their signed area. To this end, we endow the above operators with the relations

$$uu^{-1} = u^{-1}u = vv^{-1} = v^{-1}v = 1,$$

to which we add a non-commutativity relation (which expresses the fact that the elementary walk circling one lattice cell in the counterclockwise direction has signed area 1):

$$v^{-1}\, u^{-1}\, v\, u = \mathrm{Q},$$

where Q is a central element (that is, Q commutes with all operators). This entails

$$vu = \mathrm{Q}uv, \quad vu^{-1} = \mathrm{Q}^{-1}u^{-1}v, \quad v^{-1}u = \mathrm{Q}^{-1}uv^{-1}, \quad v^{-1}u^{-1} = \mathrm{Q}u^{-1}v^{-1}, \tag{1.2}$$

which allows us to reduce all terms in (1.1) into monomials of the form $u^m v^n$, $(m,n) \in \mathbb{Z}^2$ being the lattice coordinates of the end of the walk, with coefficients which are powers of Q. In particular, closed walks correspond to the monomial u^0v^0 in (1.1).

The non-commutativity relation $vu = \mathrm{Q}\, uv$ has the effect of flipping a right-up two-step segment into an up-right one, producing a factor of Q, and similarly for each relation in (1.2). In each case, the coefficient is Q to the power of the signed area of the unit lattice cells left behind by the exchange. Repeated application of these relations reduces each walk to hook-shape walk $u^m v^n$ with an overall coefficient Q^A, with A the signed area of the original walk prolonged into a closed walk by joining its end to the origin with a vertical and a horizontal straight walk. In particular, closed walks correspond to monomials $\mathrm{Q}^A u^0 v^0$, with A the signed area of the walk. This area is maximal if the walk forms a square of width $N/4$, and then the signed area is $\pm N^2/16$. Therefore, using the notation $[u^0v^0]$ for the extraction of the constant term in a Laurent polynomial in u and v, the distribution of the signed area of closed walks of length N is given by

$$[u^0v^0]\Big(u + u^{-1} + v + v^{-1}\Big)^N = \sum_{A=-\lfloor N^2/16\rfloor}^{\lfloor N^2/16\rfloor} C_N(A)\ \mathrm{Q}^A, \tag{1.3}$$

where $C_N(A)$ counts the closed walks of length N enclosing a signed area A. For example, one easily checks that $[u^0v^0]\Big(u + u^{-1} + v + v^{-1}\Big)^4 = 28 + 4\mathrm{Q} + 4\mathrm{Q}^{-1}$, indicating that among the $\binom{4}{2}^2 = 36$ closed walks making 4 steps $C_4(0) = 28$ enclose a signed area $A = 0$ and $C_4(1) = C_4(-1) = 4$ enclose a signed area $A = \pm 1$.

2. The Hofstadter model

In any irreducible representation of the operator relation $vu = Quv$, the central element Q will be represented by a number. Restricting to unitary representations, for which $u^\dagger = u^{-1}$, $v^\dagger = v^{-1}$, Q will necessarily be a complex number of norm unity, i.e., a phase. This provides a mapping between the u, v representation for walks and quantum mechanics, interpreting u and v as unitary operators acting on a quantum Hilbert space, and the operator $u + u^{-1} + v + v^{-1}$ as the Hamiltonian of a quantum system.

In fact, such a quantum system exists and corresponds to a well-known model in physics. Interpreting u and v as operators that generate hops of a quantum particle by one link on the square lattice, the non-commutativity relation $vu = Quv$ indicates that translations of the particle in the horizontal and vertical directions do not commute. This can be interpreted as that the particle is charged and coupled to a homogeneous magnetic field perpendicular to the lattice. The magnetic flux associated to this (constant) vector field is Φa^2 for any surface of area a^2.

Let us now take $\mathrm{Q} = \mathrm{e}^{\mathrm{i}2\pi\Phi/\Phi_o}$, where Φ is the magnetic flux through any unit lattice cell (i.e. for the surface of the square of width $a = 1$) and where $\Phi_o = h/c$ is the flux quantum (h is the Planck constant and c the particle's charge). The Hermitian operator

$$H = u + u^{-1} + v + v^{-1} \tag{2.1}$$

then becomes a Hamiltonian modelling a quantum particle hopping on a square lattice and coupled to a perpendicular magnetic field. This model is known as the Hofstadter model [8].

To make the physics connection completely explicit, we note that in quantum mechanics the hopping operators u and v are written as

$$u = \mathrm{e}^{\mathrm{i}(p_x - cA_x)/\hbar} \quad \text{and} \quad v = \mathrm{e}^{\mathrm{i}(p_y - cA_y)/\hbar},$$

where $A_x = -\Phi y$ and $A_y = 0$ are the two components of the vector potential of the magnetic field in the Landau gauge and $p_x = -\mathrm{i}\hbar\partial_x$ and $p_y = -\mathrm{i}\hbar\partial_y$ those of the momentum operator (where we use the standard notation $\hbar = h/(2\pi)$). Thus, assuming (1), the relation $vu = Quv$ follows from the Baker–Campbell–Hausdorff formula, using the Heisenberg commutators $[x, p_x] = [y, p_y] = \mathrm{i}\hbar$ (this identity is often called the "canonical commutation relation").

One now introduces the quantum state $\Psi_{m,n}$ representing the probability amplitude of the particle being at lattice site (m, n), on which hopping operators act as

$$u\Psi_{m,n} = \mathrm{e}^{\mathrm{i}cn\Phi/\hbar}\Psi_{m+1,n}\ , \quad v\Psi_{m,n} = \Psi_{m,n+1}.$$

As $c\Phi/\hbar = 2\pi\Phi/(h/c) = 2\pi\Phi/\Phi_o$, the factor appearing in the action of u on $\psi_{m,n}$ is Q^n. Using translation invariance in the horizontal direction we can further choose $\Psi_{m,n}$ to be an eigenstate of p_x, that is, $\Psi_{m,n} = \mathrm{e}^{\mathrm{i}mk_x}\Phi_n$. The action of u, v on Φ_n becomes

$$u\Phi_n = \mathrm{e}^{\mathrm{i}k_x}\mathrm{Q}^n\Phi_n\ ,\ v\Phi_n = \Phi_{n+1}.$$

For the Hofstadter model, the Schrödinger equation $H\Psi = E\Psi$ (which determines the eigenvalue E of the spectrum) can be rewritten as

$$(u + u^{-1} + v + v^{-1})\Psi_{m,n} = E\Psi_{m,n} \Rightarrow \Phi_{n+1} + \Phi_{n-1} + (\mathrm{Q}^n\mathrm{e}^{\mathrm{i}k_x} + \mathrm{Q}^{-n}\mathrm{e}^{-\mathrm{i}k_x})\Phi_n = E\Phi_n.$$

Going a step further, a simplification arises when the flux is *rational* (i.e. when one has $\mathrm{Q} = \mathrm{e}^{\mathrm{i}2\pi p/q}$ with p, q two coprime integers): It induces a q-periodicity of the Schrödinger equation in the vertical direction. Then, as Bloch's theorem states that solutions to the Schrödinger equation in a periodic potential take the form of a plane wave modulated by a periodic function $\tilde{\Phi}_n$, we can write

$$\Phi_n = \mathrm{e}^{\mathrm{i}nk_y}\tilde{\Phi}_n \ , \ \tilde{\Phi}_{n+q} = \tilde{\Phi}_n. \tag{2.2}$$

Indeed, for this rational flux $\mathrm{Q}^q = 1$, thus u^q, v^q become Casimirs (a physicist's term for central elements), and the choice of Bloch states (2.2) can be interpreted mathematically as choosing an irreducible representation of the u, v algebra. Acting on such states, u^q and v^q become $u^q = \mathrm{e}^{\mathrm{i}qk_x}$ and $v^q = \mathrm{e}^{\mathrm{i}qk_y}$. One ends up with u and v, acting on $\tilde{\Phi}_n$, becoming the $q \times q$ matrices

$$u = \mathrm{e}^{\mathrm{i}k_x}\begin{pmatrix} \mathrm{Q} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{Q}^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathrm{Q}^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{Q}^{q-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \mathrm{e}^{\mathrm{i}k_y}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

involving two real quantities k_x and k_y. Finding the energy spectrum (which depends on k_x and k_y) reduces to computing the eigenvalues $E_1, \ldots, E_q$ of the $q \times q$ Hamiltonian matrix $H_q := u + u^{-1} + v + v^{-1}$, i.e.

$$H_q = \begin{pmatrix} \mathrm{Q}\mathrm{e}^{\mathrm{i}k_x} + \mathrm{Q}^{-1}\mathrm{e}^{-\mathrm{i}k_x} & \mathrm{e}^{\mathrm{i}k_y} & 0 & \cdots & 0 & \mathrm{e}^{-\mathrm{i}k_y} \\ \mathrm{e}^{-\mathrm{i}k_y} & \mathrm{Q}^2\mathrm{e}^{\mathrm{i}k_x} + \mathrm{Q}^{-2}\mathrm{e}^{-\mathrm{i}k_x} & \mathrm{e}^{\mathrm{i}k_y} & \cdots & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k_y} & () & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & () & \mathrm{e}^{\mathrm{i}k_y} \\ \mathrm{e}^{\mathrm{i}k_y} & 0 & 0 & \cdots & \mathrm{e}^{-\mathrm{i}k_y} & \mathrm{Q}^q\mathrm{e}^{\mathrm{i}k_x} + \mathrm{Q}^{-q}\mathrm{e}^{-\mathrm{i}k_x} \end{pmatrix}.$$

All the machinery of quantum mechanics is now at our disposal. Selecting as in (1.3) the u^0v^0 monomial of $\left(u + u^{-1} + v + v^{-1}\right)^N$ translates in the quantum world to computing the trace of H_q^N. The quantum trace is defined as

$$\mathbf{Tr}\, H_q^N := \frac{1}{q}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\,\mathrm{tr}\, H_q^N = \frac{1}{q}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\sum_{i=1}^{q} E_i^N, \tag{2.3}$$

that is, one sums over the q eigenvalues E_i of H_q (yielding the standard matrix trace $\mathrm{tr}\, H_q^N$) and integrates over k_x and k_y while enforcing a continuous normalization in k_x, k_y and one rescales by a factor $1/q$ (thus, if one considers for example the $q \times q$ identity matrix I_q, one has $\mathrm{tr}\, I_q = q$, while $\mathbf{Tr}\, I_q = 1$). Under this definition of the trace, $\mathbf{Tr}\, u^m v^n = \delta_{m,0}\delta_{n,0}$ (integration over k_x, k_y eliminates the traces of terms involving u^{qm} and v^{qn}). We thus get a first noteworthy result (also obtained via another approach by Bellisard et al. in [1]):

Theorem 2.1. *Assuming $q > \frac{N^2}{8}$, the signed area enumeration of closed paths is given by*

$$\sum_A C_N(A)\, \mathrm{Q}^A = \mathbf{Tr}\, H_q^N. \tag{2.4}$$

3. The signed area enumeration

It is known (see e.g. William Chambers' book [3]) that the determinant of the matrix $I_q - zH_q$ satisfies

$$\det(I_q - zH_q) = \left(\sum_{n=0}^{\lfloor q/2\rfloor} (-1)^n Z(n) z^{2n}\right) - 2\Big(\cos(qk_x) + \cos(qk_y)\Big) z^q,$$

where the $Z(n)$'s are independent of k_x and k_y and $Z(0) = 1$. Christian Kreft [12] was able to rewrite $Z(n)$ in a closed form as trigonometric multiple nested sums

$$Z(n) = \sum_{k_1=1}^{q-2n+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} \prod_{j=1}^{n} s_{k_j+2n-2j}, \tag{3.1}$$

where

$$s_k = 4\sin^2(\pi kp/q) = 2 - Q^k - Q^{-k}. \tag{3.2}$$

We call s_k the spectral function of the model. This is the starting point for the signed area enumeration. We give here a summary of the procedure, more details can be found in [14, 16]. First introduce the coefficients $b(n)$ via

$$-\log\left(\sum_{n=0}^{\lfloor q/2\rfloor} (-1)^n Z(n) z^{2n}\right) = \sum_{n=1}^{\infty} b(n) z^{2n}. \tag{3.3}$$

The $b(n)$ are related to the desired traces. Start by noting that

$$\mathbf{Tr}\; H_q^{2n} = \frac{1}{q}\mathrm{tr}\, H_q^{2n} \text{ for } n < q.$$

Indeed, one has $\mathrm{tr}\, u^i v^j = q\; \delta_{i,0}\delta_{j,0}$ for $i, j < q$ and so the values of the Casimirs k_x, k_y do not appear, making the integration over them in (2.3) trivial. Then the identity

$$-\log\,\det(I_q - zH_q) = -\mathrm{tr}\,\log(I_q - zH_q) = \sum_{n=1}^{\infty} \frac{z^n}{n}\mathrm{tr}\, H_q^n = \sum_{n=1}^{\lfloor q/2\rfloor} b(n) z^{2n} + O(z^q)$$

implies that the quantum trace (2.3) is proportional to $b(n)$ for $n < q$

$$\mathbf{Tr}\, H_q^{2n} = \frac{2n}{q} b(n).$$

Now, by keeping q as a free parameter and extending it to arbitrarily big values, the quantum trace can be calculated for all n.

Note that the term of order z^n in $\det(I_q - zH_q)$ is $-(z^n/n)\mathrm{tr}\, H_q^n$ plus terms involving products of traces, $\mathrm{tr}\, H_q^{n_1}\,\mathrm{tr}\, H_q^{n_2}\cdots$ with $n_1+n_2+\cdots = n$. As each trace $\mathrm{tr}\, H_q^{n_i}$ contributes an overall factor q, $b(n)$ can also be obtained as the order q term in $Z(n)$, ignoring terms of higher order $q^2, \ldots, q^n$. Since each sum in Formula (3.1) contributes a factor of q, we get the following explicit expression for $b(n)$:

$$b(n) = \sum_{j=1}^{n} \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}, \tag{3.4}$$

$$\text{where } c(l_1, l_2, \ldots, l_j) := \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2}\; l_2\frac{\binom{l_2+l_3}{l_2}}{l_2+l_3}\; \cdots\; l_{j-1}\frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j}.$$

Here, the coefficients $c(l_1, l_2, \dots, l_j)$ are labelled by the compositions $l_1, l_2, \dots, l_j$ of n, i.e. the ordered partitions of n (thus, there are 2^{n-1} compositions of n, for example $3 = 3, 2+1, 1+2, 1+1+1$). Note that the expression for $c(l_1, l_2, \dots, l_j)$ is closely related to the enumeration of Dyck paths (up to a factor l_1); see Christian Krattenthaler's article [11, p. 516]. We have further elaborated on this relation in [6].

Putting everything together, we get

$$\mathbf{Tr}\ H_q^{2n} = \sum_{j=1}^{n} \sum_{\substack{l_1,l_2,\dots,l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j)\, \frac{2n}{q} \sum_{k=1}^{q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}.$$

What is more, the trigonometric sums involved in this formula can be computed, keeping q as a free parameter. Finally, returning to (2.4) (with $N = 2n$) the desired number of closed walks of given area is given by the following theorem.

Theorem 3.1. *The number of closed walks of length $2n$ enclosing a given signed area A is*

$$C_{2n}(A) = 2n \times \sum_{j=1}^{n} \sum_{\substack{l_1,l_2,\dots,l_j \\ \textit{composition of}\, n}} \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2}\, l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2+l_3} \cdots\ l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j} \times$$

$$\sum_{k_3=0}^{2l_3} \sum_{k_4=0}^{2l_4} \cdots \sum_{k_j=0}^{2l_j} \prod_{i=3}^{j} \binom{2l_i}{k_i} \binom{2l_1}{l_1 + A + \sum_{i=3}^{j}(i-2)(k_i - l_i)} \binom{2l_2}{l_2 - A - \sum_{i=3}^{j}(i-1)(k_i - l_i)}.$$

This formula grows quickly in complexity since one has to sum over $2^n - 1$ compositions. Its complexity is analysed in more detail in the following proposition.

Proposition 3.2. *The formula for $C_{2n}(A)$ in Theorem 3.1 involves asymptotically, up to some polynomial factor, $(4/(5-\sqrt{17}))^n \approx 4.56^n$ summands.*

Proof. This number of summands is given by

$$r(n) := \sum_{j=1}^{n} \sum_{\substack{l_1,l_2,\dots,l_j \\ \text{composition of } n}} (2l_3+1)(2l_4+1)\cdots(2l_j+1).$$

The sequence starts like $(r(n))_{n\geq 1} = (1, 2, 6, 24, 106, 480, 2186, 9968, 45466, \dots)$. Thus, $r(n)$ counts compositions of n where each summand l_i (for $i \geq 3$) can have $2l_i + 1$ colours. Let us consider first compositions of n where each summand l_i (for $i \geq 1$) can have $2l_i + 1$ colours; their generating function is

$$S(z) = \frac{1}{1 - \sum_{i\geq 1}(2i+1)z^i} = \frac{(z-1)^2}{2z^2 - 5z + 1}.$$

This corresponds to the sequence `A060801` in the On-line Encyclopedia of Integer Sequences. In our case, the $i \geq 3$ constraint modifies a little bit the generating function and one has to sum the compositions having 1 or 2 parts and those having 3 parts or more; one gets the following generating function

$$\begin{aligned} R(z) = \sum_{n\geq 1} r_n z^n &= \frac{z}{1-z} + \left(\frac{z}{1-z}\right)^2 + \left(\frac{z}{1-z}\right)^2 (S(z) - 1) \\ &= \frac{z^2 - 4z + 1}{2z^2 - 5z + 1} \frac{z}{1-z}. \end{aligned}$$

It entails $r(n) = 5r(n-1) - 2r(n-2) - 2$ (with $r(1) = 1$ and $r(2) = 2$); accordingly, $r(n)$ grows like ϕ^n with $\phi = 4/(5-\sqrt{17})$, which is coherent with the fact that $1/\phi$ is the dominant pole of $R(z)$. In conclusion, our formula for $C_{2n}(A)$ involves in total asymptotically $\phi^n \approx 4.56^n$ summands (each leading additionally to a polynomial cost in n for the product of all the binomials), which is still much less[2] than the naive generation of all $\binom{2n}{n}^2$ closed walks of length $2n$, which would be of cost $> 16^n/(\pi n)$. □

Note that it is in fact possible to compute C_{2n} in polynomial time: Using the non-commutative relations (1.2), the expansion of $(u + v + u^{-1} + v^{-1})^{2n}$ simplifies a lot and in fact has $O(n^4)$ monomials $u^i v^j Q^k$. This gives an algorithm of complexity $O(n^6)$ to compute $C_{2n}(A)$. Thus, our formula (of exponential cost) in Theorem 3.1 is not the fastest way to compute $C_{2n}(A)$, but it has the benefit of being the first explicit formula (as far as we are aware of!).

Let us end this section with a probabilistic remark. In the limit of the elementary lattice size $a \to 0$ and the walk length $2n \to \infty$ with the scaling $na^2 = t$, walks converge to Brownian motion curves and we recover the continuum limit of a particle moving on the plane in a constant magnetic field. To implement this limit, we rescale the lattice cell area to a^2, which amounts to setting $A \to A/a^2$ in $C_{2n}(A)$. Numerical simulations then suggest the following conjecture.

Conjecture 3.3. *The signed area of closed walks of length $2n$ converges, after rescaling, to the following distribution (for any $\alpha > 0$)*

$$\frac{2n\, C_{2n}(\alpha n)}{\binom{2n}{n}^2} \to \frac{\pi}{\cosh^2(\alpha\pi)}.$$

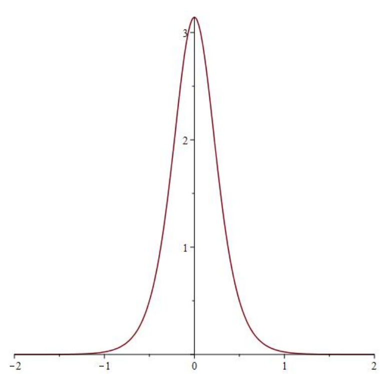


FIGURE 2. We conjecture that the distribution of the signed area asymptotically follows… not a Gaussian limit law (as it may be thought at first glance), but the $1/\cosh^2$ distribution of Paul Lévy. For small values of A, and n up to 70, we checked numerically that the convergence gets better when n increases.

This conjecture is consistent with the law for the distribution of the signed area enclosed by a Brownian curve after a time t (obtained by Paul Lévy in 1950; see [10, 13]). It can also be obtained directly in the continuum limit by considering the partition function of a quantum particle in a magnetic field with a Landau level energy spectrum.

[2] We thank one of the referees for drawing our attention to this point.

4. Exclusion statistics

The quantities $Z(n)$ and $b(n)$ introduced previously admit a statistical mechanical interpretation. Let us write the spectral function s_k in (3.2) as $s_k = \mathrm{e}^{-\beta\epsilon_k}$ (β is the inverse temperature) and interpret is as the Boltzmann factor for a quantum 1-body spectrum ϵ_k labelled by an integer k. The structure of $Z(n)$ in (3.1) then precisely corresponds to an n-body partition function for a gas of particles with 1-body spectrum ϵ_k and exclusion statistics $g = 2$: The $+2$ shifts in the spectral function arguments ensure that no two particles can occupy adjacent quantum states. Exclusion statistics is, again, a purely quantum concept which describes the statistical mechanical properties of identical particles. Ordinary particles are either bosons ($g = 0$), which can occupy the same quantum state, or fermions ($g = 1$), which cannot occupy the same quantum state. We see that square-lattice walks map to systems with statistics beyond Fermi exclusion, in which particles can occupy neither the same state nor adjacent states. In a sense, each particle excludes two quantum states, thus $g = 2$. In general, for g-exclusion particles the n-body partition function (3.1) would become

$$Z(n) = \sum_{k_1=1}^{q-gn+g} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+gn-g} s_{k_2+gn-2g} \cdots s_{k_{n-1}+g} s_{k_n}, \tag{4.1}$$

where one observes a shift g instead of 2 in the arguments of the spectral function. In line with (3.3), (3.4) the associated n-th cluster coefficient can be shown to take the form

$$b(n) = \sum_{j=1}^{n} \sum_{\substack{l_1,l_2,\ldots,l_j \\ g-\text{composition of } n}} c_g(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}, \tag{4.2}$$

where

$$c_g(l_1, l_2, \ldots, l_j) := \frac{(l_1 + \cdots + l_{g-1} - 1)!}{l_1! \cdots l_{g-1}!} \prod_{i=1}^{j-g+1} \binom{l_i + \cdots + l_{i+g-1} - 1}{l_{i+g-1}}.$$

In (4.2) one sums over all g-compositions of the integer n, obtained by inserting at will inside the usual compositions (i.e., the 2-compositions) no more than $g - 2$ zeroes in succession. For example, for $n = 3$ and $g = 3$ one has 9 such 3-compositions:

$$3, 2+1, 1+2, 1+1+1, 2+0+1, 1+0+2, 1+0+1+1, 1+1+0+1, 1+0+1+0+1.$$

For general g there are g^{n-1} such g-compositions of the integer n (see [9] for an analysis of these extended compositions, also called multicompositions).

One has reached the conclusion that the signed area enumeration for walks on the square lattice is described by a quantum gas of particles with statistical exclusion $g = 2$. To relate this explicitly to properties of the Hofstadter Hamiltonian itself, let us perform on the hopping lattice operators u and v the transformation

$$u \to -u\, v, \;\; v \to v,$$

which leave their own commutation relation invariant to get the new Hamiltonian

$$H = -u\, v - v^{-1} u^{-1} + v + v^{-1} \tag{4.3}$$

still describing the same walks but on a deformed lattice.

This new Hamiltonian (if one compares it with the initial Hamiltonian operator (2.1) of the Hofstadter model) has the advantage to lead to simpler matrix

$$I_q - zH_q = \begin{pmatrix} 1 & -(1-\mathrm{Q})z & 0 & \cdots & 0 & 0 \\ -(1-\frac{1}{\mathrm{Q}})z & 1 & -(1-\mathrm{Q}^2)z & \cdots & 0 & 0 \\ 0 & -(1-\frac{1}{\mathrm{Q}^2})z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -(1-\mathrm{Q}^{q-1})z \\ 0 & 0 & 0 & \cdots & -(1-\frac{1}{\mathrm{Q}^{q-1}})z & 1 \end{pmatrix}, \tag{4.4}$$

where we set $k_x = k_y = 0$ for simplicity (since, as we explained in Section 3, k_x and k_y do not appear in the counting formula). The Hofstadter spectral function (3.2) becomes

$$s_k = (1-\mathrm{Q}^k)(1-\frac{1}{\mathrm{Q}^k}).$$

The matrix (4.4) is a particular case of the more general class of matrices having the following shape[3]

$$I_q - zH_q = \begin{pmatrix} 1 & -f(1)z & 0 & \cdots & 0 & -g(q)z \\ -g(1)z & 1 & -f(2)z & \cdots & 0 & 0 \\ 0 & -g(2)z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -f(q-1)z \\ -f(q)z & 0 & 0 & \cdots & -g(q-1)z & 1 \end{pmatrix} \tag{4.5}$$

and associated spectral functions

$$s_k = f(k)g(k),$$

which become the building blocks of the $Z(n)$'s in (3.1) (up to spurious "umklapp" terms, a name deriving from momentum periodicity effects on lattice quantum models, which disappear if $f(q)$ and $g(q)$ both vanish).

For statistics $g = 3$, the matrix (4.5) generalizes in a natural way to

$$I_q - zH_q = \begin{pmatrix} 1 & -f(1)z & 0 & 0 & \cdots & 0 & -g(q-1)z & 0 \\ 0 & 1 & -f(2)z & 0 & \cdots & 0 & 0 & -g(q)z \\ -g(1)z & 0 & 1 & -f(3)z & \cdots & 0 & 0 & 0 \\ 0 & -g(2)z & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -f(q-2)z & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -f(q-1)z \\ -f(q)z & 0 & 0 & 0 & \cdots & -g(q-2)z & 0 & 1 \end{pmatrix}, \tag{4.6}$$

that is, with an extra vanishing paradiagonal below the unity main diagonal, which is the manifestation of the stronger $g = 3$ exclusion.

[3] We hope that the reader will easily distinguish between too similar (but unrelated!) notations: the *function* $g(k)$ for the entries of the matrix $I_q - zH_q$, and the *integer* g (the parameter of the exclusion model, a standard notation in the literature).

The spectral function corresponding to this matrix is

$$s_k = g(k)f(k)f(k+1)$$

and the determinant $\det(I_q - zH_q)$ assumes the form (4.1) with $g = 3$. For g-exclusion the generalization of (4.6) amounts to a Hamiltonian of the form

$$H = F(u)v + v^{1-g}G(u), \tag{4.7}$$

and $I - zH$ is then a matrix with $g-2$ vanishing paradiagonals below the main diagonal (here q is left arbitrary but always understood to be larger than g). The spectral parameters of this matrix are

$$F(\mathrm{Q}^k) = f(k), \quad G(\mathrm{Q}^k) = g(k),$$

and the spectral function is

$$s_k = g(k)f(k)f(k+1)\ldots f(k+g-2).$$

Clearly the Hofstadter Hamiltonian (4.3), which rewrites as $H = (1-u)v + v^{1-2}(1-u^{-1})$, is a particular case of (4.7) with $g = 2$ and $F(u) = 1-u$, $G(u) = 1-u^{-1}$.

5. Chiral walks on the triangular lattice

Let us illustrate this mechanism in the case of $g = 3$ exclusion with the specific example of chiral walks on a triangular lattice. The three hopping operators U, V and $W = \mathrm{Q}U^{-1}V^{-1}$ described in Figure 3 are such that $VU = \mathrm{Q}^2 UV$.

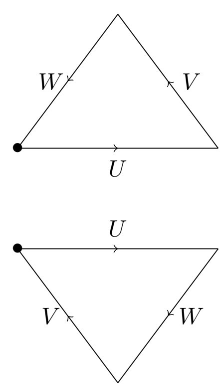
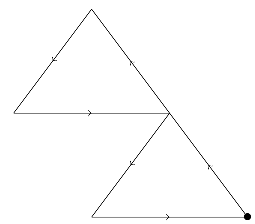


Figure 3. The three hopping operators U, V and W on the triangular lattice.

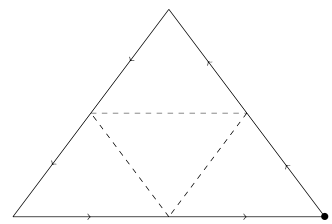
The triangular lattice Hamiltonian is

$$H = U + V + W.$$

It generates walks composed of triangles either pointing up and winding in the counterclockwise direction, or pointing down and winding in the negative direction. In this sense, the walks are chiral. The factor Q in the definition of W and the factor Q^2 in the commutation of U, V are chosen so that up-pointing (positive) triangles are assigned area $+1$. Figure 4 depicts some examples of chiral walks on the triangular lattice.



$UWUWV^2$



$U^2W^2V^2$

FIGURE 4. Examples of closed chiral walks on the triangular lattice.

To bring H to the exclusion form (4.7) one chooses the representation $U = -\mathrm{i}\,u\,v$ and $V = \mathrm{i}\,u^{-1}\,v$, with u and v as before, in which case H rewrites as

$$H = \mathrm{i}(-u + u^{-1})v + v^{-2}.$$

In this form, H is indeed a Hamiltonian of the type (4.7) for $g = 3$ exclusion, with $F(u) = \mathrm{i}(-u + u^{-1})$, $G(u) = 1$, spectral parameters

$$f(k) = -\mathrm{i}(\mathrm{Q}^k - \frac{1}{\mathrm{Q}^k}), \quad g(k) = 1,$$

spectral function

$$s_k = g(k)f(k)f(k+1) = 4\sin(2\pi pk/q)\sin\Big(2\pi p(k+1)/q\Big), \tag{5.1}$$

and matrix

$$I_q - zH_q = \begin{pmatrix} 1 & \mathrm{i}(\mathrm{Q}-\frac{1}{\mathrm{Q}})z & 0 & 0 & \cdots & 0 & -z & 0 \\ 0 & 1 & \mathrm{i}(\mathrm{Q}^2-\frac{1}{\mathrm{Q}^2})z & 0 & \cdots & 0 & 0 & -z \\ -z & 0 & 1 & \mathrm{i}(\mathrm{Q}^3-\frac{1}{\mathrm{Q}^3})z & \cdots & 0 & 0 & 0 \\ 0 & -z & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \mathrm{i}(\mathrm{Q}^{q-2}-\frac{1}{\mathrm{Q}^{q-2}})z & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \mathrm{i}(\mathrm{Q}^{q-1}-\frac{1}{\mathrm{Q}^{q-1}})z \\ 0 & 0 & 0 & 0 & \cdots & -z & 0 & 1 \end{pmatrix},$$

which is of the type (4.6) with a vanishing bottom-left entry. The non-Hermiticity of the triangular Hamiltonian, and thus of $I_q - zH_q$, is a consequence of the chiral nature of the walks.

The triangular signed area enumeration follows [14], yielding an expression similar to Theorem 3.1 with the trigonometric single sums appearing in (4.2) involving the triangular spectral function (5.1) and the sum done over all 3-compositions of the length of the triangular walks.

6. Conclusion

In conclusion, we have shown how tools from quantum and statistical physics allow for an explicit enumeration of closed walks of fixed length and signed area on planar lattices.

The enumeration formulae rely on an explicit sum over compositions, and their number of terms grows quickly with the length of the walk (although much less quickly than a brute-force counting formula). It would certainly be rewarding to rewrite it as a sum with a smaller number of terms. The use of symmetry on the lattice or alternative ways to write the generator of walks (Hamiltonian) may offer promise towards this goal. We leave this issue as well as other questions of interest to the lattice walk combinatorics community.

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LEFT-TO-RIGHT MAXIMA IN DYCK PATHS

AUBREY BLECHER[1] AND ARNOLD KNOPFMACHER[2]

[1]University of the Witwatersrand, Johannesburg, South Africa;
https://independent.academia.edu/AubreyBlecher

[2]University of the Witwatersrand, Johannesburg, South Africa;
https://scholar.google.com/citations?user=v8k5QLUAAAAJ

Abstract. In a Dyck path a peak which is (weakly) higher than all the preceding peaks is called a strict (weak) left-to-right maximum. We obtain explicit generating functions for both weak and strict left-to-right maxima in Dyck paths. The proofs of the associated asymptotics make use of analytic techniques such as Mellin transforms, singularity analysis and formal residue calculus.

Keywords: Dyck paths, generating functions, asymptotics, left-to-right maxima.

1. General introduction

A Dyck path is a lattice path in the first quadrant, that starts at the origin (0,0) with an up step ($u = (1, 1)$) and thereafter only up and down ($d = (1, -1)$) steps are allowed under the conditions that it may not go below the x-axis and that it may terminate only if the end point is on the x-axis. A Dyck path with n up steps must end at the point $(2n, 0)$; see the definition in [16]. Such a Dyck path is said to have length $2n$. For a detailed study of properties of Dyck paths see [7]. For further recent work on Dyck paths; see [1–4, 6, 9, 15].

Given an arbitrary Dyck path, we mean by a *strict left-to-right maximum*, any peak (successive pair of the form ud) in the Dyck path which is greater than the height all peaks to its left. A *weak left-to-right maximum* is a peak which is greater than or equal to the height of all peaks to its left. From here on, by left-to-right maxima we mean strict left-to-right maxima unless otherwise stated.

A standard combinatorial problem is the accounting for the number of left-to-right maxima in combinatorial structures such as permutations and words over a fixed alphabet. In this paper we focus on obtaining a generating function for the number of left-to-right maxima in Dyck paths. This is a bivariate generating function which tracks the number of up steps by z and the number of left-to-right maxima by x. We also obtain a generating function for the total number of left-to-right maxima in Dyck paths with n up steps.

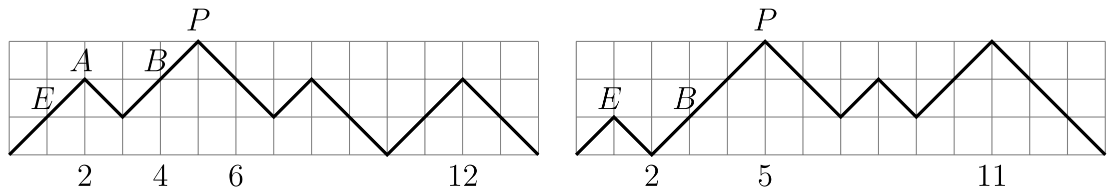

Figure 1. Two Dyck paths of length 14 and height 3

As an introduction to the method we will use for the construction of the first generating function above, here follows a sketch (Figure 1) of two Dyck paths of height 3. The left-to-right maxima are marked in the left case by A and P and in the right case by E and P. P also marks the first maximum height attained by the Dyck paths. We begin at the origin with a u step tracked in the generating function by z which leaves us at the point E. This single up step is followed by a possibly empty upside-down Dyck path of maximum height 1.

In the left example in Figure 1, this part is indeed empty (and therefore not requiring x) but not in the right example where the path between E and B is an upside-down Dyck path of height 1 which gives rise to a left-to-right maximum thus requiring an x tracker. Then we have another single u step and we proceed recursively in this way leaving us eventually at the next left-to-right maximum which is point A in the left example and P in the right. In the left example, right of A is again a possibly empty upside-down Dyck path, this time of maximum height 2 where the non empty case is tracked again by x. We are referring to the path between A and B which is actually of height 1. Once P is reached, it is followed by the rest of the path which is conceived as a *right to left portion* of a Dyck path. In the section dealing with this, the generating function for these latter Dyck paths ending at height r will be given and used, as will the generating function for Dyck paths of a fixed height h, which is used as indicated above for the possibly empty upside-down Dyck paths that occur sequentially before the point P is attained.

2. Left-to-right maxima in Dyck paths

We start this section by referring to the paper [14] by Prodinger on the first sojourn in Dyck paths. Using the notation from [14], we let $C(h)$ be the number of paths of height $\leq h$ with steps which follow all rules of Dyck paths except that they terminate at height h, and we let $A(h)$ be the number of Dyck paths of height $\leq h$ (which by definition end at height zero). It is shown in [14] that

$$C(h) := \frac{z^h\sqrt{1-4z^2}}{{\lambda_1}^{h+2} - {\lambda_2}^{h+2}}$$

and

$$A(h) := \frac{{\lambda_1}^{h+1} - {\lambda_2}^{h+1}}{{\lambda_1}^{h+2} - {\lambda_2}^{h+2}}, \tag{2.1}$$

where λ_1 and λ_2, are given by

$$\lambda_1 = \frac{1+\sqrt{1-4z^2}}{2};\ \lambda_2 = \frac{1-\sqrt{1-4z^2}}{2}.$$

As explained in the introductory section, we consider a sequence of possibly empty Dyck paths of height $\leq h$ for $h = 1, 2, \ldots$. At the end of each path in the sequence, we have a single up step that leads to the next left-to-right maximum and eventually to the first overall maximum of the entire Dyck path. We let x count the number of left-to-right maxima attained by the Dyck path. This leads to our first theorem:

Theorem 2.1. *The generating function for the number of left-to-right maxima tracked by x, for Dyck paths of maximum height r and length tracked by z is*

$$F(x,z,r) := z^r x C(r) \prod_{h=1}^{r-1} (1 + x(A(h) - 1)).$$

So, the total number of left-to-right maxima for Dyck paths of fixed height r is found by differentiating the above function with respect to x and setting $x = 1$. The derivative at this point is given by

$$\begin{aligned}
\frac{\partial}{\partial x} F(x,z,r)\Big|_{x=1} &= z^r\ C(r) \prod_{h=1}^{r-1} A(h) + z^r C(r) \prod_{h=1}^{r-1} A(h) \sum_{i=1}^{r-1} \frac{A(i)-1}{A(i)} \\
&= z^r\ C(r) \prod_{h=1}^{r-1} A(h) \left(1 + \sum_{i=1}^{r-1} \frac{A(i)-1}{A(i)}\right) \\
&= z^r C(r) \prod_{h=1}^{r-1} A(h) \left(r - \sum_{i=1}^{r-1} \frac{1}{A(i)}\right) \qquad (2.2)
\end{aligned}$$

Note that $z^r C(r) \prod_{h=1}^{r-1} A(h)$ telescopes to become

$$\frac{z^{2r}\left(1 - 4z^2\right)}{\left(-\lambda_2^{1+r} + \lambda_1^{1+r}\right)\left(-\lambda_2^{2+r} + \lambda_1^{2+r}\right)}$$

but the full generating function becomes very complicated as a function of z.

To simplify this generating function, we substitute

$$z^2 = \frac{u}{(1+u)^2}$$

in (2.2) which implies

$$\lambda_1 = \frac{1}{1+u};\ \lambda_2 = \frac{u}{1+u};\ C(r) = \frac{(1-u)(1+u)^{1+r} z^r}{1 - u^{2+r}};\ A(r) = \frac{(1+u)\left(1 - u^{1+r}\right)}{1 - u^{2+r}}$$

and obtain

$$T(r) := \frac{\partial}{\partial x} F(x,z,r)\Big|_{x=1} = \frac{(1-u)^2 u^r (1+u)}{\left(1 - u^{1+r}\right)\left(1 - u^{2+r}\right)} \left(r - \sum_{i=1}^{r-1} \frac{1 - u^{2+i}}{(1+u)\left(1 - u^{1+i}\right)}\right).$$

The full generating function for the total number of left-to-right maxima in all Dyck paths of length n is

$$\mathrm{Tot}(u) := \sum_{r=1}^{\infty} T(r).$$

Consequently, we have the following proposition:

Proposition 2.2. *The generating function* Tot(u) *for the total number of left-to-right maxima in Dyck paths of length n tracked by z is given by (using $z^2 = \frac{u}{(1+u)^2}$):*

$$\operatorname{Tot}(u) = \sum_{r=1}^{\infty} \frac{(1-u)^2 u^r (1+u)}{(1-u^{1+r})(1-u^{2+r})} \left(r - \sum_{i=1}^{r-1} \frac{1-u^{2+i}}{(1+u)(1-u^{1+i})} \right). \tag{2.3}$$

In order to obtain the series expansion for this, we use the equivalent inverse substitution for u, namely

$$u = \frac{1 - 2z^2 - \sqrt{1-4z^2}}{2z^2}, \tag{2.4}$$

and obtain in terms of z,

$$\operatorname{Tot}(u) = z^2 + 2z^4 + 6z^6 + \mathbf{19z^8} + 63z^{10} + 216z^{12} + 758z^{14} + 2705z^{16} + 9777z^{18} + 35698z^{20} + O(z^{21}).$$

We illustrate the bold term of the series by means of the black dots in Figure 2.

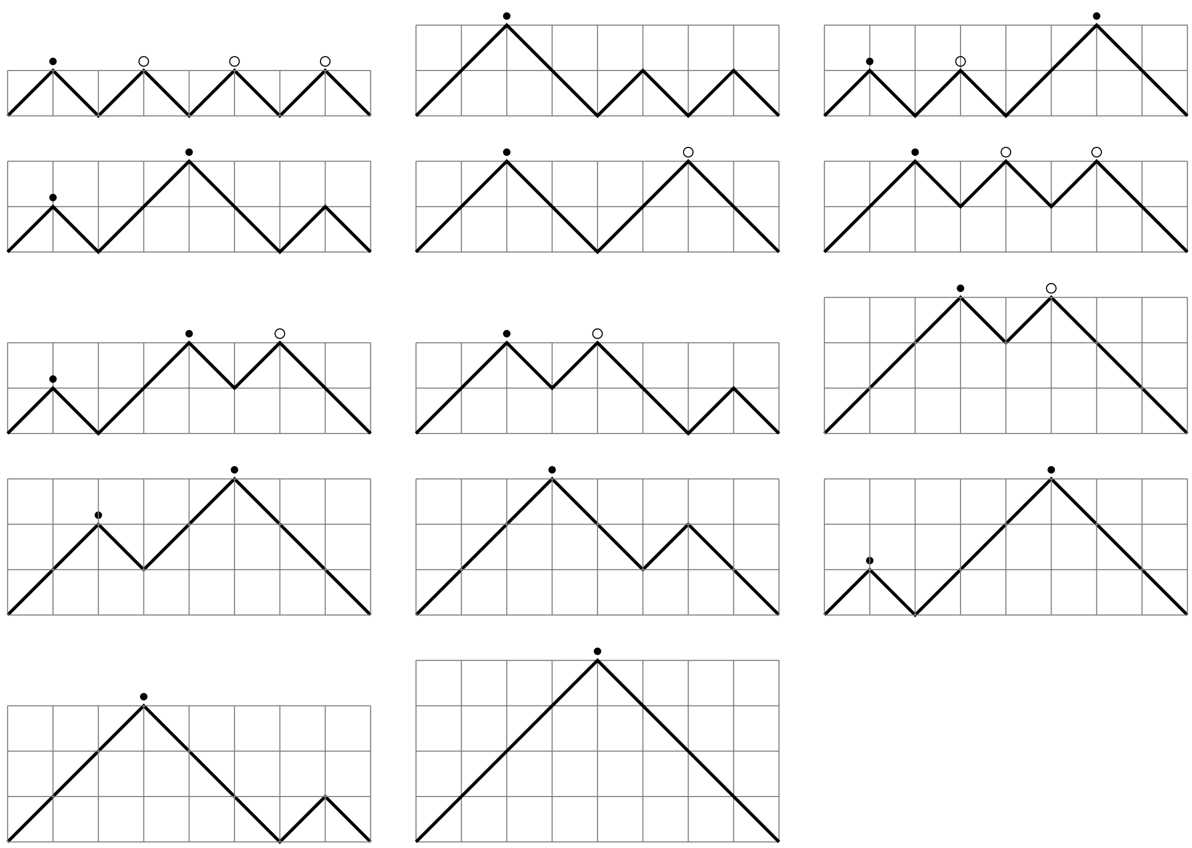

FIGURE 2. All 14 Dyck paths of length 8: they have 19 strict left-to-right maxima (indicated by black dots) and 10 weak left-to-right maxima (indicated by circles).

The type of series expansion for $\mathrm{Tot}(u)$ in Proposition 2.2 involves what is called Lambert series. There are currently no computer algebra packages that can automatically simplify expressions like Equation (2.3). Instead, it is therefore necessary to make the following lengthy calculations in order to derive Theorem 2.3.

To simplify Equation (2.3) we swap the order of the summations in the double sum, and thereafter use partial fractions on the r-indexed sum (which then telescopes as in line (2.5)) to obtain

$$\begin{aligned}
&\sum_{r=1}^{\infty} \frac{(1-u)^2 u^r (1+u)}{(1-u^{1+r})(1-u^{2+r})} \sum_{i=1}^{r-1} \frac{1-u^{2+i}}{(1+u)(1-u^{1+i})} \\
&= (1-u)^2 \sum_{i=1}^{\infty} \frac{1-u^{2+i}}{(1-u^{1+i})} \sum_{r=i+1}^{\infty} \frac{u^r}{(1-u^{1+r})(1-u^{2+r})} \\
&= (1-u)^2 \sum_{i=1}^{\infty} \frac{1-u^{2+i}}{(1-u^{1+i})} \frac{u^{1+i}}{(1-u)(1-u^{2+i})}. \qquad (2.5)
\end{aligned}$$

Now changing the index of summation from i to r,

$$(1-u) \sum_{i=1}^{\infty} \frac{1-u^{2+i}}{(1-u^{1+i})} \frac{u^{1+i}}{(1-u^{2+i})} = (1-u) \sum_{r=1}^{\infty} \frac{u^{1+r}}{(1-u^{1+r})}.$$

Altogether,

$$\begin{aligned}
\mathrm{Tot}(u) &= \sum_{r=1}^{\infty} \frac{(1-u)^2 u^r (1+u) r}{(1-u^{1+r})(1-u^{2+r})} - (1-u) \sum_{r=1}^{\infty} \frac{u^{1+r}}{(1-u^{1+r})} \\
&= \sum_{r=1}^{\infty} \frac{(1-u) u^r \left(r - u - r u^2 + u^{3+r}\right)}{(1-u^{1+r})(1-u^{2+r})} \\
&= \sum_{r=1}^{\infty} \frac{r u^r - u^{1+r} - r u^{1+r} + u^{2+r} - r u^{2+r} + r u^{3+r} + u^{3+2r} - u^{4+2r}}{(1-u^{1+r})(1-u^{2+r})}.
\end{aligned}$$

Drop the first term ru^r in the numerator above and apply partial fractions to the rest of the summand which simplifies to

$$1 - u + \frac{-1 - r + 2u - ru - u^2 + ru^2}{(1-u)(1-u^{1+r})} + \frac{r + ru - ru^2}{(1-u)(1-u^{2+r})}.$$

The separated first term with numerator ru^r after partial fractions leads to

$$\frac{ru^r}{(1-u)(1-u^{1+r})} - \frac{ru^{r+1}}{(1-u)(1-u^{2+r})}.$$

Altogether,

$$\begin{aligned}
\mathrm{Tot}(u) = &\sum_{r=1}^{\infty} \left(1 - u + \frac{-1 - r + 2u - ru - u^2 + ru^2}{(1-u)(1-u^{1+r})} + \frac{r + ru - ru^2}{(1-u)(1-u^{2+r})} \right) \\
&+ \sum_{r=1}^{\infty} \frac{ru^r}{(1-u)(1-u^{1+r})} - \sum_{r=1}^{\infty} \frac{ru^{r+1}}{(1-u)(1-u^{2+r})}.
\end{aligned}$$

To facilitate the evaluation of the infinite sums, we define a new function (where ∞ is replaced temporarily by finite M in $\mathrm{Tot}(u)$), namely:

$$\begin{aligned}\mathrm{Tot2}(u) &:= \sum_{r=1}^{M}\left(1-u+\frac{-1-r+2u-ru-u^2+ru^2}{(1-u)\left(1-u^{1+r}\right)}+\frac{r+ru-ru^2}{(1-u)\left(1-u^{2+r}\right)}\right)\\ &\quad+\sum_{r=1}^{M}\frac{ru^r}{(1-u)\left(1-u^{1+r}\right)}-\sum_{r=1}^{M}\frac{ru^{r+1}}{(1-u)\left(1-u^{2+r}\right)}.\end{aligned}$$

We now separate this into disjoint sums and shift the index of summation in the third and last sums:

$$\begin{aligned}\mathrm{Tot2}(u) &= \sum_{r=1}^{M}(1-u)+\sum_{r=1}^{M}\frac{-1-r+2u-ru-u^2+ru^2}{(1-u)\left(1-u^{1+r}\right)}+\sum_{r=2}^{M+1}\frac{(r-1)\left(1+u-u^2\right)}{(1-u)(1-u^{1+r})}\\ &\quad+\sum_{r=1}^{M}\frac{ru^r}{(1-u)\left(1-u^{1+r}\right)}-\sum_{r=2}^{M+1}\frac{(r-1)u^r}{(1-u)\left(1-u^{1+r}\right)}\\ &=\sum_{r=1}^{M}(1-u)+\frac{-2+u}{(1-u)\left(1-u^2\right)}+\sum_{r=2}^{M}\frac{-1-r+2u-ru-u^2+ru^2}{(1-u)\left(1-u^{1+r}\right)}\\ &\quad+\sum_{r=2}^{M}\frac{(r-1)\left(1+u-u^2\right)}{(1-u)\left(1-u^{1+r}\right)}+\frac{M\left(1+u-u^2\right)}{(1-u)\left(1-u^{2+M}\right)}+\frac{u}{(1-u)\left(1-u^2\right)}\\ &\quad+\sum_{r=2}^{M}\frac{ru^r}{(1-u)\left(1-u^{1+r}\right)}-\sum_{r=2}^{M}\frac{(r-1)u^r}{(1-u)\left(1-u^{1+r}\right)}-\frac{Mu^{1+M}}{(1-u)\left(1-u^{2+M}\right)}. \qquad (2.6)\end{aligned}$$

We combine the terms in the sums from r equals 2 to M in (2.6) to get

$$\frac{-2+u+u^r}{(1-u)\left(1-u^{1+r}\right)}.$$

Then we simplify the rest to get

$$\begin{aligned}\mathrm{Tot2}(u) &= \sum_{r=2}^{M}\frac{-2+u+u^r}{(1-u)\left(1-u^{1+r}\right)}-\frac{2}{1-u^2}\\ &\quad-\frac{M\left(-2+u+u^{1+M}+u^{2+M}-2u^{3+M}+u^{4+M}\right)}{(1-u)\left(1-u^{2+M}\right)}.\end{aligned}$$

Note that $\mathrm{Tot2}(u)$ and $\mathrm{Tot}(u)$ match at least for terms up to $\left[u^M\right]$. Since for the present we are only interested in the terms up to $\left[u^M\right]$, we may set all higher power terms equal to zero, to produce

$$\mathrm{Tot2b}(u)=\sum_{r=2}^{M}\frac{-2+u+u^r}{(1-u)\left(1-u^{1+r}\right)}-\frac{2}{1-u^2}+M\frac{(2-u)}{(1-u)}.$$

Noting that $M = 1 + \sum_{r=2}^{M} 1$,

$$\begin{aligned}\mathrm{Tot2b}(u) &= \sum_{r=2}^{M} \frac{-2+u+u^r}{(1-u)\,(1-u^{1+r})} - \frac{2}{1-u^2} + \frac{(2-u)}{(1-u)} + \sum_{r=2}^{M} \frac{(2-u)}{(1-u)} \\ &= \sum_{r=2}^{M} \frac{-2+u+u^r}{(1-u)\,(1-u^{1+r})} + \frac{u}{1+u} + \sum_{r=2}^{M} \frac{(2-u)}{(1-u)}.\end{aligned}$$

Combine the summands in $\sum_{r=2}^{M}$. We may now allow $M \to \infty$ to finally obtain the simplified generating function as per the next theorem:

Theorem 2.3. *The simplified generating function for the total number of left-to-right maxima in Dyck paths is*

$$\mathrm{Tot}(u) = \sum_{r=1}^{\infty} \frac{(1-u)u^r}{1-u^{1+r}}. \tag{2.7}$$

2.1. **Formula for total number of left-to-right maxima.** In this section, we will obtain an exact formula for the total number of left-to-right maxima in terms of a well-known arithmetic function, namely the divisor function $d(r)$. Compare with [5]. Note that

$$\sum_{r=1}^{\infty} \frac{u^r}{1-u^r} = \sum_{r=1}^{\infty} d(r)\,u^r.$$

To read off coefficients from equation (2.7), we observe that for any formal power series $f(z)$

$$[z^{2n}]f(z) = [u^n](1-u)(1+u)^{2n-1}f(z(u)).$$

This can be justified by using formal residue calculus; see for example [12]. Therefore

$$\begin{aligned}[z^{2n}]\,\mathrm{Tot}(z) &= [u^n](1-u)(1+u)^{2n-1} \sum_{r=1}^{\infty} \frac{(1-u)u^r}{1-u^{1+r}} \\ &= [u^n](1-u)(1+u)^{2n-1} \sum_{r=1}^{\infty} (d(r+1)-d(r))u^r \\ &= \sum_{r=1}^{n} (d(r+1)-d(r)) \left(\binom{2n-1}{n-r} - \binom{2n-1}{n-r-1} \right).\end{aligned}$$

Thus we have shown:

Theorem 2.4. *The total number of left-to-right maxima in Dyck paths of semi-length n is given by*

$$\sum_{r=1}^{n} (d(r+1)-d(r)) \left(\binom{2n-1}{n-r} - \binom{2n-1}{n-r-1} \right).$$

3. Asymptotics for strict left-to-right maxima

In this section we find the asymptotic expression for the total number of strict left-to-right maxima in Dyck paths. We will follow the approach used to study the height of planted plane trees by Prodinger in [12]. For related asymptotic calculations concerning the height of trees and lattice paths; see [10, 11, 13] and the seminal article by de Bruijn, Knuth and Rice [5].

First, we extract coefficients of z^n in $\mathrm{Tot}(u)$. That is we find

$$[z^n]\frac{1-u}{u}\sum_{r=2}^{\infty}\frac{u^r}{1-u^r}.$$

When u is in terms of z^2, by (2.4) the function $\mathrm{Tot}(u)$ has its dominant singularity at $z = 1/2$ which is mapped to $u = 1$. To study this further we set $u = e^{-t}$ and let $t \to 0$. Thus

$$\frac{1-u}{u} = e^t(1-e^{-t}) = t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots . \tag{3.1}$$

To estimate the harmonic sum $f_1(t) := \sum_{r=2}^{\infty}\frac{e^{-rt}}{1-e^{-rt}}$ as $t \to 0$, we take the Mellin transform of $f_1(t)$, see [8], which is $f_1^*(s) := \int_0^\infty f_1(t)t^{s-1}\,dt$. Thus

$$f_1^*(s) = \Gamma(s)\zeta(s)(\zeta(s)-1), \text{ for } \Re(s) > 1.$$

By using the Mellin inversion formula, we have $f_1(t) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f_1^*(s)\,t^{-s}\,ds$ (again see [8]). By computing residues this yields

$$f_1(t) \sim \frac{-1+\gamma-\log(t)}{t} + \frac{3}{4} - \frac{13t}{144} + \cdots , \tag{3.2}$$

where γ is Euler's constant.

Let

$$g_1(t) := e^t(1-e^{-t})\, f_1(t).$$

From (3.1) and (3.2)

$$g_1(t) \sim -\log(t) - 1 + \gamma + \left(\frac{3}{4} + \frac{1}{2}(-1+\gamma-\log(t))\right)t + \cdots .$$

Let $y = \sqrt{1-4z^2}$ and writing $e^{-t} = u = \frac{1-y}{1+y}$, we find $t = -\log\frac{1-y}{1+y} = 2y + \frac{2y^3}{3} + \cdots$. In terms of the y variable, we therefore need to compute $g_1(2y + \frac{2y^3}{3} + \cdots)$.

$$g_1\left(2y + \frac{2y^3}{3} + \cdots\right) \sim (-1+\gamma-\log(2)-\log(y)) + \frac{1}{2}(1+2\gamma-2\log(2)-2\log(y))y - \frac{y^2}{3} + \cdots .$$

Replacing y by $\sqrt{1-4z^2}$ gives

$$-1+\gamma-\log(2) - \frac{1}{2}\log\left(1-4z^2\right) + \frac{1}{2}\left(1+2\gamma-2\log(2)-\log\left(1-4z^2\right)\right)\sqrt{1-4z^2} + \cdots .$$

To use singularity analysis, see [8], it is convenient to put $z^2 = x$, then we find the coefficient of x^n in the above expression as $n \to \infty$. It is asymptotically equal to

$$2^{2n}\left(\frac{1}{2n} - \frac{\log(n)}{4\sqrt{\pi}n^{3/2}} + \frac{1-3\gamma}{4n^{3/2}\sqrt{\pi}} + \cdots\right). \tag{3.3}$$

To obtain the mean value we must divide by the total number of Dyck paths of semi-length n, i.e., as $n \to \infty$

$$\frac{1}{n+1}\binom{2n}{n} = 2^{2n}\left(\frac{1}{n^{3/2}\sqrt{\pi}} - \frac{9}{8\ n^{5/2}\sqrt{\pi}} + \frac{145}{128\ n^{7/2}\sqrt{\pi}}\right) + \cdots . \tag{3.4}$$

Hence, dividing (3.3) by (3.4) yields

Theorem 3.1. *The average number of strong left-to-right maxima in Dyck paths of semi-length n, as $n \to \infty$ is*

$$\frac{\sqrt{\pi n}}{2} - \frac{\log(n)}{4} + \frac{1}{4}(1-3\gamma) + O(n^{-1/2}).$$

Remark 3.2. The asymptotic formula of Theorem 3.1 when $n = 200$ yields 11.0257 for the average number of strong left-to-right maxima. Using the exact formula of Theorem 2.4 divided by the Catalan number for $n = 200$ yields 11.0503 which is indeed a very good match.

Remark 3.3. The number of strong left-to-right maxima is bounded above by the height of the path, which is known to be $\sim \sqrt{\pi n}$ as $n \to \infty$, (see, e.g., [12]). We see that asymptotically the average number is half of the height.

4. Weak left-to-right maxima in Dyck paths

For this question we first need a generating function for Dyck paths of height $\leq h$ which have only a single return to the x axis. So using the formula above from (2.1), we obtain the generating function for these where $h \geq 1$ as

$$D(h,z) = z^2 A(h-1).$$

Now in order to construct the generating function $E(h,x,z)$ for the number of times a Dyck path of height $\leq h$ and length n tracked by z, returns to 0 where the latter is tracked by a variable x in the generating function, we construct a sequence of such Dyck paths where each term in the generating function for this sequence is multiplied by x. Thus we obtain

$$E(h,x,z) = \frac{1}{1 - xD(h,z)}.$$

We now reiterate the construction in Theorem 2.1 to obtain the following theorem.

Theorem 4.1. *The generating function for the number of weak left-to-right maxima, tracked by x, for Dyck paths of maximum height r and length tracked by z is*

$$F(x,z,r) := z^{r+1}xC(r-1)\prod_{h=1}^{r} E(h,x,z). \tag{4.1}$$

To obtain the generating function for the total number of weak left-to-right maxima, we once again differentiate (4.1) with respect to x and evaluate this at $x = 1$. We obtain

Theorem 4.2. *The generating function for the total number of weak left-to-right maxima for Dyck paths of length n tracked by z is*

$$\mathrm{WTot}(u) := \sum_{r=1}^{\infty} \frac{(1-u)u^r\left(1-u^2\right)}{\left(1-u^{1+r}\right)\left(1-u^{2+r}\right)} \left(1-r+(1+u)\sum_{i=1}^{r} \frac{1-u^{1+i}}{1-u^{2+i}}\right),$$

where $z^2 = \frac{u}{(1+u)^2}$.

Proof. The derivative of (4.1) is

$$\frac{\partial}{\partial x} F(x,z,r)\Big|_{x=1} = z^{r+1} C(r-1) \prod_{h=1}^{r} E(h,1,z) \left(1 + \sum_{i=1}^{r} \frac{D(i,z)}{1-D(i,z)}\right).$$

Putting $z^2 = \frac{u}{(1+u)^2}$ in the formula above we obtain

$$z^{r+1} C(r-1) \prod_{h=1}^{r} \frac{1}{1-z^2 A(h-1)} = \frac{(1-u)u^r\left(1-u^2\right)}{\left(1-u^{1+r}\right)\left(1-u^{2+r}\right)},$$

while the remaining bracketed part becomes

$$1-r+(1+u)\sum_{i=1}^{r} \frac{1-u^{1+i}}{1-u^{2+i}}.$$

□

Now, we simplify Theorem 4.2. The double sum becomes

$$\left(1-u^2\right)^2 \sum_{i=1}^{\infty} \frac{1-u^{1+i}}{1-u^{2+i}} \sum_{r=i}^{\infty} \frac{u^r}{\left(1-u^{1+r}\right)\left(1-u^{2+r}\right)}.$$

We use partial fractions on the r-sum and then the double sum telescopes to

$$\frac{\left(1-u^2\right)^2}{(1-u)u} \sum_{i=1}^{\infty} \frac{u^{i+1}}{1-u^{i+2}}.$$

This is then combined with the single sum which simplifies to

$$\sum_{r=1}^{\infty} \left(\frac{(1-u)u^r\left(1-u^2\right)(1-r)}{\left(1-u^{1+r}\right)\left(1-u^{2+r}\right)} + \frac{\left(1-u^2\right)^2 u^{1+r}}{(1-u)u\left(1-u^{2+r}\right)} \right). \tag{4.2}$$

In order to further simplify (4.2) we replace ∞ by finite M and then apply partial fractions to the summand of the first term which splits up as

$$\frac{(-1+r)(1-u)(1+u)}{u\left(1-u^{1+r}\right)} - \frac{(-1+r+1)(1-u)(1+u)}{u\left(1-u^{2+r}\right)} - \frac{(1-u)(1+u)}{u\left(1-u^{2+r}\right)}.$$

This is telescoping and simplifies to

$$-\frac{(-1+M+1)(1-u)(1+u)}{u\left(1-u^{2+M}\right)} + \sum_{r=1}^{M} \left(\frac{\left(1-u^2\right)^2 u^{1+r}}{(1-u)u\left(1-u^{2+r}\right)} - \frac{(1-u)(1+u)}{u\left(1-u^{2+r}\right)} \right).$$

Now, replace M by $\sum_{r=1}^{M} 1$. Then, letting M tend to ∞, and finally combining all summands, we obtain

Theorem 4.3. *The simplified generating function for the total number of weak left-to-right maxima for Dyck paths of length n tracked by z is*

$$\mathrm{WTot}(u) = \sum_{r=1}^{\infty} \frac{(1-u^2)\,u^r}{1-u^{2+r}}.$$

This has series expansion

$$z^2+3z^4+9z^6+\mathbf{29z^8}+98z^{10}+341z^{12}+1210z^{14}+4356z^{16}+15860z^{18}+58276z^{20}+O\left(z^{21}\right).$$

This is illustrated in Figure 2, where the dots and circles mark all 29 of the weak left-to-right maxima in Dyck paths of length 8.

4.1. **Formula for total number of weak left-to-right maxima.** In this section, we again obtain an exact formula for the total number of left-to-right maxima in terms of the divisor function $d(r)$. To read off coefficients from Theorem 4.3, as before

$$[z^{2n}]f(z) = [u^n](1-u)(1+u)^{2n-1}f(z(u)).$$

Therefore

$$\begin{aligned}[z^{2n}]\,\mathrm{WTot}(z) &= [u^n](1-u)(1+u)^{2n-1}\sum_{r=1}^{\infty}\frac{(1-u^2)u^r}{1-u^{2+r}}\\ &= [u^n](1-u)(1+u)^{2n-1}\sum_{r=1}^{\infty}(d(r+2)-d(r))u^r.\end{aligned}$$

We thus get the following theorem.

Theorem 4.4. *The total number of weak left-to-right maxima in Dyck paths of semi-length n is given by*

$$\sum_{r=1}^{n}(d(r+2)-d(r))\left(\binom{2n-1}{n-r}-\binom{2n-1}{n-r-1}\right).$$

5. Asymptotics for weak left-to-right maxima

To find an asymptotic expression for $\mathrm{WTot}(u)$, we reiterate the approach in Section 3. This yields

Theorem 5.1. *The average number of weak left-to-right maxima in Dyck paths of semi-length n, as $n \to \infty$ is*

$$\sqrt{\pi n} - \log(n) + \frac{1}{2}(5-6\gamma) + O(n^{-1/2}).$$

Remark 5.2. The asymptotic formula of Theorem 5.1 when $n = 200$ yields 20.536 for the average number of weak left-to-right maxima. Using the exact formula of Theorem 4.4 divided by the Catalan number for $n = 200$ yields 20.368. Taking larger n improves the accuracy.

6. Open problems

Theorem 2.4 and Theorem 4.4 are very similar to each other with only a slight change in their respective summands; this suggests that there is an underlying combinatorial proof. Also from Theorems 2.3 and 4.3 we obtain

$$\mathrm{WTot}(u) = \frac{1+u}{u}\,\mathrm{Tot}(u) - 1.$$

We think that accounting for these similarities may be an interesting combinatorial problem which we leave to the reader. It might also be possible to derive Theorem 2.3 in a simpler and more direct way instead of using Proposition 2.2.

The statistic left-to-right maximum, 'lrmax' is quite important in permutations due to the fact that it is equidistributed with the 'cycle' statistic and is counted nicely by Stirling numbers of the first kind. One of the main reasons for studying lrmax in permutations is this equidistribution with the number of cycles. Hence in the current setting one should anticipate a counterpart statistic in Dyck path to be equidistributed with lrmax.

With respect to permutations, there are other statistics equidistributed with lrmax. In the Dyck path context one can also introduce such concepts. This may potentially lead to further interesting research.

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RESTRICTED DYCK PATHS ON VALLEYS SEQUENCE

RIGOBERTO FLÓREZ[1], TOUFIK MANSOUR[2], JOSÉ L. RAMÍREZ[3], FABIO A. VELANDIA[4], AND DIEGO VILLAMIZAR[5]

[1]Department of Mathematical Sciences, The Citadel, Charleston, SC, U.S.A.; https://www.rigoflorez.com/

[2]Department of Mathematics, University of Haifa, 3498838 Haifa, Israel; https://math.haifa.ac.il/toufik/

[3]Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia; https://sites.google.com/site/ramirezrjl/

[4]Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia; https://orcid.org/0000-0002-9542-0782

[5]Escuela de Ciencias Exactas e Ingeniería, Universidad Sergio Arboleda, Bogotá, Colombia; https://sites.google.com/view/dvillami/

Abstract. In this paper we study a subfamily of a classic lattice path, the *Dyck paths*, called *restricted* d*-Dyck* paths, in short d-Dyck. A valley of a Dyck path P is a local minimum of P; if the difference between the heights of two consecutive valleys (from left to right) is at least d, we say that P is a restricted d-Dyck path. The *area* of a Dyck path is the sum of the absolute values of y-components of all points in the path. We find the number of peaks and the area of all paths of a given length in the set of d-Dyck paths. We give a bivariate generating function to count the number of the d-Dyck paths with respect to the semi-length and number of peaks. After that, we analyze in detail the case $d = -1$. Among other things, we give both the generating function and a recursive relation for the total area.

Keywords: Dyck path, d-Dyck path, generating function.

1. Introduction

A classic concept, the *Dyck paths*, has been widely studied. Recently, a subfamily of these paths, non-decreasing Dyck paths, has received a certain level of interest. It is because of some statistics are given by linear combinations of Fibonacci numbers and Lucas numbers. In this paper we keep studying a generalization of the non-decreasing Dyck paths. Other generalizations of non-decreasing Dyck paths have been given for Motzkin paths and for Łukasiewicz paths [14, 15].

We now give some definitions that we use in this paper. A Dyck path is a lattice path in the first quadrant of the xy-plane that starts at the origin, ends on the x-axis, and consists of (the same number of) North-East steps $U := (1, 1)$ and South-East steps $D := (1, -1)$. The *semi-length* of a path is the total number of U's that the path has.

A *valley* (*peak*) is a subpath of the form DU (UD) and the *valley vertex* of DU is the lowest point (a local minimum) of DU. The *level* of a valley is the y-component of its valley vertex. Following [16, 17] we define the *valley vertices vector* of a Dyck path P as the vector $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$ formed by all y-coordinates (listed from left to right) of all valley vertices of P.

For a fixed $d \in \mathbb{Z}$, a Dyck path P is called *restricted d-Dyck* or *d-Dyck* (for simplicity), if either P has at most one valley, or if its valley vertex vector ν satisfies that $\nu_{i+1} - \nu_i \geq d$, where $1 \leq i < k$. The set of all d-Dyck paths of semi-length n is denoted $\mathcal{D}_d(n)$, where $r_d(n)$ denotes its cardinality, and the set of all d-Dyck paths is denoted by $\mathcal{D}_d$.

The first well-known example of these paths is the set of 0-Dyck paths; in the literature, see [4, 6, 7, 9, 10, 12], this family is known as non-decreasing Dyck paths. The whole family of Dyck paths can be seen as a limit of d-Dyck and it occurs when $d \to -\infty$. Another example, from Figure 1 we observe that $\nu = (0, 1, 0, 3, 4, 3, 2)$ and that $\nu_{i+1} - \nu_i \geq -1$, for $i = 1, \ldots, 6$, so the figure depicts a (-1)-Dyck path of length 28 (or semi-length 14).

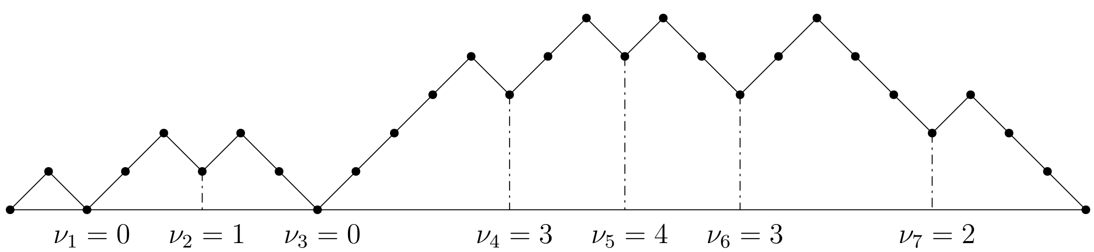

Figure 1. A (-1)-Dyck path of length 28.

The recurrence relations and/or the generating functions for d-Dyck when $d \geq 0$ have different behavior than the case $d < 0$. For example, generating functions accounting for the number of valleys, the number of peaks, and the area, for d-Dyck when $d \geq 0$, are all rational for all variables (see [4, 6, 7, 10, 12, 16, 17]). However, when we analyze in this paper several aspects for $d < 0$ (the number of paths, the area of the paths, and the number of peaks) we find that the generating functions are all algebraic (non-rational).

In this paper we give a bivariate generating function to count the number of paths in $\mathcal{D}_d(n)$, for $d \leq 0$, with respect to the number of peaks and semi-length. We also give a relationship between the total number of d-Dyck paths and the Catalan numbers. Additionally, we give an explicit symbolic expression for the generating function with respect to the semi-length. For the particular case $d = -1$ we give a combinatorial expression and a recursive relation for the total number of paths. We also analyze the asymptotic behavior for the sequence $r_{-1}(n)$.

It is well known that there are many bijections between Dyck paths and other combinatorial objects, we are wondering if there are other bijections between d-Dyck paths for $d < -1$ and other object of combinatorics.

The *area* of a Dyck path P is the sum of the values of y-components of all points in the path. That is, the area of P, denoted by $\texttt{area}(P)$, corresponds to the surface area under P and above of the x-axis. For example, if P is the path in Figure 1, then $\texttt{area}(P) = 70$. We use generating functions and recursive relations to analyze the distribution of the area of all paths in $\mathcal{D}_{-1}(n)$.

The problem of enumerating the area in directed lattice paths, in a general setting, was solved by Banderier and Gittenberger [3], building on the enumerative and asymptotics results from [2], where Dyck, Motzkin, and Łukasiewicz paths are particular cases.

A summary of notation used throughout the paper appears in Table 1 in the appendix.

2. Number of d-Dyck paths and Peaks Statistic

Given a family of lattice paths, a classic question is how many lattice paths are there of certain length, and a second classic question is how many peaks are there depending on the length of the path. These questions have been completely answered, for instance, for Dyck paths [8], d-Dyck paths for $d \geq 0$ [4, 17], and Motzkin paths [20] among others. In this section we give a bivariate generating function according to the semi-length and the number of peaks of the d-Dyck paths with $d < 0$.

Given a d-Dyck path P, we denote the semi-length of P by $\ell(P)$ and denote the number of peaks of P by $\rho(P)$. So, the bivariate generating function to count the number of paths and peaks of d-Dyck paths is defined by

$$L_d(x, y) := \sum_{P \in \mathcal{D}_d} x^{\ell(P)} y^{\rho(P)}.$$

2.1. Some facts known when $d \geq 0$.

These results can be found in [17].

- If $d \geq 0$, then the generating function $F_d(x, y)$ is given by
$$L_d(x, y) = 1 + \frac{xy(1 - 2x + x^2 + xy - x^{d+1}y)}{(1 - x)(1 - 2x + x^2 - x^{d+1}y)}.$$
- If $d \geq 1$,
$$r_d(n) = \sum_{k=0}^{\lfloor \frac{n+d-2}{d} \rfloor} \binom{n - (d-1)(k-1)}{2k}.$$
- If $n > d$, then we have the recursive relation
$$r_d(n) = 2r_d(n-1) - r_d(n-2) + r_d(n-d-1),$$
with the initial values $r_d(n) = \binom{n}{2} + 1$, for $0 \leq n \leq d$.
- Let $p_d(n, k)$ be the number of d-Dyck paths of semi-length n, having exactly k peaks. If $d \geq 0$, then
$$p_d(n, k) = \binom{n + k - d(k-2) - 2}{2(k-1)}.$$
For the whole set of Dyck paths, the number $p_{-\infty}(n, k)$, is given by the Narayana numbers $N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

2.2. Peaks statistic for d a negative integer.

For the remaining part of the paper we consider only the case $d < 0$ and use e to denote $|d|$. A *pyramid* of semi-length $h \geq 1$ is a subpath of the form X^hY^h; it is *maximal*, denote by Δ_h, if it can not be extended to a pyramid $X^{h+1}Y^{h+1}$.

Theorem 2.1. *If d is a negative integer and $e := |d|$, then the generating function $L_e(x, y)$ satisfies the functional equation*

$$L_e(x, y) = xy + xL_e(x, y) + xS_e(x, y)L_e(x, y), \tag{2.1}$$

where $S_e(x)$ satisfies the algebraic equation

$$(1 - xS_e(x, y))^e(y + (1 - y)xS_e(x, y)) - S_e(x, y)(1 - xS_e(x, y))^{e+1} - \frac{x^{e+2}y}{1 - x}S_e(x, y) = 0.$$

Proof. We start this proof by introducing some notation. The set $\mathcal{Q}_{d,i} \subseteq \mathcal{D}_d$ denotes the family of non-empty paths where the last valley is at level i. We consider the generating function

$$Q_i^{(e)}(x, y) := \sum_{P \in \mathcal{Q}_{d,i}} x^{\ell(P)}y^{\rho(P)}.$$

It is convenient to consider the sum over the $Q_i^{(e)}(x, y)$. We also consider the generating function, with respect to the lengths and peaks, that counts the d-Dyck paths that have either no valleys or the last valley is at level less than e. That is,

$$S_e(x, y) = \frac{y}{1 - x} + \sum_{j=0}^{e-1} Q_j^{(e)}(x, y). \tag{2.2}$$

A path P can be uniquely decomposed as either UD, UTD, or $UQDT$ (by considering the first return decomposition), where $T \in \mathcal{D}_d$ and Q is either a pyramid or is a path in $\cup_{i=0}^{e-1} \mathcal{Q}_{d,i}$ (see Figure 2, for a graphical representation of this decomposition). Notice that $\nu_{i+1} - \nu_i \geq d$ and the decomposition $UQDT$ ensures that Q holds as in the former line.

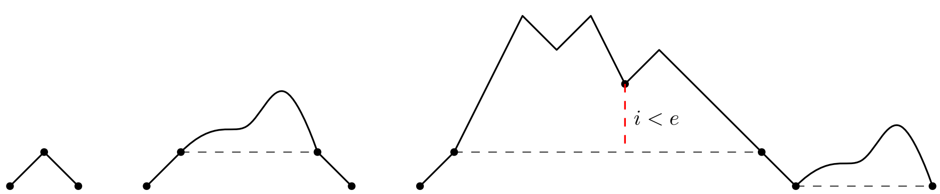

Figure 2. Decomposition of a d-Dyck path.

From the symbolic method we obtain the functional equation

$$L_e(x, y) = xy + xL_e(x, y) + xS_e(x, y)L_e(x, y).$$

Now we are going to obtain a system of equations for the generating functions $Q_i(x, y)$. Let Q be a path in the set $\mathcal{Q}_{d,i}$. If $i = 0$, then the path Q can be decomposed uniquely as either $UQ'D\Delta$ or $UQ'DR$, where Δ is a pyramid, R is a path in $\mathcal{Q}_{d,0}$, and Q' is either a pyramid or $Q' \in \cup_{i=0}^{e-1} \mathcal{Q}_{d,i}$. Therefore, we have the functional equation

$$Q_0^{(e)}(x, y) = xS_e(x, y)\frac{xy}{1 - x} + xS_e(x, y)Q_0^{(e)}(x, y).$$

For $i > 0$, any path Q can be decomposed uniquely in one of these two forms UR_1D or $UQDR_2$, where $R_1 \in \mathcal{Q}_{d,i-1}, R_2 \in \mathcal{Q}_{d,i}$, and Q is either a pyramid or $Q \in \cup_{i=0}^{e-1} \mathcal{Q}_{d,i}$. So, we have the functional equation

$$Q_i^{(e)}(x, y) = xQ_{i-1}^{(e)}(x, y) + xS_e(x, y)Q_i^{(e)}(x, y). \tag{2.3}$$

Summarizing the discussion above, we obtain the system of equations:

$$\begin{cases} Q_0^{(e)}(x,y) & = xS_e(x,y)\frac{xy}{1-x} + xS_e(x,y)Q_0^{(e)}(x,y) \\ Q_1^{(e)}(x,y) & = xQ_0^{(e)}(x,y) + xS_e(x,y)Q_1^{(e)}(x,y) \\ & \vdots \\ Q_i^{(e)}(x,y) & = xQ_{i-1}^{(e)}(x,y) + xS_e(x,y)Q_i^{(e)}(x,y) \\ & \vdots \\ Q_{e-1}^{(e)}(x,y) & = xQ_{e-2}^{(e)}(x,y) + xS_e(x,y)Q_{e-1}^{(e)}(x,y). \end{cases} \tag{2.4}$$

Summing up the equations in (2.4), we obtain that

$$\sum_{j=0}^{e-1} Q_j^{(e)}(x,y) = xS_e(x,y)\left(\sum_{j=0}^{e-1} Q_j^{(e)}(x,y) + \frac{xy}{1-x}\right) + x\sum_{j=0}^{e-2} Q_j^{(e)}(x,y).$$

From this and (2.2) we have

$$\begin{aligned} S_e(x,y) - \frac{y}{1-x} &= x\left(S_e(x,y) - \frac{y}{1-x} - Q_{e-1}^{(e)}(x,y)\right) \\ &\quad + xS_e(x,y)\left(S_e(x,y) - \frac{y}{1-x}\right) + \frac{x^2y}{1-x}S_e(x,y). \end{aligned} \tag{2.5}$$

Iterating (2.3), we have $Q_i^{(e)}(x,y)$, with $i \geq 0$, can be expressed as

$$Q_i^{(e)}(x,y) = \frac{x^{i+2}yS_e(x,y)}{(1-x)(1-xS_e(x,y))^{i+1}}. \tag{2.6}$$

Substituting (2.6) into (2.5) we obtain the desired functional equation. □

Solving (2.5) for $S_e(x,y)$ we have

$$S_e(x,y) = \frac{1-x+xy-\sqrt{1-2x+x^2-2xy-2x^2y+x^2y^2+4x^2Q_{e-1}^{(e)}(x,y)}}{2x}. \tag{2.7}$$

We observe that substituting (2.7) into (2.1), we have

$$\begin{aligned} L_e(x,y) &= \frac{xy}{1-x-xS_e(x,y)} \\ &= \frac{xy}{1-x-\dfrac{1-x+xy-\sqrt{1-2x+x^2-2xy-2x^2y+x^2y^2+4x^2Q_{e-1}^{(e)}(x,y)}}{2}}. \end{aligned}$$

Since $S_e(x,y)$ is a power series and by (2.6), we obtain that $Q_{e-1}^{(e)}(x,y) \to 0$ as $e \to \infty$, where here we assumed that $|x| < 1$ (for details on convergence of generating functions; see [11, p. 731]). Therefore,

$$\lim_{e\to\infty} L_e(x,y) = \frac{1-x-xy-\sqrt{1-2x+x^2-2xy-2x^2y+x^2y^2}}{2x}.$$

This last generating function is the distribution of the Narayana sequence. This corroborates with the fact that the restricted $(-\infty)$-Dyck paths coincide with the non-empty Dyck paths.

Theorem 2.2. *If* $1 \leq k \leq |d| + 3$*, then the* k*-th coefficient of the generating function* $L_e(x, 1)$ *coincides with the Catalan number* $C_k = \frac{1}{k+1}\binom{2k}{k}$.

Proof. We first observe that the shortest Dyck path that contains a forbidden sequence of valleys is $P = U^{e+2}DUD^{e+2}UD$ (clearly, $\ell(P) = e + 4$) with $e = |d|$. Therefore, if $d < 0$, then $r_d(n) = C_n$, for $n = 1, 2, \ldots, |d| + 3$. □

The first few values for the sequence $r_d(n)$, for $d \in \{-1, -2, -3, -4\}$ are

$$\begin{aligned}
\{r_{-1}(n)\}_{n\geq 1} &= \{\mathbf{1},\, \mathbf{2},\, \mathbf{5},\, \mathbf{14},\, 41,\, 123,\, 375,\, 1157,\, 3603, \ldots\},\\
\{r_{-2}(n)\}_{n\geq 1} &= \{\mathbf{1},\, \mathbf{2},\, \mathbf{5},\, \mathbf{14},\, \mathbf{42},\, 131,\, 419,\, 1365,\, 4511, \ldots\},\\
\{r_{-3}(n)\}_{n\geq 1} &= \{\mathbf{1},\, \mathbf{2},\, \mathbf{5},\, \mathbf{14},\, \mathbf{42},\, \mathbf{132},\, 428,\, 1419,\, 4785, \ldots\},\\
\{r_{-4}(n)\}_{n\geq 1} &= \{\mathbf{1},\, \mathbf{2},\, \mathbf{5},\, \mathbf{14},\, \mathbf{42},\, \mathbf{132},\, \mathbf{429},\, 1429,\, 4850, \ldots\}.
\end{aligned}$$

For example, there are 41 (-1)-Dyck paths out of the 42 Dyck paths of length 10. Figure 3 depicts the only Dyck path of length 10 that is not a (-1)-Dyck path.

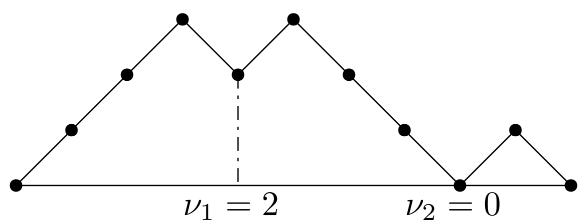

FIGURE 3. The only Dyck path of length 10 that is not a (-1)-Dyck path.

Recall that d is a negative integer and that $e := |d|$. Then by Theorem 2.1, we have

$$(L_e(x,y) + y)^e \left(xL_e^2(x,y) + (xy + x - 1)L_e(x,y) + xy\right) - \frac{x}{1-x}((1-x)L_e(x,y) - xy)(L_e(x,y))^{e+1} = 0.$$

This implies that

$$\sum_{j=2}^{e+1} x\binom{e}{j-2} y^{e+2-j}(L_e(x,y))^j + \sum_{j=1}^{e+1}(xy + x - 1)\binom{e}{j-1} y^{e+1-j}(L_e(x,y))^j + \sum_{j=0}^{e} x\binom{e}{j} y^{e+1-j}(L_e(x,y))^j + \frac{x^2y}{1-x}(L_e(x,y))^{e+1} = 0.$$

Hence, by taking $y = 1$ and collecting powers of $L_e(x, 1)$, we have

$$L_e(x,1) = Z\left(a_0 + \sum_{j=2}^{e+1} a_j(x)(L_e(x,1))^j\right),$$

where $Z = 1$, and

$$\begin{aligned}
a_0 &= \frac{x}{1-(e+2)x},\\
a_j &= \frac{1}{1-(e+2)x}\left(x\binom{e+2}{j} - \binom{e}{j-1}\right), \quad j = 2,3,\dots,e,\\
a_{e+1} &= \frac{(e+2)x(1-x)-1+x(1+x)}{(1-x)(1-(e+2)x)}.
\end{aligned}$$

Hence, by the Lagrange inversion formula, we expand the generating function $L_e(x,1)$ as a power series in Z to obtain

$$L_e(x,1) = \sum_{n\geq 1} \frac{[Z^{n-1}]}{n} \sum_{i_0+i_2+i_3+\cdots+i_{e+1}=n} \frac{n!}{i_0!i_2!\cdots i_{e+1}!} a_0^{i_0} Z^{2i_2+\cdots+(e+1)i_{e+1}} \prod_{j=2}^{e+1} a_j^{i_j},$$

that leads to the following result.

Theorem 2.3. *We have*

$$L_e(x,1) = \sum_{n\geq 1} \frac{\sum_{2i_2+\cdots+(e+1)i_{e+1}=n-1} \binom{n}{i_2,\dots,i_{e+1}} x^{n-i_2-\cdots-i_{e+1}} t^{i_{e+1}} \prod_{j=2}^{e}\left(x\binom{e+2}{j} - \binom{e}{j-1}\right)^{i_j}}{n(1-(e+2)x)^n},$$

where

$$\binom{n}{i_2,\dots,i_{e+1}} = \frac{n!}{i_2!\cdots i_{e+1}!(n-i_2-\cdots-i_{e+1})!} \text{ and } t = \frac{(e+2)x(1-x)-1+x(1+x)}{1-x}.$$

For example, Theorem 2.3 with $e = 2$ gives

$$L_2(x,1) = \sum_{n\geq 1} \frac{\sum_{2i_2+3i_3=n-1} \binom{n}{i_2,i_3} x^{n-i_2-i_3}(6x-2)^{i_2}(\frac{-3x^2+5x-1}{1-x})^{i_3}}{n(1-4x)^n}.$$

Thus,

$$\begin{aligned}
L_2(x,1) &= \frac{x}{1-4x} + \frac{x^2(6x-2)}{(1-4x)^3} + \frac{x^3t}{(1-4x)^4} + \frac{2x^3(6x-2)^2}{(1-4x)^5} + \frac{5x^4t(6x-2)}{(1-4x)^6}\\
&+ \frac{5x^4(6x-2)^3+3x^5t^2}{(1-4x)^7} + \frac{21x^5(6x-2)^2t}{(1-4x)^8} + \frac{28x^6(-2+6x)t^2+14x^5(-2+6x)^4}{(1-4x)^9} + \cdots,
\end{aligned}$$

where $t = (-3x^2+5x-1)/(1-x)$.

3. Some results for the case $d = -1$

In this section we keep analyzing the bivariate generating function given in the previous section for the particular case $d = -1$. For this case, we provide more detailed results. We denote by $\mathcal{Q}$ the set of all non-empty paths in $\mathcal{D}_{-1}$ having at least one valley, where the last valley is at ground level. We denote by $\mathcal{Q}_n$ the subset of $\mathcal{Q}$ formed by all paths of semi-length n and denote by q_n the cardinality of $\mathcal{Q}_n$. For simplicity, when $d = -1$ (or $e = 1$) we use $L(x,y)$ instead of $L_1(x,y)$. As a consequence of Theorem 2.1, taking $d = -1$, we obtain this theorem.

Theorem 3.1. *The bivariate generating function* $L(x,y)$ *is given by*

$$L(x,y)=\frac{(x-1)y\left(1-x(2+y)-\sqrt{(1-x-2xy-2x^2y+x^2y^2-x^3y^2)/(1-x)}\right)}{2(1-2x+x^2-2xy+x^2y)}.$$

Notice that a path $Q\in\mathcal{Q}$ can be uniquely decomposed as either $U\Delta DU\Delta' D$, $U\Delta DR$, UR_1DR_2, or $URDU\Delta D$, where Δ,Δ' are pyramids, and $R,R_1,R_2\in\mathcal{Q}$ (see Figure 4 for a graphical representation of this decomposition).

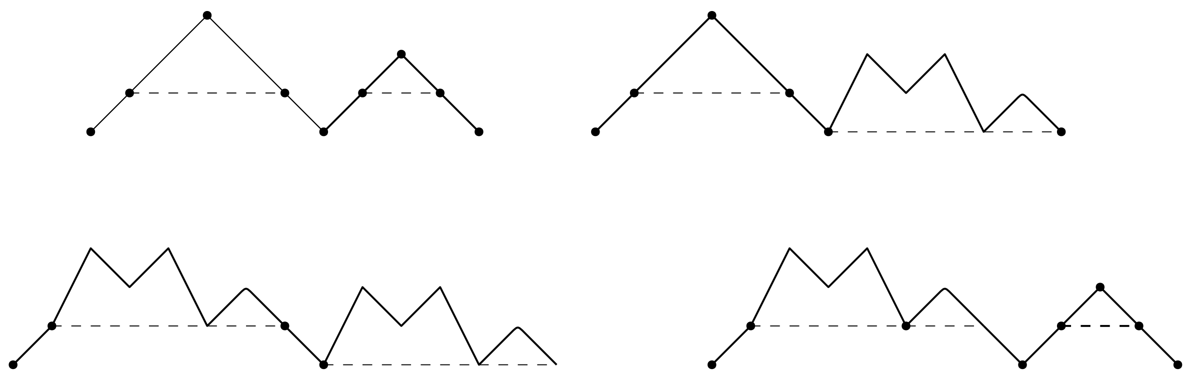

FIGURE 4. Decomposition of a (-1)-Dyck path in $\mathcal{Q}$.

Therefore, if

$$Q(x,y):=\sum_{Q\in\mathcal{Q}}x^{\ell(Q)}y^{\rho(P)},$$

then

$$Q(x,y)=x^2\left(\frac{y}{1-x}\right)^2+x\left(\frac{y}{1-x}\right)Q(x,y)+x(Q(x,y))^2+x^2\left(\frac{y}{1-x}\right)Q(x,y).$$

Solving the equation above for $Q(x,y)$, we find that

$$Q(x,y)=\frac{1-x-xy-x^2y-\sqrt{(1-x)(1-x-2xy-2x^2y+x^2y^2-x^3y^2)}}{2(1-x)x}. \tag{3.1}$$

Expressing $L(x,y)$ as a series expansion we obtain these first few terms:

$$L(x,y)=xy+x^2\left(y^2+y\right)+x^3\left(y^3+3y^2+y\right)+x^4\left(y^4+\mathbf{6}y^3+6y^2+y\right)$$
$$+x^5\left(y^5+10y^4+19y^3+10y^2+y\right)+x^6\left(y^6+15y^5+46y^4+45y^3+15y^2+y\right)+\cdots.$$

Figure 5 depicts all six paths in $\mathcal{D}_{-1}(4)$ with exactly 3 peaks. Notice that this is the coefficient of x^4y^3, in boldface type, in the above series.

The generating function for the (-1)-Dyck paths is given by

$$L(x):=L(x,1)=\frac{-1+4x-3x^2+\sqrt{1-4x+2x^2+x^4}}{2(1-4x+2x^2)}. \tag{3.2}$$

Thus,

$$L(x)=x+2x^2+5x^3+14x^4+41x^5+123x^6+375x^7+1157x^8+\cdots.$$

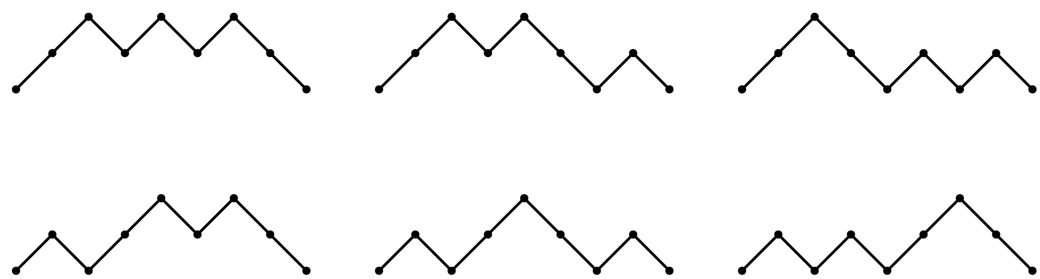

FIGURE 5. All six paths in $\mathcal{D}_{-1}(4)$ with exactly 3 peaks.

For the sake of simplicity, if there is not ambiguity, for the remaining part of the paper we use $r(n)$ instead of $r_{-1}(n)$. Our interest here is to give a combinatorial formula for this sequence. First of all, we give some preliminary results. Let $b(n)$ be the number of (-1)-Dyck paths of semi-length n that either have no valleys or the last valley is at ground level. Note that $b(n)-1$ is the n-th coefficient of the generating function $Q(x,1)$; see (3.1), or equivalently

$$\begin{aligned}\sum_{n\geq 0} b(n)x^n &= Q(x,1) + \frac{1}{1-x} = \frac{1-x^2-\sqrt{1-4x+2x^2+x^4}}{2(1-x)x}\\ &= 1 + x + 2x^2 + 4x^3 + 9x^4 + 22x^5 + 57x^6 + 154x^7 + 429x^8 + \cdots .\end{aligned}$$

This generating function coincides with the generating function of the number of Dyck paths of semi-length n that avoid the subpath $UUDU$. From Proposition 5 of [19] and [5, p. 10] we conclude the following proposition.

Proposition 3.2. *For all $n \geq 0$ we have*

$$b(n) = 1 + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^j}{n-j}\binom{n-j}{j}\binom{2n-3j}{n-j+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\sum_{j=0}^{n-k}\binom{n-k}{j}N(j,k),$$

where $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ are the Narayana numbers, with $N(0,0)=1$.

We tried to find a combinatorial proof of the previous proposition. However, we were not able to do it. This remark summarizes our observations toward a potential proof. It will be interesting to see such a combinatorial proof.

Remark 3.3. Let $\mathcal{B}(n) = \mathcal{Q}_n \cup \{\Delta_n\}$ denote the set of (-1)-Dyck paths having either no valleys or the last valley is at ground level. That is, $b(n) = |\mathcal{B}(n)|$. A South-East step in $P \in \mathcal{B}(n)$, satisfies one of these two conditions.

- The step belongs to a pyramid.
- The step is part of a valley, say for example, the m-th valley, such that $\nu_m - \nu_{m-1} = -1$. In this case, the valley with height ν_m is called (-1)-*valley*.

We denote by $\mathcal{B}_{j,k}(n)$ the set of paths in $\mathcal{B}(n)$ with exactly j valleys, where k of them are (-1)-valley. Now, a path $P \in \mathcal{B}_{j,k}(n)$ can be decomposed as

$$P = U^{s_0}\Delta_{t_1}U^{s_1}\Delta_{t_2}D^{r_1}U^{s_2}\cdots\Delta_{t_{j-1}}D^{r_{j-2}}U^{s_{j-1}}\Delta_{t_j}D^{r_{j-1}}\Delta_{t_{j+1}},$$

where $r_i \in \{0,1\}$, $t_i \geq 1$, $s_i \geq 0$, and with the additional property that there are exactly k indices i for which $r_i = 1$.

There are $n-k$ South-East steps that belong to one of the $j+1$ pyramids in the path. So, we can represent the possible choices of the t_i as an integer composition of $n-k$ into $j+1$ parts in $\binom{n-k-1}{j+1-1} = \binom{n-k-1}{j}$ ways. This means that setting $t_i = 1$ for all i, in the spirit of Proposition 3.2, the Narayana numbers $N(j,k+1)$ should correspond to the number of (-1)-Dyck paths of semi-length $j+1+k$ containing exactly j valleys and $k \leq j-1$ (-1)-valleys where the last valley is at ground level. That is $|\mathcal{B}_{j,k}(j+k+1)| = N(j,k+1)$ for $j > 0$.

Theorem 3.4. *The total number of paths in $\mathcal{D}_{-1}(n)$ is given by*

$$r(n) = \sum_{k=0}^{n}\sum_{i=0}^{n-k-1}\binom{n-k-1}{i}q^{(i)}(k),$$

where

$$q^{(i)}(n) = \sum_{n_1+n_2+\cdots+n_i=n} b(n_1)b(n_2)\cdots b(n_i).$$

Proof. Recall that $\ell(P)$ denotes the semi-length of a path P. Let us denote by $\mathcal{Q}^{(i)}(n)$ the set of i-tuples $(P_1,\dots,P_i)$ of paths $P_j \in \mathcal{B} = \bigcup_{m\geq 0}\mathcal{B}(m)$, such that $\ell(P_1)+\cdots+\ell(P_i) = n$. It is clear that $\left|\mathcal{Q}^{(i)}(n)\right| = q^{(i)}(n)$. Note that the empty path $\lambda \in \mathcal{B}$. Let $\mathcal{C}_i(n)$ be the set of integer compositions of n with i parts. The cardinality of this set is given by the binomial coefficient $\binom{n-1}{i-1}$.
Let $\mathcal{QC}(n)$ be the set of $(2i+1)$-tuples $(c_1, P_1, \dots, c_i, P_i, c_{i+1})$ such that the element $((P_1,\dots,P_i),(c_1,\dots,c_{i+1}))$ is in $\mathcal{Q}^{(i)}(j)\times\mathcal{C}_{i+1}(n-j)$. (Note that $\mathcal{QC}(n)$ is isomorphic to $\bigcup_{i,j}(\mathcal{Q}^{(i)}(j)\times\mathcal{C}_{i+1}(n-j))$.) We now consider the function

$$\varphi : \mathcal{QC}(n) \longrightarrow \mathcal{D}_{-1}(n),$$

defined by

$$\varphi\left((c_1,P_1,c_2,P_2,\dots,c_i,P_i,c_{i+1})\right) = U^{c_1}M_1U^{c_2}\cdots M_iU^{c_{i+1}}D^{g},$$

where the integer $g \geq c_{i+1}$ is the number of necessary down-steps to reach the x-axis, and M_j is given by

$$M_j = \begin{cases} D^{c_j}, & \text{if } P_j = \lambda; \\ P_j, & \text{if } P_j = \Delta; \\ P_jD, & \text{otherwise.}\end{cases}$$

Figure 6 depicts two examples on the application of the function φ.

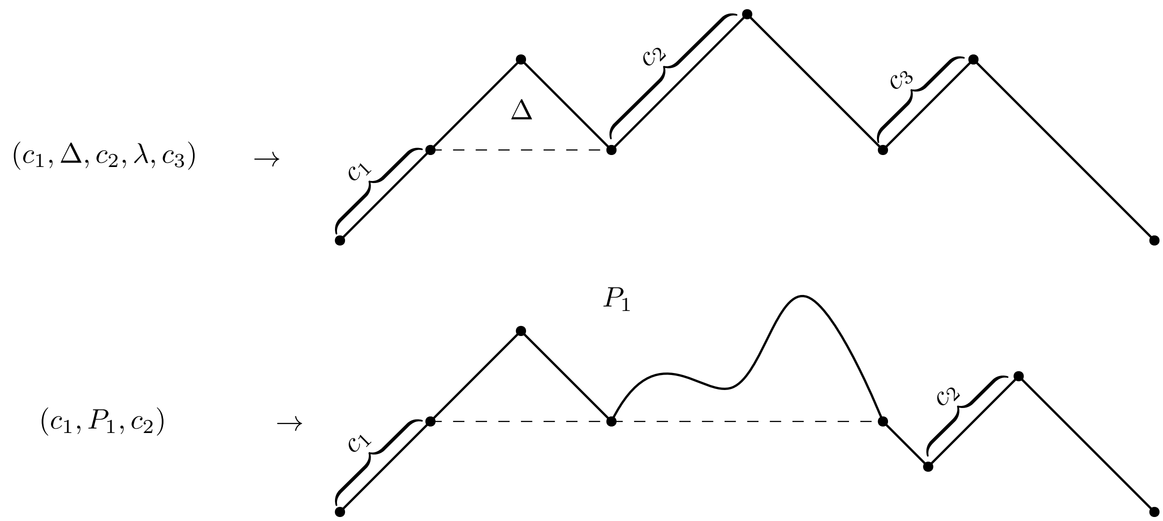

FIGURE 6. Function φ applied to the vectors $(c_1, \Delta, c_2, \lambda, c_3)$ and (c_1, P_1, c_2).

For the remaining part of this proof we use $\Delta_0 = \lambda$. We define ϕ from $\mathcal{D}_{-1}(n)$ to $\mathcal{QC}(n)$ via the Algorithm 1 below.

Algorithm 1 Function ϕ

---

(1) Let $i = 1$.

(2) If there are (-1)-valleys, go to step (3), else, the path is non-decreasing, and for some integers $s_m \geq 0$, $g \geq 0$, and $t_m > 0$, it can be decomposed as
$$P = U^{s_0}\Delta_{t_1}U^{s_1}\Delta_{t_2}\cdots U^{s_{j-1}}\Delta_{t_j}\Delta_{t_{j+1}}D^g.$$
- If no valleys, that is $j = s_0 = g = 0$, then **return** the vector (t_1).
- If there is just one valley, that is $j = 1$, set
$$(s'_0, t'_1) = \begin{cases} (s_0, t_1), & \text{if } s_0 > 0; \\ (t_1, 0), & \text{otherwise;} \end{cases}$$
and then **return** the vector $(s'_0, \Delta_{t'_1}, t_2)$.
- In the general case, set
$$(s'_m, t'_{m+1}) = \begin{cases} (s_m, t_{m+1}), & \text{if } s_m > 0; \\ (t_{m+1}, 0), & \text{otherwise;} \end{cases}$$
for $m < j$ and then **return** the vector $(s'_0, \Delta_{t'_1}, s'_1, \ldots, \Delta_{t'_j}, t_{j+1})$.

(3) Find the **rightmost** occurrence of an (-1)-valley, that is, a subpath of the form $D\Delta_k DU$, with $k > 0$. Denote the height of this valley as h_i. Decompose the path P as $P = \widetilde{P}_0 P_i D \widetilde{P}_1$, where P_i is the maximal subpath that is a Dyck path to the left of the aforementioned (-1)-valley. Notice that P_i ends on $D\Delta_k$ and so, it belongs to $\mathcal{B}$. Let $\hat{P}_0 = \widetilde{P}_0 D^{h_i+1}$ and $\hat{P}_1 = \widetilde{P}_1 D^{-h_i}$, where D^{-r} means deleting the last r South-East steps of the path. By assumption, the path $\hat{P}_1$ is non-decreasing: go to step (2) using $\hat{P}_1$ and call the returned tuple τ_1. Increase the value of i by one and go to step (2) with the path $\hat{P}_0$ and call the returning tuple τ_0. **Return** (τ_0, P_i, τ_1).

We now give an example of the application of the Algorithm 1. Consider, for instance, the path given in Figure 7. First of all, search for the rightmost (-1)-valley (in the first case, it is denoted by dashed circle after P_1), they are decorated by a red circle around them, the algorithm applied to $\hat{P}_1 = UD$ (where $\widetilde{P}_1 = UDD$) gives $\tau_1 = (1)$. We extract the path P_1 and we locate the next (-1)-valley, the right part on this instance, given by $UDUDU$, corresponds to $\tau_1 = (1, \lambda, 1, \lambda, 1)$, and the left part of P_2 corresponds to $(1, \Delta_1, 1)$, and so the whole path is encoded by the vector $(1, \Delta_1, 1, P_2, 1, \lambda, 1, \lambda, 1, P_1, 1)$.

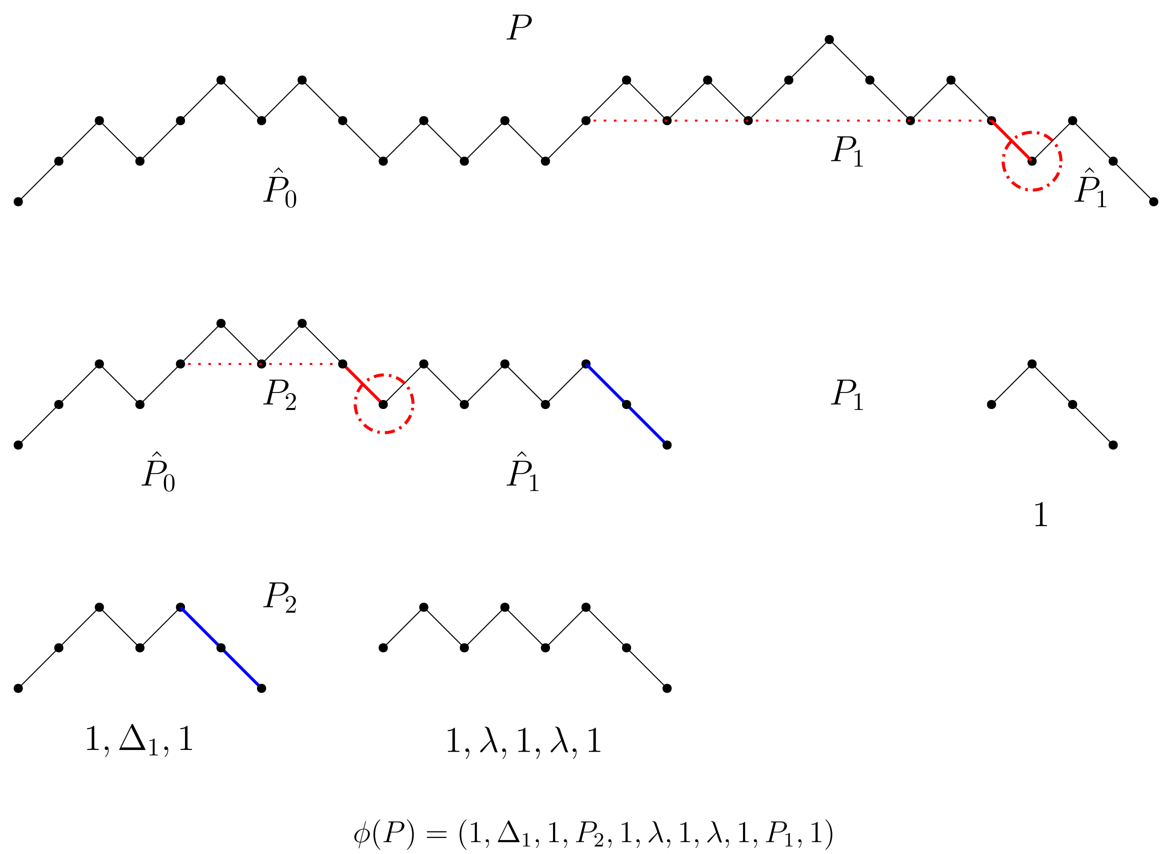

FIGURE 7. Example inverse function.

Using these decompositions, we show by induction that $\phi \circ \varphi|_{\mathcal{QC}_k(n)} = id_{\mathcal{QC}_k(n)}$ and $\varphi \circ \phi|_{\mathcal{B}_k(n)} = id_{\mathcal{B}_k(n)}$ for every $k \geq 0$. These equalities and the functionality of ϕ, given by choosing the paths P_i in a **maximal** way, imply that φ is a bijection and ϕ is its inverse. Let $S(k)$ be the statement

$$\phi \circ \varphi|_{\mathcal{QC}_k(n)} = id_{\mathcal{QC}_k(n)} \quad \text{and} \quad \varphi \circ \phi|_{\mathcal{B}_k(n)} = id_{\mathcal{B}_k(n)}.$$

For the basis step, $S(0)$, first notice that if $P \in \mathcal{B}_0(n)$, then the Algorithm 1 Part (2) guarantees that $\phi(P)$ contains only paths of the form Δ_t for $t \geq 0$. On the other hand, φ returns a path without (-1)-valley when given a tuple $\tau \in \mathcal{QC}_0(n)$ by definition of the M_j's, and so the functions and their compositions are well defined when restricted to $\mathcal{B}_0(n)$ and $\mathcal{QC}_0(n)$. Let $P = U^{s_0}\Delta_{t_1} \cdots \Delta_{t_i}\Delta_{t_{i+1}}D^g \in \mathcal{B}_0(n)$, using Algorithm 1 we get $\phi(P) = (s_0', \Delta_{t_1'}, \dots, \Delta_{t_i'}, t_{i+1})$.

Now, we have

$$\varphi(\phi(P)) = \varphi(s'_0, \Delta_{t'_1}, \ldots, \Delta_{t'_i}, t_{i+1}) = U^{s'_0} M_1 \cdots M_i U^{t_{i+1}} D^{g'},$$

with $M_m = D^{s'_{m-1}} = D^{t_m}$ if $t'_m = 0$ and $s'_{m-1} = t_m$, or $M_m = \Delta_{t'_m} = \Delta_{t_m}$ if $s_{m-1} > 0$, for $1 \leq m \leq i$. This gives $\varphi(\phi(P)) = P$.
Let $\tau = (c_1, \Delta_{t_1}, \ldots, c_i, \Delta_{t_i}, c_{i+1})$ with $c_m > 0$ and $t_m \geq 0$, then

$$\varphi(\tau) = U^{c_1} M_1 \cdots U^{c_i} M_i U^{c_{i+1}} D^g,$$

where

$$M_m = \begin{cases} D^{c_m}, & \text{if } t_m = 0; \\ \Delta_{t_m}, & \text{otherwise}; \end{cases}$$

and let $1 \leq x_1 < x_2 < \cdots < x_q \leq i$ be such that $t_{x_m} = 0$, that is the paths $\Delta_{t_{x_m}}$ in τ that are of the form $\Delta_0 = \lambda$. Notice that $c_m > 0$ and the definition of M_m imply that either $U^{c_m} M_m = \Delta_{c_m}$ exactly for $t_m = 0$ or $U^{c_m} \Delta_{t_m}$ for $t_m > 0$, which allows us to decompose $\varphi(\tau)$ as

$$\varphi(\tau) = \left(U^{c_1}\Delta_{t_1} \cdots U^{c_{x_1-1}}\Delta_{t_{x_1-1}}\right) U^0 \Delta_{c_{x_1}} \cdots U^0 \Delta_{c_{x_q}} \left(U^{c_{x_q+1}}\Delta_{t_{x_q+1}} \cdots \Delta_{t_i}\right) \Delta_{c_{i+1}} D^{g-c_{i+1}},$$

where every pyramid in the decomposition is non-empty and so the decomposition is unique.
We now have that $\phi(\varphi(\tau)) = (c_1, \Delta_{t_1}, \ldots, c_{x_1-1}, \Delta_{t_{x_i-1}}, c_{x_1}, \lambda, \ldots, c_{x_q}, \lambda, \ldots, c_{i+1}) = \tau$.
For the inductive step $S(k)$, we assume that we have the desired equalities for $\ell < k$. Notice that any tuple $\tau \in \mathcal{QC}_k(n)$ can be decomposed as $\tau = (\tau_0, P_1, \tau_1)$ with τ_0 containing $\ell < k$ paths that are not pyramids, $P_1 \neq \Delta_t$ for any $t \geq 0$, and $\tau_1 \in \mathcal{QC}_0(n')$. Notice, further, that

$$\varphi((\tau_0, P_1, \tau_1)) = \varphi(\tau_0) D^- P_1 D \varphi(\tau_1) D^{g_2},$$

where $\varphi(\tau_0)D^-$ means deleting the South-East steps suffix of $\varphi(\tau_0)$. By the recursive step in the Algorithm 1, we have that $\phi(\varphi(\tau)) = \tau$ by using the inductive hypothesis. Analogously, we can decompose a path as in the recursive step of Algorithm 1, and the inductive hypothesis give $\varphi(\phi(P)) = P$. □

Proposition 3.5 is a direct consequence of the decomposition given in the proof of Theorem 3.1. The first result follows from Figure 4 and the second result uses the first part of this proposition and the decomposition UTD, $U\Delta DT$, or $UQDT$ as given in the proof of Theorem 3.1.

Proposition 3.5. *If $n > 1$, then these hold*

(1) If $q_n = |\mathcal{Q}_n|$, then

$$q_n = 2q_{n-1} + q_{n-2} + q_{n-3} + \sum_{i=2}^{n-4} q_i(q_{n-i-1} - q_{n-i-2}) + 1,$$

for $n > 3$, with the initial values $q_1 = 0$, $q_2 = 1$, and $q_3 = 3$.

(2) If $r(n) = |\mathcal{D}_d(n)|$, then

$$r(n) = 3r(n-1) - r(n-2) + q_{n-2} + \sum_{i=2}^{n-3} q_i(r(n-i-1) - r(n-i-2)),$$

for $n > 3$, with the initial values $r(1) = 1$, $r(2) = 2$, and $r(3) = 5$.

The generating function of the sequence $r(n)$ is algebraic of order two, then $r(n)$ satisfies a recurrence relation with polynomial coefficients; see [1, Proposition 4]. This can be automatically solved with implementation of Kauers in *Mathematica* [18]. In particular we obtain that $r(n)$ satisfies the recurrence relation:

$$\begin{aligned}2nr(n) - 4nr(n+1) + (12+5n)r(n+2) - 4(15+4n)r(n+3)\\ + 10(9+2n)r(n+4) - 2(21+4n)r(n+5) + (6+n)r(n+6) = 0, \quad \text{with } n \geq 6\end{aligned}$$

and the initial values $r(0) = 0, r(1) = 1, r(2) = 2, r(3) = 5, r(4) = 14$, and $r(5) = 41$.

In Theorem 3.6 we give an asymptotic approximation for the sequence $r(n)$. To accomplish this goal, we use the singularity analysis method to find the asymptotes of the coefficients of a generating function (see, for example, [11] for the details).

We recall that in literature $f_n \sim g_n$ means that f_n and g_n are asymptotic equivalent. That is, $f_n/g_n \to 1$ as $n \to \infty$.

Theorem 3.6. *If ρ is the smallest real positive root of $1 - 4x + 2x^2 + x^4$, then the number of (-1)-Dyck paths has this asymptotic approximation*

$$r(n) \sim \frac{\rho^{-n}}{\sqrt{n^3\pi}} \cdot \frac{\sqrt{\rho(4-4\rho-4\rho^3)}}{4(-1+4\rho-2\rho^2)},$$

where ρ is called the dominant singularity of the generating function $L(x)$.

Proof. From a symbolic computation we find that

$$\rho = \frac{1}{3}\left(-1 - \frac{4\ 2^{2/3}}{\sqrt[3]{13+3\sqrt{33}}} + \sqrt[3]{2\left(13+3\sqrt{33}\right)}\right) \approx 0.295598.$$

From the expression given in (3.2) for $L(x)$ we have

$$L(x) = \frac{-1+4x-3x^2}{2(1-4x+2x^2)} + \frac{\sqrt{1-4x+2x^2+x^4}}{2(1-4x+2x^2)} \sim (\rho - x)^{1/2}\frac{\sqrt{\rho(4-4\rho-4\rho^3)}}{2(1-4\rho+2\rho^2)} \quad \text{as } x \to \rho^-.$$

Therefore,

$$r(n) \sim \frac{n^{-1/2-1}}{\rho^n(-2\sqrt{\pi})}\frac{\sqrt{\rho(4-4\rho-4\rho^3)}}{2(1-4\rho+2\rho^2)} = \frac{\rho^{-n}}{\sqrt{n^3\pi}}\frac{\sqrt{\rho(4-4\rho-4\rho^3)}}{4(-1+4\rho-2\rho^2)}. \qquad \square$$

4. The area of the (-1)-Dyck paths

In this section we use generating functions and recursive relations to analyze the distribution of the area of the paths in the set of restricted (-1)-Dyck paths. We recall that the *area* of a Dyck path is the sum of the absolute values of y-components of all points in the path. We use $\texttt{area}(P)$ to denote the area of a path P. From Figure 1 on Page 118, we can see that $\texttt{area}(P) = 70$. We use $a(n)$ to denote the total area of all paths in $\mathcal{D}_{-1}(n)$. In Theorem 4.1 we give a generating function for the sequence $a(n)$. We now introduce a bivariate generating function depending on this previous parameter and $\ell(P)$ (the semi-length of P). So,

$$A(x,q) := \sum_{P \in \mathcal{D}_{-1}} x^{\ell(P)} q^{\texttt{area}(P)}.$$

Let $\mathcal{Q} \subset \mathcal{D}_{-1}(n)$ be the set formed by all paths having at least one valley, where the last valley is at ground level; let $\mathcal{Q}_n \subset \mathcal{Q}$ be the set formed by all paths of semi-length n, and let $q_n = |\mathcal{Q}_n|$.

Theorem 4.1. *The generating function for the sequence $a(n)$ is given by*

$$V(x) = \sum_{n\geq 0} a(n)x^n = \frac{b(x) - c(x)\sqrt{1-4x+2x^2+x^4}}{(1-x)^2(1-4x+2x^2)^3(1-3x-x^2-x^3)},$$

where

$$b(x) = 2x - 23x^2 + 107x^3 - 262x^4 + 359x^5 - 256x^6 + 82x^7 - 5x^8 - 10x^9 + 6x^{10},$$
$$c(x) = x - 10x^2 + 41x^3 - 89x^4 + 108x^5 - 73x^6 + 18x^7 + 2x^8.$$

Proof. From the decomposition UD, UTD, $U\Delta DT$, or $UQDT$ given in the proof of Theorem 3.1 we obtain the functional equation

$$A(x,q) = xq + xqA(xq^2,q) + E(x,q)A(x,q) + xqB(xq^2,q)A(x,q), \tag{4.1}$$

where $E(x,q) := \sum_{j\geq 1} x^j q^{j^2}$ and $B(x,q) := \sum_{P\in\mathcal{Q}} x^{\ell(P)} q^{\texttt{area}(P)}$. Note that $E(x,q)$ corresponds to the generating function that counts the total number of non-empty pyramids in the given decomposition.

From the decomposition given in Figure 4, we obtain the functional equation

$$B(x,q) = E(x,q)^2 + E(x,q)B(x,q) + xqB(q^2x,q)B(x,q) + xqB(q^2x,q)E(x,q). \tag{4.2}$$

Let $M(x)$ be the generating function of the total area of the (-1)-Dyck paths in $\mathcal{Q}$. From the definition of $A(x,q)$ we have

$$V(x) = \left.\frac{\partial A(x,q)}{\partial q}\right|_{q=1}.$$

Substituting x by xq^2 in (4.2), and then differentiating with respect to q and taking $q = 1$, we obtain

$$\begin{aligned} W(x) := \left.\frac{\partial B(xq^2,q)}{\partial q}\right|_{q=1} &= \frac{2(3-x)x^2}{(1-x)^4} + \frac{(3-x)x}{(1-x)^3}Q(x) + \frac{x}{1-x}\left(W(x) + 2x\frac{\partial Q(x)}{\partial x}\right) \\ &\quad + 3xQ(x)^2 + xQ(x)\left(W(x) + 4x\frac{\partial Q(x)}{\partial x}\right) + xQ(x)\left(W(x) + 2x\frac{\partial Q(x)}{\partial x}\right) \\ &\quad + \frac{3x^2}{1-x}Q(x) + \frac{x^2}{1-x}\left(W(x) + 4x\frac{\partial Q(x)}{\partial x}\right) + \frac{x^2(3-x)}{(1-x)^3}Q(x), \end{aligned} \tag{4.3}$$

where $Q(x) := Q(x,1)$ and $Q(x,y)$ is the generating function given in (3.1) on Page 124.

Now, differentiating (4.1) with respect to q and then taking $q = 1$ we obtain,

$$\begin{aligned} V(x) &= x + xL(x) + x\left(V(x) + 2x\frac{\partial L(x)}{\partial x}\right) + \frac{x(x+1)}{(1-x)^3}L(x) \\ &\quad + \frac{x}{1-x}V(x) + xQ(x)L(x) + xW(x)L(x) + xQ(x)V(x). \end{aligned} \tag{4.4}$$

Solving (4.3) for $W(x)$ and substituting into (4.4) and then solving the resulting expression for $V(x)$ we obtain the desired result. $\square$

The first few values of the series of $V(x)$ are

$$V(x) = \sum_{n\geq 1} a(n)x^n = x + 6x^2 + 29x^3 + 130x^4 + 547x^5 + 2198x^6 + 8551x^7 + \cdots .$$

We now give a recursive relation for $a(n)$. Again for the sake of simplicity, the proof here is based on a geometric decomposition of the paths. So, we avoid some details. However, in [13] there are detailed proofs of Proposition 4.2 and Theorem 4.3. We recall that $q_n = |\mathcal{Q}_n|$ and that for simplicity we use $r(n)$ instead of $r_{-1}(n)$.

The following two results may follow as a direct application of (4.2). However, we include here a different combinatorial proof.

Proposition 4.2. *If A_n with $n \geq 1$ is the total area of all paths in $\mathcal{Q}_n$, then*

$$A_n = 2A_{n-1} + A_{n-2} + 2A_{n-3} + q_n - q_{n-1} + 2nq_{n-2} + 2(n-5)q_{n-3} + 4n^2 - 14n + 13 + \sum_{i=2}^{n-4} 2(A_i + iq_i + i(i+1))(q_{n-i-1} - q_{n-i-2}), \quad \text{with } n > 4,$$

and the initial values $A_1 = 0$, $A_2 = 2$, $A_3 = 13$, and $A_4 = 58$.

Proof. From Figure 4 we know that a path in $\mathcal{Q}_n$ can be decomposed in one of these four cases; $\Delta_i\Delta_{n-i}$, $\Delta_i Q$, $XQY\Delta_i$, $XQ'YQ$ where $Q, Q' \in \mathcal{Q}$

Case 1. The area of $\Delta_i\Delta_{n-i}$ is $i^2 + (n-i)^2$. Since for a fixed $i \in \{1, 2, \dots, n-1\}$, there is exactly one path of the form $\Delta_i\Delta_{n-i}$ in $\mathcal{Q}_n$, we have that the total area of this type of paths is $\sum_{i=1}^{n-1}(i^2 + (n-i)^2) = n(n-1)(2n-1)/3$.

Case 2. The area of $P_i := \Delta_i Q$ is $i^2 + A_{n-i}$. Since for every $i \in \{1, 2, \dots, n-2\}$ there are q_{n-i} paths of the form P_i, we have that the total area of all paths of the form P_i is given by $i^2 q_{n-i} + A_i$. Therefore, the total area of this type of paths is $\sum_{i=1}^{n-2} i^2 q_{n-i} + \sum_{j=2}^{n-1} A_j$.

Case 3. For a fixed i, the area of a path of the form $XQ'YQ''$ is given by $2i + 1 + A_i + A_{n-i-1}$, where $Q' \in \mathcal{Q}_i$, $Q'' \in \mathcal{Q}_{n-i-1}$ and $i \in \{2, 3, \dots, n-3\}$. Note that for a fixed i and a fixed $Q \in \mathcal{Q}_{n-i-1}$ there q_i paths of the form $XQ'YQ$ with $Q \in \mathcal{Q}_i$. This implies that for a fixed $i \in \{2, 3, \dots, n-3\}$ the total area of this type of paths is $A_{n-i-1}q_i + (2i+1)q_i q_{n-i-1} + A_i q_{n-i-1}$. We conclude for i varying from 2 to $n-3$, we obtain that the total area of this type of paths is

$$\sum_{i=2}^{n-3} A_{n-i-1}q_i + \sum_{i=2}^{n-3}((2i+1)q_i q_{n-i-1} + A_i q_{n-i-1}).$$

Case 4. The area of $H_i := XQ_\ell Y\Delta_i$ is given by area of Δ_i (which is i^2) plus the area of $XQ_\ell Y$ (this is given by A_ℓ, the area of Q_ℓ, plus $2i+1$ which is the area of the trapezoid generated by X and Y). Since for every $i \in \{1, 2, \dots, n-3\}$ there are q_{n-i-1} paths of the form H_i with $Q \in \mathcal{Q}_{n-i}$, we conclude that the total area of this type of paths is

$$\sum_{i=1}^{n-3} i^2 q_{n-i-1} + \sum_{i=2}^{n-2}((2i+1)q_i + A_i).$$

Adding the results from Cases 1-4, we obtain that the recursive relation for the area A_n is given by

$$A_n = \sum_{i=1}^{n-1}\left(i^2+(n-i)^2\right)+\sum_{i=1}^{n-2} i^2 q_{n-i}+\sum_{i=2}^{n-1} A_i+\sum_{i=2}^{n-3}(2i+1)q_i q_{n-(i+1)}+\sum_{i=2}^{n-3} A_i q_{n-(i+1)}+ \sum_{i=2}^{n-3} A_i q_{n-(i+1)}+\sum_{i=2}^{n-2} A_i+\sum_{i=1}^{n-3} i^2 q_{n-(i+1)}+\sum_{i=2}^{n-2}(2i+1)q_i.$$

Subtracting A_n from A_{n+1} and simplifying we have

$$A_n = 2A_{n-1}+A_{n-2}+2A_{n-3}+(2n-5)q_{n-3}+(2n-4)q_{n-2}+q_{n-1}+4n^2-14n+15+ \sum_{i=2}^{n-4}(2A_i+(2i+1)q_i)(q_{n-i-1}-q_{n-i-2})+\sum_{i=2}^{n-3}\left(2i^2-2i+1\right)(q_{n-i}-q_{n-i-1}).$$

We now rearrange this expression to obtain q_n (see the expression within brackets) given in Proposition 3.5

$$\begin{aligned} A_n = {}& 2A_{n-1}+A_{n-2}+2A_{n-3}+(2n-6)q_{n-3}+(2n-4)q_{n-2}-q_{n-1}+4n^2-14n+13+ \\ & \sum_{i=2}^{n-4} 2(A_i+iq_i)(q_{n-i-1}-q_{n-i-2})+\sum_{i=2}^{n-3} 2\left(i^2-i\right)(q_{n-i}-q_{n-i-1}) \\ & +[2q_{n-1}+q_{n-2}+q_{n-3}+\sum_{i=2}^{n-4} q_i(q_{-i+n-1}-q_{-i+n-2})+1]. \end{aligned}$$

After some simplifications we obtain the desired recursive relation. □

The proof of the following theorem is similar to the proof of Proposition 4.2. We recall that $r(i)=|\mathcal{D}_{-1}(i)|$ and $q_j=|\mathcal{Q}_j|$.

Theorem 4.3. *If $a(n)$ is the total area of all paths in $\mathcal{D}_{-1}(n)$, for $n \geq 1$, then $a(n)$ satisfies the recursive relation*

$$\begin{aligned} a(n) = {}& 3a(n-1)-a(n-2)+A_{n-2}+2(n-1)q_{n-2}+2nr(n-1)+2(3-n)r(n-2) \\ & -4r(n-3)+(n-1)^2+\sum_{i=3}^{n-2} q_{i-1}(a(n-i)-a(n-i-1)) \\ & +\sum_{i=3}^{n-2}\left(A_{i-1}+(2i-1)q_{i-1}+i^2\right)(r(n-i)-r(n-i-1)). \end{aligned}$$

Proof. First of all, we note that a path in $\mathcal{D}_{-1}(n)$ can be decomposed as XQ_1Y, $\Delta_i Q_{n-i}$, and $XQ'YD$, where $Q_j, D \in \mathcal{D}_{-1}$, and $Q' \in \mathcal{Q}_j$. This decomposition gives these three cases to consider.

Case 1. The area of XQY is $(2n-1)+a(n-1)$, where $a(n-1)$ is the area of $Q \in \mathcal{D}_{-1}(n-1)$ and $2n-1$ is the are of the trapezoid generated by X and Y. This gives that the total area of all paths of the form XQY with $Q \in \mathcal{D}_{-1}(n-1)$ is $(2n-1)r(n-1)+a(n-1)$.

Case 2. The area of $K_i := X^iY^iQ_\ell$ is $i^2+a(n-i)$, where $Q_\ell \in \mathcal{D}_{-1}(n-i)$. Since for a fixed $i \in \{1,2,\dots,n-1\}$ there are $r(n-i)$ paths of form K_i, we conclude that the total area of all these paths is $\sum_{i=1}^{n-1} i^2 r(n-i)+a(n-i)$.

Case 3. The area of $M_i := XQ'YD$ is $((2i+1) + A_i + a(n-i-1)$, where $Q' \in \mathcal{Q}_i$ and $D \in \mathcal{D}_{-1}(n-i-1)$. Note that for a given path $D \in \mathcal{D}_{-1}(n-i-1)$, there are as many paths of the form $XQ'YD$ as paths in $\mathcal{Q}_i$. It is easy to see that for a fixed $i \in \{2, 3, \dots, n-2\}$ there are $r(n-i-1)$ subpaths of the form $XQ'Y$. Note that X and Y give rise to a trapezoid, where the two parallel sides have lengths $2i$ and $2i+2$, giving rise to an area of $2i+1$. So, the contribution to the area given by the first subpaths of the form $XQ'Y$ is equal to the area of the trapezoids plus the area of all paths of the form Q' (these are on top of the trapezoids). Thus, the area of a trapezoid multiplied by the total number of the paths of the form Q' plus the area of all paths of the form Q' and then all of these multiplied by the total number of paths of the form D. Thus, the contribution to the area given by the first subpaths of the form $XQ'Y$ (overall paths of the form M_i for a fixed i), is $((2i+1)q_i r(n-i-1) + A_i r(n-i-1))$.
We conclude that the total area of this type of paths is

$$\sum_{i=2}^{n-2} A_i r(n-i-1) + \sum_{i=2}^{n-2} (2i+1) q_i r(n-i-1).$$

Adding the results from Cases 1-3, we obtain that the recursive relation for the area $a(n)$ is given by

$$\begin{aligned} a(n) = a(n-1) + (2n-1)r(n-1) + \sum_{i=1}^{n-1} i^2 r(n-i) + \sum_{i=1}^{n-1} a(n-i) \\ + \sum_{i=2}^{n-2} q_i a(n-i-1) + \sum_{i=2}^{n-2} A_i r(n-i-1) + \sum_{i=2}^{n-2} (2i+1) q_i r(n-i-1). \end{aligned}$$

Subtracting $a(n)$ from $a(n+1)$ and simplifying we have

$$\begin{aligned} a(n) &= 3a(n-1) - a(n-2) + A_{n-2} + 2(n-1)q_{n-2} + (2n-1)r(n-1) + (3-2n)r(n-2) + (n-1)^2 \\ &+ \sum_{i=3}^{n-2} q_{i-1}(a(n-i) - a(n-i-1)) + \sum_{i=3}^{n-2} A_{i-1}(r(n-i) - r(n-i-1)) \\ &+ \sum_{i=3}^{n-2} (2i-1) q_{i-1}(r(n-i) - r(n-i-1)) + \sum_{i=1}^{n-2} i^2 (r(n-i) - r(n-i-1)). \end{aligned}$$

After some other simplifications we have that

$$\begin{aligned} a(n) &= 3a(n-1) - a(n-2) + A_{n-2} + 2(n-1)q_{n-2} + 2nr(n-1) \\ &+ 2(3-n)r(n-2) - 4r(n-3) + (n-1)^2 + \sum_{i=3}^{n-2} q_{i-1}(a(n-i) - a(n-i-1)) \\ &+ \sum_{i=3}^{n-2} \left(A_{i-1} + (2i-1)q_{i-1} + i^2\right)(r(n-i) - r(n-i-1)). \end{aligned}$$

This completes the proof. $\square$

Notice that the total area of the Dyck paths (cf. [21]) is given by $4^n - \binom{2n+1}{n}$.

5. Appendix. Notation table

Concept	Notation
Set restricted d-Dyck paths	$\mathcal{D}_d$
Set restricted d-Dyck paths of length n	$\mathcal{D}_n$
Cardinality of $\mathcal{D}_d(n)$	$r_d(n)$
Cardinality of $\mathcal{D}_{-1}(n)$	$r_{-1}(n)$ or $r(n)$
Area of a path P	$\texttt{area}(\mathrm{P})$
Semi-length of P	$\ell(P)$
Number of peaks of P	$\rho(P)$
Number of paths in $\mathcal{D}_d(n)$ having exactly k peaks.	$p_d(n,k)$
Paths with the last valley at level i	$\mathcal{Q}_{d,i}$
General pyramid	Δ
Pyramid $(XY)^k$	Δ_k

Table 1. Summary of notation.

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COUNTING LATTICE PATHS BY CROSSINGS AND MAJOR INDEX II: TRACKING DESCENTS VIA TWO-ROWED ARRAYS

SERGI ELIZALDE[1]

[1]Dartmouth College, Hanover, USA; https://math.dartmouth.edu/~sergi

Abstract. We present refined enumeration formulas for lattice paths in $\mathbb{Z}^2$ with two kinds of steps, by keeping track of the number of descents (i.e., turns in a given direction), the major index (i.e., the sum of the positions of the descents), and the number of crossings. One formula considers crossings between a path and a fixed line; the other considers crossings between two paths. Building on the first paper of the series, which used lattice path bijections to give the enumeration with respect to major index and crossings, we obtain a refinement that keeps track of the number of descents. The proof is based on new bijections which rely on certain two-rowed arrays that were introduced by Krattenthaler.

Keywords: lattice path, major index, crossings, descents, bijection.

1. Introduction

1.1. **Background.** Lattice paths in the plane with two kinds of steps have played an important role in combinatorics and mathematical statistics for decades [14, 19]. The statistic giving the number of times that a path crosses a fixed line has been studied at least since the sixties [4–6, 10, 21, 23], often in connection to random walks. For tuples of paths, the enumeration in the special case of non-crossing tuples, in its closely related non-intersecting variant, is given by the celebrated Lindström–Gessel–Viennot determinant [9, 17], and has applications to symmetric functions, plane partitions, tilings, and statistical physics [7].

On the other hand, a very different statistic, the sum of the positions of the turns in a given direction, has been studied in [12, 15, 20]. This statistic is called the *major index* because it arises naturally when interpreting the paths as binary words, and it was introduced by MacMahon [18].

In the first paper of this series [3], we enumerated paths with respect to the number of crossings of a line and the major index, as well as pairs of paths with respect to the number of times they cross each other and the sum of their major indices. The goal of the present paper is to refine the results from [3] by another important statistic, which is related to the major index and arguably more natural: the number of turns in a given direction, or equivalently, the number of descents of the associated binary word.

The number of turns arises when studying the distribution of runs in random walks [19], the coefficients of Hilbert polynomials of determinantal and Pfaffian rings [16], and summations for Schur functions [12]. A thorough investigation of this parameter on lattice paths was provided by Krattenthaler [13]. In particular, a refinement by this statistic of the classical determinantal formula of Lindström–Gessel–Viennot [9, 17] counting tuples of non-intersecting paths was given in [11, Thm. 1] and [13, Thm. 3.6.1]. In related work, Krattenthaler and Mohanty [15] enumerated lattice paths constrained to a strip with respect to the number of descents and the major index.

The tools that were used in [3] to deal with crossings and the major index consisted of bijections with a neat description in terms of lattice paths. While these bijections were suited to study the major index, unfortunately they do not behave well with respect to the number of descents, which is why the results obtained in [3] do not include this statistic. Instead, in this paper we will construct different bijections that are not described in terms of paths, but rather in terms of two-rowed arrays.

Such arrays, which are more general than paths, have been used by Krattenthaler and Mohanty to study descents and major index on lattice paths in a strip [15], and by Krattenthaler to enumerate tuples of non-intersecting paths with respect to the number of turns [11, 13] and to the major index [12]. However, to our knowledge, they have never been used while also keeping track of the number of crossings. While two-rowed arrays allow us to track simultaneously track multiple statistics, including the number of descents, the trade-off is that they make the proofs more involved and less intuitive than those in [3].

Paralleling the results in [3], this paper solves two problems: the enumeration of single paths with respect to the number of times that they cross a fixed line, and the enumeration of pairs of paths with respect to the number of times that they cross each other, refined in both cases by the number of descents and the major index. This paper is self-contained and does not rely on any material from [3].

Our work is partially motivated by the simplicity of the resulting formulas in both cases. For single paths with given endpoints, crossing a line at least a certain number of times and having a fixed number of descents, we will show that the polynomial enumerating them with respect to the major index is given by a product of two q-binomial coefficients and a power of q. For pairs of paths crossing each other, the formulas we obtain involve a product of two generating functions whose coefficients have again the same form.

The second source of motivation is that our results for paths crossing a line have applications to the refined enumeration of integer partitions according to the number of sign changes of their successive ranks (or off-diagonal ranks). These applications, which generalize results of Seo and Yee [22], will be explored in [2] in connection to the study of partitions with constrained ranks.

1.2. **Preliminaries.** For points $A, B \in \mathbb{Z}^2$, we denote by $\mathcal{P}_{A\to B}$ the set of lattice paths with steps $N = (0, 1)$ (north) and $E = (1, 0)$ (east) that start at A and end at B. Sometimes it will be convenient to consider paths with steps $U = (1, 1)$ (up) and $D = (1, -1)$ (down) instead. For nonnegative integers a, b, we denote by $\mathcal{G}_{a,b}$ set of paths with a steps U and b steps D starting at the origin.

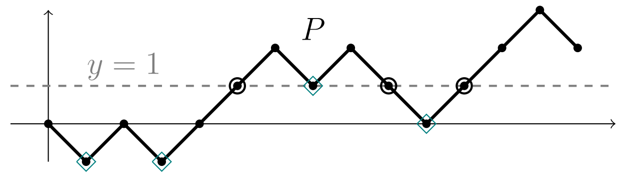


FIGURE 1. A path $P \in \mathcal{G}_{8,6}^{\geq 3,1}$ with $\mathrm{maj}(P) = 1 + 3 + 7 + 10 = 21$. The four valleys are marked with teal diamonds, and the three crossings of the line $y = 1$ are circled in black. The middle crossing is a downward crossing, whereas the other two are upward crossings.

In both cases, encoding paths as binary words, with 0s recording N (resp. U) steps, and 1s recording E (resp. D) steps, we define a *descent* (also called a *valley*) of the path to be a vertex preceded by an E and followed by an N (resp. preceded by a D and followed by a U). The number of descents of a path P is denoted by $\mathrm{des}(P)$. The major index of P, denoted by $\mathrm{maj}(P)$, is defined to be the sum of the positions of the descents, where the position is determined by numbering the vertices along the path, starting at 0. See Figure 1 for an example. We also define a *peak* of the path to be a vertex preceded by an N and followed by an E (resp. preceded by a U and followed by a D).

The enumeration of binary words by the number of descents and the major index is implicit in work of MacMahon [18]. An explicit proof was given by Fürlinger and Hofbauer [8]. To state this result in its lattice path version, recall that the *q-binomial coefficients* are defined as

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \prod_{k=0}^{n-1} \frac{1-q^{m-k}}{1-q^{n-k}}$$

if $0 \leq n \leq m$, and as 0 otherwise.

Lemma 1.1 ([8, 18]). *For $a, b \geq 0$,*

$$\sum_{P \in \mathcal{G}_{a,b}} t^{\mathrm{des}(P)} q^{\mathrm{maj}(P)} = \sum_{n \geq 0} t^n q^{n^2} \begin{bmatrix} a \\ n \end{bmatrix}_q \begin{bmatrix} b \\ n \end{bmatrix}_q .$$

Equivalently, for $x, y, u, v \in \mathbb{Z}$,

$$\sum_{P \in \mathcal{P}_{(x,y)\to(u,v)}} t^{\mathrm{des}(P)} q^{\mathrm{maj}(P)} = \sum_{n \geq 0} t^n q^{n^2} \begin{bmatrix} u-x \\ n \end{bmatrix}_q \begin{bmatrix} v-y \\ n \end{bmatrix}_q .$$

A self-contained proof of this lemma will be included in Section 3.1. The rest of the paper is structured as follows. In Section 2 we state our results, both for single paths crossing a line and for pairs of paths crossing each other. In Section 3 we prove them in the case of single paths crossing a line, by introducing two-rowed arrays to encode paths, generalizing the notion of crossings to such arrays, and then describing certain bijections on them. In Section 4 we prove our results for pairs of paths crossing each other, by generalizing crossings to pairs of two-rowed arrays, and then defining bijections on such pairs.

2. Main results

2.1. **Paths crossing a line.** First we consider the enumeration of paths with U and D steps according to the number of times that they cross a fixed horizontal line. For integers ℓ, r with $r \geq 0$, let $\mathcal{G}_{a,b}^{\geq r,\ell}$ denote the set of paths in $\mathcal{G}_{a,b}$ that cross the line $y = \ell$ at least r times. A vertex of the path on the line $y = \ell$ is a *crossing* if it is either preceded and followed by a D —in which case it is called a *downward crossing*—, or preceded and followed by a U —called an *upward crossing*. See Figure 1 for an example.

We will provide expressions for the polynomials

$$G_{a,b}^{\geq r,\ell}(t,q) = \sum_{P \in \mathcal{G}_{a,b}^{\geq r,\ell}} t^{\operatorname{des}(P)} q^{\operatorname{maj}(P)}$$

for arbitrary integers a, b, r, ℓ with $a, b, r \geq 0$. Note that the polynomials for paths crossing the line $y = \ell$ *exactly* r times can be obtained from the above simply as $G_{a,b}^{\geq r,\ell}(t,q) - G_{a,b}^{\geq r+1,\ell}(t,q)$.

An expression for $G_{a,b}^{\geq r,\ell}(1,q)$ was given in [3, Thms. 2.1 and 2.2]. The following result refines these theorems by incorporating the statistic des.

Theorem 2.1. *Let $a, b, m \geq 0$, and let $\ell \in \mathbb{Z}$.*

I. *If $0 < \ell < a - b$, then*

$$G_{a,b}^{\geq 2m+1,\ell}(t,q) = G_{a,b}^{\geq 2m,\ell}(t,q) = \sum_{n \geq 0} t^n q^{n^2+m(m+\ell+1)} \begin{bmatrix} a \\ n-m \end{bmatrix}_q \begin{bmatrix} b \\ n+m \end{bmatrix}_q . \tag{2.1}$$

II. *If $0 > \ell > a - b$, then*

$$G_{a,b}^{\geq 2m+1,\ell}(t,q) = G_{a,b}^{\geq 2m,\ell}(t,q) = \sum_{n \geq 0} t^n q^{n^2+m(m-\ell-1)} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} b \\ n-m \end{bmatrix}_q . \tag{2.2}$$

III. *If $0 > \ell < a - b$, then*

$$G_{a,b}^{\geq 2m+2,\ell}(t,q) = G_{a,b}^{\geq 2m+1,\ell}(t,q) = \sum_{n \geq 0} t^n q^{n^2+(m+1)(m-\ell)} \begin{bmatrix} a-\ell-1 \\ n-m-1 \end{bmatrix}_q \begin{bmatrix} b+\ell+1 \\ n+m+1 \end{bmatrix}_q . \tag{2.3}$$

IV. *If $0 < \ell > a - b$, then*

$$G_{a,b}^{\geq 2m+2,\ell}(t,q) = G_{a,b}^{\geq 2m+1,\ell}(t,q) = \sum_{n \geq 0} t^n q^{n^2+m(m+\ell+1)} \begin{bmatrix} a-\ell-1 \\ n+m \end{bmatrix}_q \begin{bmatrix} b+\ell+1 \\ n-m \end{bmatrix}_q . \tag{2.4}$$

V. *If $0 = \ell < a - b$, then*

$$G_{a,b}^{\geq 2m,\ell}(t,q) = \sum_{n \geq 0} t^n q^{n^2+m(m+1)} \begin{bmatrix} a \\ n-m \end{bmatrix}_q \begin{bmatrix} b \\ n+m \end{bmatrix}_q , \tag{2.5}$$

$$G_{a,b}^{\geq 2m+1,\ell}(t,q) = \sum_{n \geq 0} t^n q^{n^2+m(m+1)} \begin{bmatrix} a-1 \\ n-m-1 \end{bmatrix}_q \begin{bmatrix} b+1 \\ n+m+1 \end{bmatrix}_q . \tag{2.6}$$

VI. *If* $0 = \ell > a - b$, *then*

$$G_{a,b}^{\geq 2m,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+m(m-1)} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} b \\ n-m \end{bmatrix}_q, \tag{2.7}$$

$$G_{a,b}^{\geq 2m+1,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+m(m+1)} \begin{bmatrix} a-1 \\ n+m \end{bmatrix}_q \begin{bmatrix} b+1 \\ n-m \end{bmatrix}_q. \tag{2.8}$$

VII. *If* $0 < \ell = a - b$, *then*

$$G_{a,b}^{\geq 2m,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+m(m+\ell+1)} \begin{bmatrix} a \\ n-m \end{bmatrix}_q \begin{bmatrix} b \\ n+m \end{bmatrix}_q, \tag{2.9}$$

$$G_{a,b}^{\geq 2m+1,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+m(m+\ell+1)} \begin{bmatrix} a+1 \\ n-m \end{bmatrix}_q \begin{bmatrix} b-1 \\ n+m \end{bmatrix}_q. \tag{2.10}$$

VIII. *If* $0 > \ell = a - b$, *then*

$$G_{a,b}^{\geq 2m,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+m(m-\ell-1)} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} b \\ n-m \end{bmatrix}_q, \tag{2.11}$$

$$G_{a,b}^{\geq 2m+1,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+(m+1)(m-\ell)} \begin{bmatrix} a+1 \\ n+m+1 \end{bmatrix}_q \begin{bmatrix} b-1 \\ n-m-1 \end{bmatrix}_q. \tag{2.12}$$

IX. *If* $0 = \ell = a - b$, *then*

$$G_{a,b}^{\geq 2m,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+m(m+1)} \frac{1-q^{a-2m}}{1-q^a} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} a \\ n-m \end{bmatrix}_q, \tag{2.13}$$

$$G_{a,b}^{\geq 2m+1,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+m(m+1)} \frac{1-q^{a+2(m+1)}}{1-q^a} \begin{bmatrix} a \\ n+m+1 \end{bmatrix}_q \begin{bmatrix} a \\ n-m-1 \end{bmatrix}_q. \tag{2.14}$$

2.2. **Pairs of paths crossing each other.** Next we consider the enumeration pairs of paths with respect to the number of crossings between them. For this problem it is convenient to consider paths with N and E steps. Let P and Q be two such paths, and suppose that $V_1, V_2, \dots, V_s$ (where $s \geq 1$) is a maximal sequence of consecutive common vertices such that

- neither V_1 nor V_s are endpoints of P or Q;
- for each of P and Q, its step arriving at V_1 is of the same type (N or E) as its step leaving V_s.

In this case, vertex V_s is called a *crossing* of P and Q. This definition differs slightly from the one used in [3], where the term crossing refers to the first vertex V_1 of the sequence. Of course, the number of crossings of P and Q does not depend on this convention, but defining the crossing to be V_s will be more convenient in the proofs in Section 4. Figure 2 shows some examples of crossings.

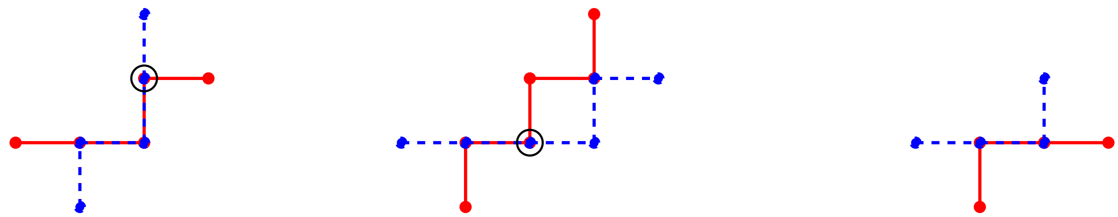


FIGURE 2. Two examples of crossings, circled in black, and a pair of paths that do not cross (right).

Let $\chi(P,Q)$ denote the number of crossings of P and Q; see Figure 3 for an example.

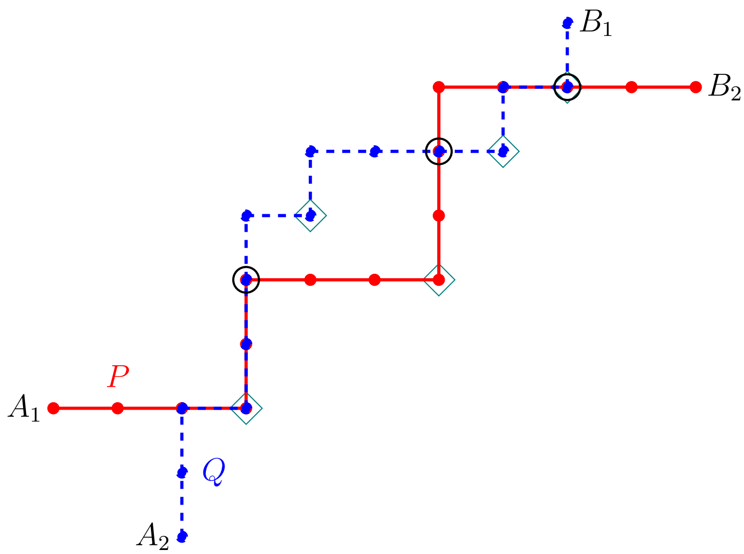


FIGURE 3. A pair of paths with $\chi(P,Q) = 3$, $\mathrm{des}(P) + \mathrm{des}(Q) = 6$, and $\mathrm{maj}(P) + \mathrm{maj}(Q) = 45$.

For $A_1, A_2, B_1, B_2 \in \mathbb{Z}^2$, $r \geq 0$, and $\{\circ, \bullet\} = \{1, 2\}$, let

$$\mathcal{P}^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet} = \{(P,Q) : P \in \mathcal{P}_{A_1\to B_\circ}, Q \in \mathcal{P}_{A_2\to B_\bullet}, \chi(P,Q) \geq r\}.$$

To enumerate such pairs of paths with respect to the sum of their numbers of descents (the *total descent number*) and the sum of their major indices (the *total major index*), we define the polynomials

$$H^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet}(t,q) = \sum_{(P,Q)\in\mathcal{P}^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet}} t^{\mathrm{des}(P)+\mathrm{des}(Q)} q^{\mathrm{maj}(P)+\mathrm{maj}(Q)}.$$

Note that the polynomials for pairs of paths that cross each other *exactly* r times are given by the difference $H^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet}(t,q) - H^{\geq r+1}_{A_1\to B_\circ, A_2\to B_\bullet}(t,q)$.

To state our formulas, let us first define the following polynomial in t and q that depends on the points $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $B_1 = (u_1, v_1)$, $B_2 = (u_2, v_2)$, and a parameter $k \in \mathbb{Z}$:

$$f_{k,A_1,A_2,B_2,B_1}(t,q)$$
$$= q^{k(k+x_2-x_1)} \left(\sum_{n \geq 0} t^n q^{n(n+k)} \begin{bmatrix} u_2 - x_1 \\ n \end{bmatrix}_q \begin{bmatrix} v_2 - y_1 \\ n+k \end{bmatrix}_q \right) \left(\sum_{n \geq 0} t^n q^{n(n-k)} \begin{bmatrix} u_1 - x_2 \\ n \end{bmatrix}_q \begin{bmatrix} v_1 - y_2 \\ n-k \end{bmatrix}_q \right).$$

We use the notation $A_1 \prec A_2$ to mean that $x_1 < x_2$ and $y_1 > y_2$. The theorem below refines [3, Thm. 2.4].

Theorem 2.2. *Let $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $B_1 = (u_1, v_1)$ and $B_2 = (u_2, v_2)$ be points in $\mathbb{Z}^2$ such that $A_1 \prec A_2$ and $B_1 \prec B_2$. Suppose additionally that*

$$x_1 + y_1 = x_2 + y_2. \tag{2.15}$$

Then, for all $m \geq 0$,

$$H^{\geq 2m+1}_{A_1 \to B_2, A_2 \to B_1}(t,q) = H^{\geq 2m}_{A_1 \to B_2, A_2 \to B_1}(t,q) = f_{2m,A_1,A_2,B_2,B_1}(t,q), \tag{2.16}$$

$$H^{\geq 2m+2}_{A_1 \to B_1, A_2 \to B_2}(t,q) = H^{\geq 2m+1}_{A_1 \to B_1, A_2 \to B_2}(t,q) = f_{2m+1,A_1,A_2,B_2,B_1}(t,q). \tag{2.17}$$

Let now $A = (x, y)$ and $B = (u, v)$ be points in $\mathbb{Z}^2$. Then, for all $r \geq 0$,

$$H^{\geq r}_{A \to B_1, A \to B_2}(t,q) = f_{r,A,A,B_2,B_1}(t,q), \tag{2.18}$$

$$H^{\geq r}_{A_1 \to B, A_2 \to B}(t,q) = f_{r,A_1,A_2,B,B}(t,q), \tag{2.19}$$

$$H^{\geq r}_{A \to B, A \to B}(t,q) = \begin{cases} f_{0,A,A,B,B}(t,q) & \text{if } r = 0, \\ 2\sum_{j \geq 1} (-1)^{j-1} f_{r+j,A,A,B,B}(t,q) & \text{if } r \geq 1. \end{cases} \tag{2.20}$$

Let us now detail the proofs of these results.

3. Proofs for paths crossing a line

In this section we prove Theorem 2.1. Before diving into the details, we remark that it would be possible to give an alternative proof by induction on the length (number of steps) of the path, by first separating each of the nine cases of the theorem into two subcases, according to whether the last step of the path is a U or a D. For example, if $0 < \ell < a - b$, the refinement to be proved by induction would state that the generating function for paths in $\mathcal{G}_{a,b}^{\geq r,\ell}$ that end with a D, where $r = 2m$ or $r = 2m + 1$, equals

$$G_{a,b-1}^{\geq r,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+m(m+1+\ell)} \begin{bmatrix} a \\ n-m \end{bmatrix}_q \begin{bmatrix} b-1 \\ n+m \end{bmatrix}_q,$$

and so the generating function for those that end with a U equals

$$G_{a,b}^{\geq r,\ell}(t,q) - G_{a,b-1}^{\geq r,\ell}(t,q) = \sum_{n\geq 0} t^n q^{n^2+m(m+\ell)+b-n} \begin{bmatrix} a \\ n-m \end{bmatrix}_q \begin{bmatrix} b-1 \\ n+m-1 \end{bmatrix}_q.$$

Then, to prove each one of these formulas, we would remove the last step of the path, and deduce them from the formulas for shorter paths that hold by the induction hypothesis. This often requires additional subcases; for example, for the above paths ending in U, the cases $\ell + 1 < a - b$ and $\ell + 1 = a - b$ would be considered separately.

Instead of such a tedious induction proof, we have chosen to present a proof that relies on certain two-rowed arrays that have been used by Krattenthaler and Mohanty [15]. One advantage of our proof is that it is bijective. Additionally, the methodology of two-rowed arrays that we introduce here will later allow us to prove Theorem 2.2 for pairs of paths, where a potential proof by induction is much less clear.

3.1. **Two-rowed arrays.** Let $x, y, u, v, k \in \mathbb{Z}$ and $n, j \geq 0$ throughout the section. We use the notation

$$\begin{aligned}
(x,u]_j &= \{(c_1,\ldots,c_j) : x < c_1 < c_2 < \cdots < c_j \leq u\},\\
[y,v)_j &= \{(d_1,\ldots,d_j) : y \leq d_1 < d_2 < \cdots < d_j < v\},\\
(x,v)_j &= \{(c_1,\ldots,c_j) : x < c_1 < c_2 < \cdots < c_j < v\},\\
[y,u]_j &= \{(d_1,\ldots,d_j) : y \leq d_1 < d_2 < \cdots < d_j \leq u\}.
\end{aligned}$$

We consider pairs of such sequences arranged in a particular way, which we call *two-rowed arrays*, following [11–13, 15]. We denote by $\left\{\begin{smallmatrix}(x,u]_{n+k}\\ [y,v)_{n-k}\end{smallmatrix}\right\}$, or $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}$ for short, the set of arrays of the form

$$\begin{array}{ccccccccccccc}
 & & x & < & c_1 & < & c_2 & < & \cdots & < & c_{n+k} & \leq & u\\
y & \leq & d_1 & < & d_2 & < & \cdots & < & d_{n-k} & < & v & &
\end{array},$$

with the convention that this set is empty unless $|k| \leq n$. The two rows are interlaced from the left, starting with the leftmost element in the bottom row. Elements in this set are denoted by $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$, where $\mathbf{c} = (c_1,\ldots,c_{n+k}) \in (x,u]_{n+k}$ and $\mathbf{d} = (d_1,\ldots,d_{n-k}) \in [y,v)_{n-k}$.

Similarly, we denote by $\left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\pm k}$ the set of arrays of the form

$$\begin{array}{ccccccccccccc}
 & & x & < & c_1 & < & c_2 & < & \cdots & < & c_{n+k} & < & v\\
y & \leq & d_1 & < & d_2 & < & \cdots & < & d_{n-k} & \leq & u & &
\end{array}.$$

The reason two-rowed arrays are useful for our problem is that elements of $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm 0}$, which we denote simply by $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n}$, encode lattice paths in $\mathcal{P}_{(x,y)\to(u,v)}$. This is because such paths are uniquely determined by the coordinates of their valleys. There exists a path in $\mathcal{P}_{(x,y)\to(u,v)}$ whose valleys are at coordinates $(c_1,d_1),(c_2,d_2),\dots,(c_n,d_n)$ if and only if

$$x < c_1 < c_2 < \cdots < c_n \le u \quad \text{and} \quad y \le d_1 < d_2 < \cdots < d_n < v,$$

that is, $\mathbf{c} = (c_1,\dots,c_n) \in (x,u]_n$ and $\mathbf{d} = (d_1,\dots,d_n) \in [y,v)_n$. Thus, this encoding is a bijection

$$\{P \in \mathcal{P}_{(x,y)\to(u,v)} : \operatorname{des}(P) = n\} \to \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n}. \tag{3.1}$$

It has the property that, if P is encoded by $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$, then

$$\operatorname{maj}(P) = \sum_{i=1}^{n}(c_i + d_i - x - y) = \|\mathbf{c}\| + \|\mathbf{d}\| - n(x+y), \tag{3.2}$$

where $\|\mathbf{c}\|$ denotes the sum of the entries of $\mathbf{c}$. Next we enumerate two-rowed arrays with respect to this statistic.

Lemma 3.1. *(i) We have*

$$\sum_{\mathbf{c}\in(x,u]_j} q^{\|\mathbf{c}\|} = q^{\binom{j+1}{2}+jx}\begin{bmatrix}u-x\\ j\end{bmatrix}_q, \qquad \sum_{\mathbf{d}\in[y,v)_j} q^{\|\mathbf{d}\|} = q^{\binom{j+1}{2}+j(y-1)}\begin{bmatrix}v-y\\ j\end{bmatrix}_q,$$

$$\sum_{\mathbf{c}\in(x,v)_j} q^{\|\mathbf{c}\|} = q^{\binom{j+1}{2}+jx}\begin{bmatrix}v-x-1\\ j\end{bmatrix}_q, \qquad \sum_{\mathbf{d}\in[y,u]_j} q^{\|\mathbf{d}\|} = q^{\binom{j+1}{2}+j(y-1)}\begin{bmatrix}u-y+1\\ j\end{bmatrix}_q.$$

(ii) We have

$$\sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\in\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} = q^{n^2+k(k+x-y+1)}\begin{bmatrix}u-x\\ n+k\end{bmatrix}_q\begin{bmatrix}v-y\\ n-k\end{bmatrix}_q, \tag{3.3}$$

$$\sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\in\left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\pm k}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} = q^{n^2+k(k+x-y+1)}\begin{bmatrix}v-x-1\\ n+k\end{bmatrix}_q\begin{bmatrix}u-y+1\\ n-k\end{bmatrix}_q. \tag{3.4}$$

Proof. We prove the first identity in part (i), since the other three are analogous. Writing $c_i' = c_i - i - x$ for $1 \le i \le j$, the left-hand side is equal to

$$\sum_{x<c_1<c_2<\cdots<c_j\le u} q^{c_1+\cdots+c_j} = q^{\binom{j+1}{2}+jx}\sum_{0\le c_1'\le c_2'\le\cdots\le c_j'\le u-x-j} q^{c_1'+\cdots+c_j'}.$$

This sum counts partitions with at most j parts with largest part at most $u-x-j$, which is a well-known interpretation of the q-binomial coefficients (see e.g. [1, Thm. 3.1]).

Part (ii) follows easily from part (i) using the simplification

$$\binom{n+k+1}{2} + \binom{n-k+1}{2} + (n+k)x + (n-k)(y-1) - n(x+y) = n^2 + k(k+x-y+1)$$

in the exponent of q. $\square$

To see how Lemma 3.1 will be applied, let us first use it to give a proof of Lemma 1.1.

Proof of Lemma 1.1. The two statements are clearly equivalent, so we prove the second one. Using the encoding (3.1), together with Equations (3.2) and (3.3) for $k=0$, we get

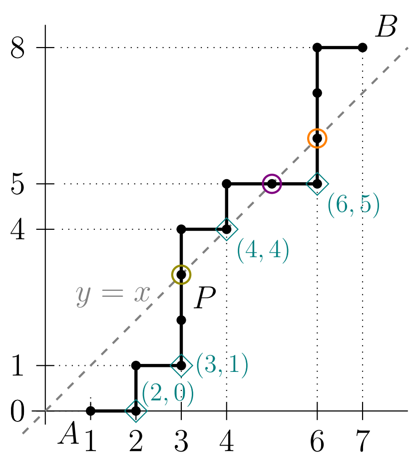
$$\sum_{P\in\mathcal{P}_{(x,y)\to(u,v)}} t^{\operatorname{des}(P)}q^{\operatorname{maj}(P)} = \sum_{n\geq 0} t^n \sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x,u]\\ [y,v)}\right\}_n} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} = \sum_{n\geq 0} t^n q^{n^2} \begin{bmatrix} u-x \\ n\end{bmatrix}_q \begin{bmatrix} v-y \\ n\end{bmatrix}_q .$$

□

3.2. **Crossings in single two-rowed arrays.** To encode paths in $\mathcal{G}_{a,b}$ as two-rowed arrays, we first turn the U and D steps into N and E steps, respectively. Additionally, to study crossings of the line $y=\ell$ in the original path, we move the starting point to $(\ell,0)$, so that these crossings become crossings of the diagonal $y=x$ for the resulting path. Denoting by $\mathcal{P}^{\geq r}_{A\to B}$ the set of paths in $\mathcal{P}_{A\to B}$ that cross the diagonal at least r times, this transformation is a bijection

$$\mathcal{G}^{\geq r,\ell}_{a,b} \to \mathcal{P}^{\geq r}_{(\ell,0)\to(b+\ell,a)}. \tag{3.5}$$

See Figure 4 for an example. In analogy to the definitions for paths in $\mathcal{G}_{a,b}$ crossing a line $y=\ell$, we define upward (resp. downward) crossings of paths in $\mathcal{P}_{A\to B}$ to be vertices in the diagonal $y=x$ that are preceded and followed by an N (resp. by an E).



$$\longleftrightarrow \quad \begin{matrix} 1<2<3<4<6\leq 7 \\ 0\leq 0<1<4<5<8 \end{matrix} \in \left\{\substack{(1,7]\\ [0,8)}\right\}_4$$

FIGURE 4. The path in $\mathcal{P}^{\geq 3}_{(1,0)\to(7,8)}$ obtained by applying the transformation (3.5) to the path in Figure 1, and the corresponding two-rowed array given by the encoding (3.1), where the crossings have been circled.

Next we show how these crossings of the diagonal can be read from the encoding (3.1) of the path as a two-rowed array. Indeed, suppose that $P\in\mathcal{P}_{(x,y)\to(u,v)}$ is encoded by $\substack{\mathbf{c}\\ \mathbf{d}} \in \left\{\substack{(x,u]\\ [y,v)}\right\}_n$, and let $c_0:=x$, $d_0:=y$, $c_{n+1}:=u$, $d_{n+1}:=v$ by convention. An upward crossing of P occurs when, for some $0\leq i\leq n$, the vertex (c_i,d_i) —which is a valley or the first vertex of the path— lies below the diagonal and the vertex (c_i,d_{i+1}) —which is a peak or the last vertex of the path— lies above the diagonal. This happens precisely when $d_i<c_i<d_{i+1}$ for some $0\leq i\leq n$. Similarly, a downward crossing occurs when, for some $1\leq i\leq n+1$, the vertex (c_{i-1},d_i) —which is a peak or the starting point of the path— lies above the diagonal and the vertex (c_i,d_i) —which is a valley or the last vertex of the path— lies below the diagonal. This happens precisely when $c_{i-1}<d_i<c_i$ for some $1\leq i\leq n+1$.

This description allows us to extend the notion of crossings to two-rowed arrays $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}$ with $k \in \mathbb{Z}$, whose rows may have different lengths. Using the convention $c_0 := x$, $d_0 := y$, $c_{n+k+1} := u$, $d_{n-k+1} := v$, say that $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ has an *upward crossing* at c_i if $0 \le i \le n-|k|$ and

$$d_i < c_i < d_{i+1}, \tag{3.6}$$

and that it has a *downward crossing* at d_i if $1 \le i \le n-|k|+1$ and

$$c_{i-1} < d_i < c_i.$$

For two-rowed arrays of the form $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\pm k}$, the definition of upward and downward crossings is the same, now using the convention $c_0 := x$, $d_0 := y$, $c_{n+k+1} := v$, $d_{n-k+1} := u$. Figure 5 shows two examples, where the crossings have been circled. As usual, the term *crossings* refers to both upward and downward crossings.

$$\begin{matrix} 0 < 1 < 4 < 5 < 7 \le 7 \\ 2 \le 2 < 3 < 5 \end{matrix} \in \left\{\begin{smallmatrix}(0,7)\\ [2,5)\end{smallmatrix}\right\}_{3\pm 1} \qquad \begin{matrix} 0 < 3 < 4 < 6 \\ 0 \le 1 < 2 < 5 < 7 \le 8 \end{matrix} \in \left\{\begin{smallmatrix}(0,6)\\ [0,8]\end{smallmatrix}\right\}_{3\mp 1}$$

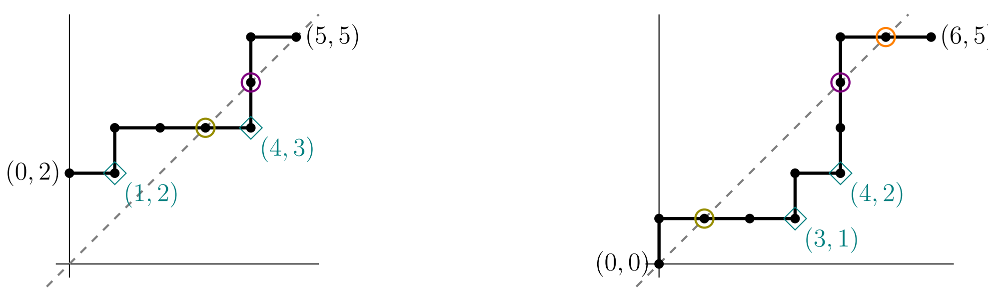


FIGURE 5. Two two-rowed arrays $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ with their crossings circled, and the corresponding paths $T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix})$. Note that for the array on the right, $c_3 = 6$ is not a crossing because it violates the condition $i \le n - |k|$.

In both of the above cases, let $T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}) \in \mathcal{P}_{(x,y)\to(c_{n-|k|+1},d_{n-|k|+1})}$ be the path whose valleys are at coordinates (c_i, d_i) for $1 \le i \le n - |k|$ (with the caveat that, in the special case when $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_n$ and $d_n = u$, the vertex (c_n, d_n) is not actually a valley of this path). Then the upward and downward crossings of the two-rowed array $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ can be identified with the upward and downward crossings of $T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix})$; see the examples in Figure 5. Note that $T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix})$ is essentially the path corresponding to the two-rowed array obtained by truncating the longer row of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ so that both rows have equal length. To be precise, this path depends not only on $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ but also on the endpoints x, y, u, v,

Throughout the paper, the *rth crossing* of a two-rowed array refers to the rth crossing *from the left*, in the order in which the entries are placed, namely $y, x, d_1, c_1, d_2, c_2, \ldots$ We note that this convention is different from the one used in [3], where path crossings were numbered from the right. The unusual convention in [3] was needed because the path bijections in that paper, in order to track the major index, changed the portion of the paths to the *left* of a crossing. On the other hand, the notation in this paper becomes slightly simpler by defining bijections for two-rowed arrays (in Sections 3.3 and 4.3) that change the portion of the arrays to the *right* of a crossing instead.

For nonnegative r, the superscript $\geq r$ on a set of two-rowed arrays denotes the subset of those that have at least r crossings. When $r \geq 1$, a symbol $\uparrow$ (resp. $\downarrow$) next to this superscript denotes the subset where the rth crossing is an upward (resp. downward) crossing. For example, $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}^{\geq r\uparrow}$ consists of two-rowed arrays in $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}^{\geq r}$ where the rth crossing is an upward crossing. In the case $r = 0$, we simply define

$$\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}^{\geq 0\uparrow} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}^{\geq 0\downarrow} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}^{\geq 0} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k} \tag{3.7}$$

by convention.

The encoding (3.1) restricts to a bijection

$$\{P \in \mathcal{P}^{\geq r}_{(x,y)\to(u,v)} : \operatorname{des}(P) = n\} \to \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n}^{\geq r}. \tag{3.8}$$

Composing this with the bijection (3.5) and using Equation (3.2), it follows that

$$G^{\geq r,\ell}_{a,b}(t,q) = \sum_{n\geq 0} t^n \sum_{\substack{\mathbf{c}\\ \mathbf{d}} \in \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n}^{\geq r}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)}, \tag{3.9}$$

where $(x,y) = (\ell,0)$ and $(u,v) = (b+\ell,a)$.

To prove Theorem 2.1, we will construct bijections between $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n}^{\geq r}$ and sets of the form $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}$ or $\left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\pm k}$ for some $k \in \mathbb{Z}$, which will depend on the relations between x and y and between u and v, and then apply Lemma 3.1.

Lemma 3.2. *Let $r \geq 1$, and let $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ be a two-rowed array in either $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k}^{\geq r}$ or $\left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\pm k}^{\geq r}$.*
If $x > y$ or $x = y = d_1$, then the rth crossing of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ is an upward crossing if r is odd, and a downward crossing if r is even.
If $x < y$ or $x = y < d_1$, then the rth crossing of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ is a downward crossing if r is odd, and an upward crossing if r is even.

Proof. As noted above, upward and downward crossings of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ are the same as those of the path $T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}) \in \mathcal{P}_{(x,y)\to(c_{n-|k|+1},d_{n-|k|+1})}$. If $x > y$ (resp. $x < y$), this path starts below (resp. above) the diagonal, which forces the first crossing to be upward (resp. downward), with successive crossings alternating between upward and downward. If $x = y$, then $T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix})$ starts with an E if $y = d_1$, and with an N if $y < d_1$, from which the same conclusions follow. $\square$

The next lemma shows that the relationships between x and y and between u and v often force the number of crossings of a two-rowed array to have a given parity. We use the notation $n \mp s$ to mean $n \pm (-s)$.

Lemma 3.3. *Let $s, m \geq 0$.*

(a) *If $x > y$ and $u < v$, then*

$$\left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+1} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+1\uparrow} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m\downarrow} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m}, \tag{3.10}$$

$$\left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+2} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+2\downarrow} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+1\uparrow} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+1}. \tag{3.11}$$

(b) *If $x > y$ and $u > v$, then*

$$\left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+2} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+2\downarrow} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+1\uparrow} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+1}, \tag{3.12}$$

$$\left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+1} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+1\uparrow} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m\downarrow} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m}. \tag{3.13}$$

(c) *If $x < y$ and $u < v$, then*

$$\left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+2} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+2\uparrow} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+1\downarrow} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+1}, \tag{3.14}$$

$$\left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+1} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+1\downarrow} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m\uparrow} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m}. \tag{3.15}$$

(d) *If $x < y$ and $u > v$, then*

$$\left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+1} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m+1\downarrow} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m\uparrow} = \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\mp s}^{\geq 2m}, \tag{3.16}$$

$$\left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+2} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+2\uparrow} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+1\downarrow} = \left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\pm s}^{\geq 2m+1}. \tag{3.17}$$

(e) *If $x > y$ and $u = v$, then* (3.10)–(3.12) *hold for $s \geq 1$, and* (3.13) *holds for $s \geq 0$.*

(f) *If $x < y$ and $u = v$, then* (3.14)–(3.16) *hold for $s \geq 1$, and* (3.17) *holds for $s \geq 0$.*

(g) *Statements* (a), (b), (e) *also hold if we replace $x > y$ with $x = y$ and restrict to two-rowed arrays $\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix}$ with $y = d_1$.*

Proof. In each equation, the outer equalities follow from Lemma 3.2 (using the convention (3.7) as needed), and the left-hand side is trivially contained in the right-hand side. To prove the reverse containment, we will show that the parity of the number of crossings of the relevant two-rowed arrays is determined by the relation between x and y and between u and v in each case.

Recall that if $\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix}$ is a two-rowed array in either $\left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm k}$ or $\left\{\begin{smallmatrix}(x,v) \\ [y,u]\end{smallmatrix}\right\}_{n\pm k}$, for some $k \in \mathbb{Z}$, then $T(\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix})$ is a path from (x, y) to $(c_{n-|k|+1}, d_{n-|k|+1})$ which has the same upward and downward crossings as $\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix}$. The parity of the number of crossings is determined by what side of the diagonal the endpoints of the path are on. If $x > y$, $T(\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix})$ starts below the diagonal; if $x < y$, it starts above the diagonal; and if $x = y = d_1$, it starts with an E leaving the diagonal, so it behaves as in the $x > y$ case.

Suppose first that $\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,u] \\ [y,v)\end{smallmatrix}\right\}_{n\pm s}$, where $s \geq 0$ and $u < v$. Then the last vertex of $T(\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix})$ is (c_{n-s+1}, v), which lies above the diagonal, since $c_{n-s+1} \leq u < v$. Thus, if $x > y$, then $T(\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix})$ starts below the diagonal and ends above the diagonal, so it must have an odd number of crossings, proving Equation (3.10). If $x = y = d_1$, the same conclusion holds. On the other hand, if $x < y$, then $T(\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix})$ starts and ends above the diagonal, so it must have an even number of crossings, proving Equation (3.14). Modifying the hypotheses so that $s \geq 1$ and $u \geq v$, the last vertex of $T(\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix})$ still lies above the diagonal, since $c_{n-s+1} < c_{n-s+2} \leq u \leq v$, so Equations (3.10) and (3.14) also hold in this case.

If $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{ \genfrac{}{}{0pt}{}{(x,v)}{[y,u]} \right\}_{n\mp s}$, where $s \geq 0$ and $u < v$, then the last vertex of $T\binom{\mathbf{c}}{\mathbf{d}}$ is (v, d_{n-s+1}), which lies below the diagonal, since $d_{n-s+1} \leq u < v$. Thus, $T\binom{\mathbf{c}}{\mathbf{d}}$ must have an even number of crossings if $x > y$ or $x = y = d_1$, proving Equation (3.11), and an odd number of crossings if $x < y$, proving Equation (3.15). These two equations still hold with the modified hypotheses $s \geq 1$ and $u \leq v$, since $d_{n-s+1} < d_{n-s+2} \leq u \leq v$ in this case, so the last vertex of $T\binom{\mathbf{c}}{\mathbf{d}}$ still lies below the diagonal.

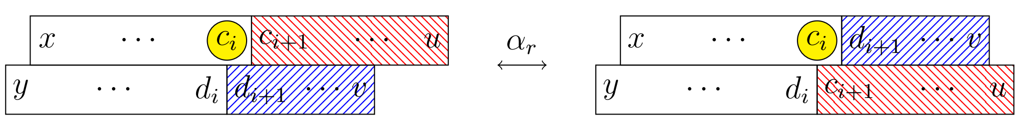
If $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{ \genfrac{}{}{0pt}{}{(x,u]}{[y,v)} \right\}_{n\mp s}$, where $s \geq 0$ and $u > v$, then the last vertex of $T\binom{\mathbf{c}}{\mathbf{d}}$ is (u, d_{n-s+1}), which lies below the diagonal, since $d_{n-s+1} \leq v < u$. This vertex also lies below the diagonal when $s \geq 1$ and $u \geq v$, since $d_{n-s+1} < d_{n-s+2} \leq v \leq u$. This proves Equations (3.12) and (3.16).

If $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{ \genfrac{}{}{0pt}{}{(x,v)}{[y,u]} \right\}_{n\pm s}$, where $s \geq 0$ and $u > v$, then the last vertex of $T\binom{\mathbf{c}}{\mathbf{d}}$ is (c_{n-s+1}, u), which lies above the diagonal, since $c_{n-s+1} \leq v < u$. This vertex also lies above the diagonal when $s \geq 1$ and $u \geq v$, since $c_{n-s+1} < c_{n-s+2} \leq v \leq u$. Finally, when $s = 0$ and $u = v$, the path $T\binom{\mathbf{c}}{\mathbf{d}}$ ends on the diagonal (at (v, u)), but its last step is an E step, since $c_{n-s} < v$. This proves Equations (3.13) and (3.17) for all $s \geq 0$ and $u \geq v$. □

3.3. **The bijections α_r and β_r.** We are almost ready to define the key bijections α_r and β_r. These are reminiscent of the bijections σ_r and τ_r defined in [3] for paths. An important difference, however, is that the image by β_r of a two-rowed array that encodes a path does not encode a path in general, so one cannot view β_r as a map on paths.

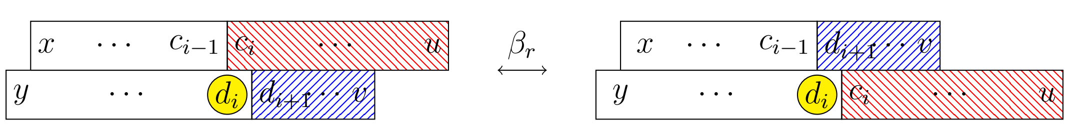
Let $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}$ be a two-rowed array in either $\left\{ \genfrac{}{}{0pt}{}{(x,u]}{[y,v)} \right\}_{n\pm k}$ or $\left\{ \genfrac{}{}{0pt}{}{(x,v)}{[y,u]} \right\}_{n\pm k}$. We say that a crossing of $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}$ at c_i (resp. d_i) is *proper* if $c_i \notin \{u, v\}$ (resp. $d_i \notin \{u, v\}$).

For $r \geq 1$, the map α_r applies to two-rowed arrays $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}$ whose rth crossing is a proper upward crossing, and it swaps the parts of the top and the bottom rows of the array to the right of this crossing. Schematically, if the rth crossing is at c_i, we have



The properness of the crossing guarantees that c_{i+1} exists and that $c_i < c_{i+1}$. Additionally, we have $c_i < d_{i+1}$ and $d_i < c_{i+1}$, so the rows of $\alpha_r\binom{\mathbf{c}}{\mathbf{d}}$ are increasing. The two-rowed array $\alpha_r\binom{\mathbf{c}}{\mathbf{d}}$ has a crossing at c_i, since $d_i < c_i < c_{i+1}$, and this crossing is still proper. This is in fact the rth crossing of $\alpha_r\binom{\mathbf{c}}{\mathbf{d}}$, because the portion of the arrays to the left of c_i is not affected by α_r. It follows that α_r is an involution.

Similarly, the map β_r applies to two-rowed arrays $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}$ whose rth crossing is a proper downward crossing, and it also swaps the top and the bottom rows of the array to the right of this crossing. Schematically, if the rth crossing is at d_i, we have



Again, the rth crossing of $\beta_r\binom{\mathbf{c}}{\mathbf{d}}$ is still at d_i and is a proper crossing, and the map β_r is an involution.

Lemma 3.4. *Let $x, y, u, v, k \in \mathbb{Z}$, $n \geq 0$ and $r \geq 1$, satisfying that, if $u > v$, then $k \leq 0$, and if $u < v$, then $k \geq 1$. The map α_r restricts to a bijection*

$$\left\{ {(x,u) \atop [y,v]} \right\}^{\geq r\uparrow}_{n\pm k} \overset{\alpha_r}{\longleftrightarrow} \left\{ {(x,v) \atop [y,u]} \right\}^{\geq r\uparrow}_{n\mp k},$$

and the map β_r restricts to a bijection

$$\left\{ {(x,u) \atop [y,v]} \right\}^{\geq r\downarrow}_{n\pm(k-1)} \overset{\beta_r}{\longleftrightarrow} \left\{ {(x,v) \atop [y,u]} \right\}^{\geq r\downarrow}_{n\mp k}.$$

Both α_r and β_r preserve the sum of the entries of the arrays.

Proof. The conditions on k, which depend on the relationship between u and v, guarantee that the rth crossing of a two-rowed array in any of the four sets above is always proper, and so the maps α_r and β_r are defined. Indeed, an improper upward crossing of ${\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,u) \atop [y,v]} \right\}_{n\pm k}$ at c_i could only occur if $u = c_i < d_{i+1} \leq v$ and $k \leq 0$. Arrays ${\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,v) \atop [y,u]} \right\}_{n\mp k}$ cannot have improper upward crossings, since $c_i = v$ is incompatible with $i \leq n - |k|$. An improper downward crossing of ${\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,u) \atop [y,v]} \right\}_{n\pm(k-1)}$ at d_i could only occur if $v = d_i < c_i \leq u$ and $k - 1 \geq 0$. And an improper downward crossing of ${\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,v) \atop [y,u]} \right\}_{n\mp k}$ at d_i could only occur if $u = d_i < c_i \leq v$ and $k \leq 0$.

Having already seen that α_r and β_r are involutions, it remains to describe their images. Given ${\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,u) \atop [y,v]} \right\}^{\geq r\uparrow}_{n\pm k}$ whose rth crossing is at c_i, if we write ${\mathbf{c} \atop \mathbf{d}}$ as

$$\begin{array}{l} x < c_1 < c_2 < \cdots < c_i < c_{i+1} < \quad \cdots \quad < c_{n+k} \leq u \\ y \leq d_1 < d_2 < \cdots < d_i < d_{i+1} < d_{i+2} < \cdots < d_{n-k} < v \end{array},$$

then $\alpha_r({\mathbf{c} \atop \mathbf{d}})$ is the two-rowed array

$$\begin{array}{l} x < c_1 < c_2 < \cdots < c_i < d_{i+1} < d_{i+2} < \cdots < d_{n-k} < v \\ y \leq d_1 < d_2 < \cdots < d_i < c_{i+1} < \quad \cdots \quad < c_{n+k} \leq u \end{array},$$

which has an upward crossing at c_i and thus belongs to $\left\{ {(x,v) \atop [y,u]} \right\}^{\geq r\uparrow}_{n\mp k}$.

Similarly, given ${\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,u) \atop [y,v]} \right\}^{\geq r\downarrow}_{n\pm(k-1)}$ whose rth crossing is at d_i, if we write ${\mathbf{c} \atop \mathbf{d}}$ as

$$\begin{array}{l} x < c_1 < c_2 < \cdots < c_{i-1} < c_i < c_{i+1} < \quad \cdots \quad < c_{n+k-1} \leq u \\ y \leq d_1 < d_2 < \quad \cdots \quad < d_i < d_{i+1} < \cdots < d_{n-k+1} < v \end{array},$$

then $\beta_r({\mathbf{c} \atop \mathbf{d}})$ is the two-rowed array

$$\begin{array}{l} x < c_1 < c_2 < \cdots < c_{i-1} < d_{i+1} < \cdots < d_{n-k+1} < v \\ y \leq d_1 < d_2 < \quad \cdots \quad < d_i < c_i < c_{i+1} < \quad \cdots \quad < c_{n+k-1} \leq u \end{array},$$

which has a downward crossing at d_i and thus belongs to $\left\{ {(x,v) \atop [y,u]} \right\}^{\geq r\downarrow}_{n\mp k}$.

It is clear by construction that both α_r and β_r preserve the sum of the entries of the arrays. □

3.4. **Proof of Theorem 2.1.** For $a,b,r\geq 0$ and $\ell\in\mathbb{Z}$, we interpret elements of $\mathcal{G}^{\geq r,\ell}_{a,b}$ as paths in $\mathcal{P}^{\geq r}_{(x,y)\to(u,v)}$, where $(x,y)=(\ell,0)$ and $(u,v)=(b+\ell,a)$, using the transformation (3.5). For any $n\geq 0$, the subset of paths having n descents is in bijection with the set $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq r}_n$, using the encoding (3.8).

The proof is divided into nine cases according to whether the paths start below ($0<\ell$, equivalently $x>y$), on ($0=\ell$, equivalently $x=y$), or above ($0>\ell$, equivalently $x<y$) the line being crossed, and whether they end below ($\ell>a-b$, equivalently $u>v$), on ($\ell=a-b$, equivalently $u=v$), or above ($\ell<a-b$, equivalently $u<v$) this line. In each case, we determine $G^{\geq r,\ell}_{a,b}(t,q)$ by first using Equation (3.9) to rewrite it in terms of two-rowed arrays, then repeatedly applying the maps from Lemma 3.4 to construct bijections between $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq r}_n$ and certain sets of two-rowed arrays with no requirement on the number of crossings, and finally using Lemma 3.1. The cases are labeled as in [3] for consistency, but we will prove them in a slightly different order.

Case I: $0<\ell<a-b$, equivalently $x>y$ and $u<v$. By Equation (3.10) with $s=0$,

$$\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1}_n=\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m}_n,$$

and so $G^{\geq 2m+1,\ell}_{a,b}(t,q)=G^{\geq 2m,\ell}_{a,b}(t,q)$. Using Lemmas 3.3(a) and 3.4, noting that the condition $k\geq 1$ in the latter holds at each step, we construct a composition of bijections $\alpha_1\circ\beta_2\circ\cdots\circ\alpha_{2m-1}\circ\beta_{2m}$:

$$\begin{aligned}\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m}_n=\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m\downarrow}_n\xrightarrow{\beta_{2m}}\left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 2m\downarrow}_{n\mp 1}=\left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 2m-1\uparrow}_{n\mp 1}\xrightarrow{\alpha_{2m-1}}\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m-1\uparrow}_{n\pm 1}=\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m-2\downarrow}_{n\pm 1}\\ \xrightarrow{\beta_{2m-2}}\cdots\xrightarrow{\alpha_1}\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 1\uparrow}_{n\pm m}=\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n\pm m}=\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm m}.\end{aligned}\tag{3.18}$$

See Figure 6 for an example. Since these bijections preserve the sum of the entries of the two-rowed arrays, Equation (3.3) gives

$$\begin{aligned}\sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m}_n}q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)}&=\sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm m}}q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)}\\ &=q^{n^2+m(m+x-y+1)}\begin{bmatrix}u-x\\ n+m\end{bmatrix}_q\begin{bmatrix}v-y\\ n-m\end{bmatrix}_q=q^{n^2+m(m+\ell+1)}\begin{bmatrix}a\\ n-m\end{bmatrix}_q\begin{bmatrix}b\\ n+m\end{bmatrix}_q.\end{aligned}\tag{3.19}$$

Using Equation (3.9), this proves Equation (2.1).

$$\left\{\begin{smallmatrix}(1,7]\\ [0,8)\end{smallmatrix}\right\}^{\geq 2}_4=\left\{\begin{smallmatrix}(1,7]\\ [0,8)\end{smallmatrix}\right\}^{\geq 2\downarrow}_4\xrightarrow{\beta_2}\left\{\begin{smallmatrix}(1,8)\\ [0,7]\end{smallmatrix}\right\}^{\geq 2\downarrow}_{4\mp 1}=\left\{\begin{smallmatrix}(1,8)\\ [0,7]\end{smallmatrix}\right\}^{\geq 1\uparrow}_{4\mp 1}\xrightarrow{\alpha_1}\left\{\begin{smallmatrix}(1,7]\\ [0,8)\end{smallmatrix}\right\}^{\geq 1\uparrow}_{4\pm 1}=\left\{\begin{smallmatrix}(1,7]\\ [0,8)\end{smallmatrix}\right\}_{4\pm 1}$$

$$\begin{matrix}1<2<3<4<6\leq 7 & \qquad 1<2<3<4<8 & \qquad 1<2<3<4<5<6\leq 7\\ 0\leq 0<1<4<5<8 & \qquad 0\leq 0<1<4<5<6\leq 7 & \qquad 0\leq 0<1<4<8\end{matrix}$$

FIGURE 6. An example of the bijection (3.18), where $(x,y)=(1,0)$, $(u,v)=(7,8)$, $m=1$ and $n=4$.

Case II: $0 > \ell > a-b$, equivalently $x < y$ and $u > v$. Similarly to Case I, the equality $G^{\geq 2m+1,\ell}_{a,b}(t,q) = G^{\geq 2m,\ell}_{a,b}(t,q)$ follows now from Equation (3.16) with $s = 0$. Again Lemmas 3.3(d) and 3.4, noting that the condition $k \leq 0$ holds at each step, allow us to build a sequence of bijections $\beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m-1} \circ \alpha_{2m}$:

$$\begin{aligned}
&\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m}_n = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_n \\
\xrightarrow{\alpha_{2m}}\ &\left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 2m\uparrow}_n = \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_n \quad \xrightarrow{\beta_{2m-1}} \quad \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_{n\mp 1} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m-2\uparrow}_{n\mp 1} \\
\xrightarrow{\alpha_{2m-2}}\ &\cdots \quad \xrightarrow{\beta_1} \quad \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 1\downarrow}_{n\mp m} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n\mp m} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\mp m}.
\end{aligned}$$

Then, by Equation (3.3),

$$\begin{aligned}
\sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m}_n} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} &= \sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\mp m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} \\
&= q^{n^2-m(-m+x-y+1)} \begin{bmatrix} u-x \\ n-m \end{bmatrix}_q \begin{bmatrix} v-y \\ n+m \end{bmatrix}_q = q^{n^2+m(m-\ell-1)} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} b \\ n-m \end{bmatrix}_q, \qquad (3.20)
\end{aligned}$$

proving Equation (2.2).

Case III: $0 > \ell < a-b$, equivalently $x < y$ and $u < v$. The equality $G^{\geq 2m+2,\ell}_{a,b}(t,q) = G^{\geq 2m+1,\ell}_{a,b}(t,q)$ follows now from Equation (3.14) with $s = 0$. Lemmas 3.3(c) and 3.4 produce a sequence of bijections $\beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m+1}$:

$$\begin{aligned}
&\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1}_n = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_n \\
\xrightarrow{\beta_{2m+1}}\ &\left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n\mp 1} = \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n\mp 1} \quad \xrightarrow{\alpha_{2m}} \quad \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n\pm 1} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_{n\pm 1} \\
\xrightarrow{\beta_{2m-1}}\ &\cdots \quad \xrightarrow{\beta_1} \quad \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 1\downarrow}_{n\mp(m+1)} = \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n\mp(m+1)} = \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\mp(m+1)}.
\end{aligned}$$

By Equation (3.4),

$$\begin{aligned}
\sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1}_n} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} &= \sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\mp(m+1)}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} \\
&= q^{n^2-(m+1)(-m+x-y)} \begin{bmatrix} v-x-1 \\ n-m-1 \end{bmatrix}_q \begin{bmatrix} u-y+1 \\ n+m+1 \end{bmatrix}_q \\
&= q^{n^2+(m+1)(m-\ell)} \begin{bmatrix} a-\ell-1 \\ n-m-1 \end{bmatrix}_q \begin{bmatrix} b+\ell+1 \\ n+m+1 \end{bmatrix}_q, \qquad (3.21)
\end{aligned}$$

proving Equation (2.3).

Case IV: $0 < \ell > a-b$, equivalently $x > y$ and $u > v$. Here $G^{\geq 2m+2,\ell}_{a,b}(t,q) = G^{\geq 2m+1,\ell}_{a,b}(t,q)$ because of Equation (3.12) with $s = 0$. Lemmas 3.3(b) and 3.4 give a sequence of bijections $\alpha_1 \circ \beta_2 \circ \cdots \circ \alpha_{2m+1}$:

$$\begin{aligned} & && \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1}_n = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1\uparrow}_n \\ &\xrightarrow{\alpha_{2m+1}} && \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 2m+1\uparrow}_n = \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 2m\downarrow}_n \xrightarrow{\beta_{2m}} \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m\downarrow}_{n\mp 1} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m-1\uparrow}_{n\mp 1} \\ &\xrightarrow{\alpha_{2m-1}} && \cdots \xrightarrow{\alpha_1} \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 1\uparrow}_{n\pm m} = \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n\pm m} = \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\pm m}. \end{aligned}$$

Thus, using Equation (3.4), we get Equation (2.4):

$$\begin{aligned} &\sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1}_n} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} = \sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\pm m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} \\ &= q^{n^2+m(m+x-y+1)} \begin{bmatrix} v-x-1 \\ n+m \end{bmatrix}_q \begin{bmatrix} u-y+1 \\ n-m \end{bmatrix}_q = q^{n^2+m(m+\ell+1)} \begin{bmatrix} a-\ell-1 \\ n+m \end{bmatrix}_q \begin{bmatrix} b+\ell+1 \\ n-m \end{bmatrix}_q. \end{aligned} \tag{3.22}$$

Case VII: $0 < \ell = a-b$, equivalently $x > y$ and $u = v$. In this case, the parity of the total number of crossings is not forced by the endpoints, so we consider the cases $r = 2m$ and $r = 2m+1$ separately. The case $r = 2m$ is proved like Case I, constructing a sequence of bijections $\alpha_1 \circ \beta_2 \circ \cdots \circ \alpha_{2m-1} \circ \beta_{2m}$:

$$\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m}_n = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m\downarrow}_n \longrightarrow \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 1\uparrow}_{n\pm m} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n\pm m} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm m}, \tag{3.23}$$

where we use Lemma 3.2 for the left equality, and Lemmas 3.3(e) and 3.4 to compose the bijections. Equation (2.9) now follows using Equation (3.19) again.

The case $r = 2m+1$ is proved like Case IV, constructing a sequence of bijections $\alpha_1 \circ \beta_2 \circ \cdots \circ \alpha_{2m+1}$:

$$\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1}_n = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1\uparrow}_n \longrightarrow \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 1\uparrow}_{n\pm m} = \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n\pm m} = \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\pm m}. \tag{3.24}$$

Now we use Equation (3.22) and the fact that $\ell = a-b$ to prove Equation (2.10).

Case VIII: $0 > \ell = a-b$, equivalently $x < y$ and $u = v$. This case is analogous to Case VII. When $r = 2m$, we use the same sequence bijections as in Case II,

$$\beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m-1} \circ \alpha_{2m} : \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m}_n \longrightarrow \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\mp m}, \tag{3.25}$$

using Lemma 3.3(f). Equation (2.11) now follows from Equation (3.20).

When $r = 2m+1$, we use the same sequence of bijections as in Case III,

$$\beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m+1} : \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1}_n \longrightarrow \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\mp(m+1)}. \tag{3.26}$$

Equation (2.12) follows from Equation (3.21) after the substitution $\ell = a-b$.

Case V: $0 = \ell < a - b$, equivalently $x = y$ and $u < v$. We will reduce this case to Case VIII by applying an involution ν on two-rowed arrays that changes the sign of each entry, reverses each row (so that the negated entries increase from left to right), and swaps the top and the bottom rows. The map ν restricts to bijections

$$\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm k} \overset{\nu}{\longleftrightarrow} \left\{\begin{smallmatrix}(-v,-y]\\ [-u,-x)\end{smallmatrix}\right\}_{n\mp k}, \qquad \left\{\begin{smallmatrix}(x,v)\\ [y,u]\end{smallmatrix}\right\}_{n\pm k} \overset{\nu}{\longleftrightarrow} \left\{\begin{smallmatrix}[-u,-y]\\ (-v,-x)\end{smallmatrix}\right\}_{n\mp k} \tag{3.27}$$

for any $k \in \mathbb{Z}$. Additionally, in the case $k = 0$, it restricts to a bijection

$$\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n}^{\geq r} \overset{\nu}{\longleftrightarrow} \left\{\begin{smallmatrix}(-v,-y]\\ [-u,-x)\end{smallmatrix}\right\}_{n}^{\geq r}, \tag{3.28}$$

since it preserves the number of crossings; specifically, upward crossings turn into downward crossings, and vice versa. Indeed, the two-rowed array

$$\begin{array}{ccccccccccc} & x & < & c_1 & < & c_2 & < & \cdots & < & c_n \leq & u \\ y \leq & d_1 & < & d_2 & < & \cdots & < & d_n & < & v & \end{array}$$

is mapped by ν to

$$\begin{array}{ccccccccccccc} & -v & < & -d_n & < & -d_{n-1} & < & \cdots & < & -d_1 & \leq & -y & \\ -u & \leq & -c_n & < & -c_{n-1} & < & \cdots & < & -c_1 & < & -x & & \end{array}.$$

Thus, the first array has an upward crossing at c_i if and only if the second one has a downward crossing at $-c_i$, since condition (3.6) is equivalent to $-d_{i+1} < -c_i < -d_i$, and similarly for the other type of crossing. In terms of the corresponding lattice paths given by the encoding (3.8), the involution ν translates to a reflection along the line $x + y = 0$.

The conditions $x = y$ and $u < v$ are equivalent to $-v < -u$ and $-y = -x$, so we can apply the bijections from Case VIII to the set on the right-hand side of (3.28). When $r = 2m$, Equation (3.25) gives a bijection

$$\beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m-1} \circ \alpha_{2m} : \left\{\begin{smallmatrix}(-v,-y]\\ [-u,-x)\end{smallmatrix}\right\}_{n}^{\geq 2m} \longrightarrow \left\{\begin{smallmatrix}(-v,-y]\\ [-u,-x)\end{smallmatrix}\right\}_{n\mp m}.$$

Conjugating by ν, we get a bijection

$$\nu \circ \beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m-1} \circ \alpha_{2m} \circ \nu : \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n}^{\geq 2m} \longrightarrow \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n\pm m}$$

that preserves the sum of the entries. Using Equation (3.19) with $\ell = 0$, we deduce Equation (2.5).

When $r = 2m + 1$, Equation (3.26) gives a bijection

$$\beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m+1} : \left\{\begin{smallmatrix}(-v,-y]\\ [-u,-x)\end{smallmatrix}\right\}_{n}^{\geq 2m+1} \longrightarrow \left\{\begin{smallmatrix}(-v,-x)\\ [-u,-y]\end{smallmatrix}\right\}_{n\mp(m+1)},$$

and conjugating by ν we get

$$\nu \circ \beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m+1} \circ \nu : \left\{\begin{smallmatrix}(x,u]\\ [u,v)\end{smallmatrix}\right\}_{n}^{\geq 2m} \longrightarrow \left\{\begin{smallmatrix}[y,u]\\ (x,v)\end{smallmatrix}\right\}_{n\pm(m+1)}.$$

Swapping the top and bottom rows and using Equation (3.21) with $\ell = 0$, we deduce Equation (2.6) .

Case VI: $0 = \ell > a - b$, equivalently $x = y$ and $u > v$. By applying the map ν, this case reduces to Case VII, since the conditions $x = y$ and $u > v$ are equivalent to $-v > -u$ and $-y = -x$. When $r = 2m$, conjugating the bijection (3.23) with ν gives a bijection

$$\nu \circ \alpha_1 \circ \beta_2 \circ \cdots \circ \alpha_{2m-1} \circ \beta_{2m} \circ \nu : \left\{ {(x,u] \atop [y,v)} \right\}_n^{\geq 2m} \longrightarrow \left\{ {(x,u] \atop [y,v)} \right\}_{n \mp m}.$$

Using Equation (3.20) with $\ell = 0$, we deduce Equation (2.7).

When $r = 2m + 1$, conjugating the bijection (3.24) with ν gives a bijection

$$\nu \circ \alpha_1 \circ \beta_2 \circ \cdots \circ \alpha_{2m+1} \circ \nu : \left\{ {(x,u] \atop [y,v)} \right\}_n^{\geq 2m+1} \longrightarrow \left\{ {[y,u] \atop (x,v)} \right\}_{n \mp m}.$$

Swapping the top and bottom rows and using Equation (3.22) with $\ell = 0$, we deduce Equation (2.8).

Case IX: $0 = \ell = a - b$, equivalently $x = y$ and $u = v$. We consider two cases according to the first step of the path. Via the bijection (3.8), paths in the left-hand side starting with an N are encoded by two-rowed arrays ${\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,u] \atop [y,v)} \right\}_n^{\geq r}$ with $y < d_1$; equivalently, by ${\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,u] \atop [y+1,v)} \right\}_n^{\geq r}$. Note that replacing the lower bound y with $y + 1$ does not affect the number of crossings of the array, since ${\mathbf{c} \atop \mathbf{d}}$ cannot have a crossing at c_0 in either case. Since $x < y+1$, the conditions in Case VIII hold with $y+1$ playing the role of y. Equation (3.25) gives a bijection

$$\beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m-1} \circ \alpha_{2m} : \left\{ {(x,u] \atop [y+1,v)} \right\}_n^{\geq 2m} \longrightarrow \left\{ {(x,u] \atop [y+1,v)} \right\}_{n \mp m}.$$

Then, using Equation (3.3), it follows that

$$\begin{aligned} \sum_{{\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,u] \atop [y+1,v)} \right\}_n^{\geq 2m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} &= q^n \sum_{{\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,u] \atop [y+1,v)} \right\}_{n \mp m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y+1)} \\ &= q^n q^{n^2-m(-m+x-y)} \begin{bmatrix} u-x \\ n-m \end{bmatrix}_q \begin{bmatrix} v-y-1 \\ n+m \end{bmatrix}_q = q^{n^2+n+m^2} \begin{bmatrix} a \\ n-m \end{bmatrix}_q \begin{bmatrix} a-1 \\ n+m \end{bmatrix}_q. \end{aligned} \tag{3.29}$$

Similarly, Equation (3.26) gives a bijection

$$\beta_1 \circ \alpha_2 \circ \cdots \circ \beta_{2m+1} : \left\{ {(x,u] \atop [y+1,v)} \right\}_n^{\geq 2m+1} \longrightarrow \left\{ {(x,v) \atop [y+1,u]} \right\}_{n \mp (m+1)},$$

and Equation (3.4) implies that

$$\begin{aligned} \sum_{{\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,u] \atop [y+1,v)} \right\}_n^{\geq 2m+1}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} &= q^n \sum_{{\mathbf{c} \atop \mathbf{d}} \in \left\{ {(x,v) \atop [y+1,u]} \right\}_{n \mp (m+1)}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y+1)} \\ &= q^n q^{n^2-(m+1)(-m-1+x-y)} \begin{bmatrix} v-x-1 \\ n-m-1 \end{bmatrix}_q \begin{bmatrix} u-y \\ n+m+1 \end{bmatrix}_q \\ &= q^{n^2+n+(m+1)^2} \begin{bmatrix} a-1 \\ n-m-1 \end{bmatrix}_q \begin{bmatrix} a \\ n+m+1 \end{bmatrix}_q. \end{aligned} \tag{3.30}$$

On the other hand, paths in the left-hand side of (3.8) starting with an E are encoded by two-rowed arrays $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{\genfrac{}{}{0pt}{}{(x,u]}{[y,v)}\right\}_n^{\geq r}$ with $y = d_1$. Let us use the notation $\left\{\genfrac{}{}{0pt}{}{(x,u]_n}{\llbracket y,v)_n}\right\}^{\geq r}$ for such arrays, where the double bracket indicates that the first element in the bottom row is forced to equal its lower bound. By Lemma 3.3(g), we can use the same bijections as in Case VII, noting that the condition $y = d_1$ is preserved when applying the maps from Lemma 3.4. For $r = 2m$, we get a bijection

$$\alpha_1 \circ \beta_2 \circ \cdots \circ \alpha_{2m-1} \circ \beta_{2m} : \left\{\genfrac{}{}{0pt}{}{(x,u]}{\llbracket y,v)}\right\}_n^{\geq 2m} \longrightarrow \left\{\genfrac{}{}{0pt}{}{(x,u]}{\llbracket y,v)}\right\}_{n\pm m},$$

and Equation (3.3) implies that

$$\begin{aligned}
\sum_{\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{\genfrac{}{}{0pt}{}{(x,u]}{\llbracket y,v)}\right\}_n^{\geq 2m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} &= \sum_{\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{\genfrac{}{}{0pt}{}{(x,u]}{\llbracket y,v)}\right\}_{n\pm m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} \\
&= \sum_{\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{\genfrac{}{}{0pt}{}{(x,u]}{[y,v)}\right\}_{n\pm m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} - q^n \sum_{\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{\genfrac{}{}{0pt}{}{(x,u]}{[y+1,v)}\right\}_{n\pm m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y+1)} \\
&= q^{n^2+m(m+x-y+1)} \begin{bmatrix} u-x \\ n+m \end{bmatrix}_q \begin{bmatrix} v-y \\ n-m \end{bmatrix}_q - q^n q^{n^2+m(m+x-y)} \begin{bmatrix} u-x \\ n+m \end{bmatrix}_q \begin{bmatrix} v-y-1 \\ n-m \end{bmatrix}_q \\
&= q^{n^2+m(m+1)} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} a \\ n-m \end{bmatrix}_q - q^n q^{n^2+m^2} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} a-1 \\ n-m \end{bmatrix}_q \\
&= q^{n^2+m(m+1)} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \left(\begin{bmatrix} a \\ n-m \end{bmatrix}_q - q^{n-m} \begin{bmatrix} a-1 \\ n-m \end{bmatrix}_q \right) \\
&= q^{n^2+m(m+1)} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} a-1 \\ n-m-1 \end{bmatrix}_q . \qquad (3.31)
\end{aligned}$$

Similarly, for $r = 2m+1$, we get a bijection (see the example in Figure 7):

$$\alpha_1 \circ \beta_2 \circ \cdots \circ \alpha_{2m+1} : \left\{\genfrac{}{}{0pt}{}{(x,u]}{\llbracket y,v)}\right\}_n^{\geq 2m+1} \longrightarrow \left\{\genfrac{}{}{0pt}{}{(x,v)}{\llbracket y,u]}\right\}_{n\pm m}. \qquad (3.32)$$

$$\left\{\genfrac{}{}{0pt}{}{(0,7]}{\llbracket 0,7)}\right\}_3^{\geq 3} = \left\{\genfrac{}{}{0pt}{}{(0,7]}{\llbracket 0,7)}\right\}_3^{\geq 3\uparrow} \xrightarrow{\alpha_3} \left\{\genfrac{}{}{0pt}{}{(0,7)}{\llbracket 0,7]}\right\}_3^{\geq 3\uparrow} = \left\{\genfrac{}{}{0pt}{}{(0,7)}{\llbracket 0,7]}\right\}_3^{\geq 2\downarrow} \xrightarrow{\alpha_1} \left\{\genfrac{}{}{0pt}{}{(0,7]}{\llbracket 0,7)}\right\}_{3\mp 1}^{\geq 2\downarrow} = \left\{\genfrac{}{}{0pt}{}{(0,7]}{\llbracket 0,7)}\right\}_{3\mp 1}^{\geq 1\uparrow} \xrightarrow{\alpha_1} \left\{\genfrac{}{}{0pt}{}{(0,7)}{\llbracket 0,7]}\right\}_{3\pm 1}^{\geq 1\uparrow} = \left\{\genfrac{}{}{0pt}{}{(0,7)}{\llbracket 0,7]}\right\}_{3\pm 1}$$

$$\begin{array}{cccc}
0<2<3<6\leq 7 & 0<2<3<6<7 & 0<2<3\leq 7 & 0<2<3<5<6<7 \\
0=0<3<5<7 & 0=0<3<5\leq 7 & 0=0<3<5<6<7 & 0=0<3\leq 7
\end{array}$$

FIGURE 7. An example of the bijection (3.32), where $(x,y) = (0,0)$, $(u,v) = (7,7)$, $m = 1$ and $n = 3$.

Then, Equation (3.4) implies that

$$\sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x,u]\\ [\![y,v)}\right\}_n^{\geq 2m+1}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} = \sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x,v)\\ [\![y,u]}\right\}_{n\pm m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)}$$

$$= \sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x,v)\\ [y,u]}\right\}_{n\pm m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} - q^n \sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x,v)\\ [y+1,u]}\right\}_{n\pm m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y+1)}$$

$$= q^{n^2+m(m+x-y+1)} \begin{bmatrix} v-x-1 \\ n+m \end{bmatrix}_q \begin{bmatrix} u-y+1 \\ n-m \end{bmatrix}_q - q^n q^{n^2+m(m+x-y)} \begin{bmatrix} v-x-1 \\ n+m \end{bmatrix}_q \begin{bmatrix} u-y \\ n-m \end{bmatrix}_q$$

$$= q^{n^2+m(m+1)} \begin{bmatrix} a-1 \\ n+m \end{bmatrix}_q \begin{bmatrix} a+1 \\ n-m \end{bmatrix}_q - q^n q^{n^2+m^2} \begin{bmatrix} a-1 \\ n+m \end{bmatrix}_q \begin{bmatrix} a \\ n-m \end{bmatrix}_q$$

$$= q^{n^2+m(m+1)} \begin{bmatrix} a-1 \\ n+m \end{bmatrix}_q \begin{bmatrix} a \\ n-m-1 \end{bmatrix}_q . \tag{3.33}$$

Adding Equations (3.29) and (3.31) to account for all paths with at least $2m$ crossings, we get

$$\sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x,u]\\ [y,v)}\right\}_n^{\geq 2m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)}$$

$$= q^{n^2+m(m+1)} \left(q^{n-m} \begin{bmatrix} a \\ n-m \end{bmatrix}_q \begin{bmatrix} a-1 \\ n+m \end{bmatrix}_q + \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} a-1 \\ n-m-1 \end{bmatrix}_q \right)$$

$$= q^{n^2+m(m+1)} \frac{1-q^{a-2m}}{1-q^a} \begin{bmatrix} a \\ n+m \end{bmatrix}_q \begin{bmatrix} a \\ n-m \end{bmatrix}_q ,$$

which proves Equation (2.13).

Similarly, adding Equations (3.30) and (3.33) to account for all paths with at least $2m+1$ crossings, we get

$$\sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x,u]\\ [y,v)}\right\}_n^{\geq 2m+1}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n(x+y)} = q^{n^2+m(m+1)} \frac{1-q^{a+2(m+1)}}{1-q^a} \begin{bmatrix} a \\ n+m+1 \end{bmatrix}_q \begin{bmatrix} a \\ n-m-1 \end{bmatrix}_q ,$$

which proves Equation (2.14).

4. Proofs for paths crossing each other

In this section we prove Theorem 2.2. Using the bijection (3.1), we will encode pairs of lattice paths as pairs of two-rowed arrays, describe crossings in this setting, and then define certain bijections on pairs of arrays.

4.1. **Pairs of two-rowed arrays.** Throughout the section, let $k \in \mathbb{Z}$ and $n \geq 0$, let $\{\circ, \bullet\} = \{1, 2\}$, and let $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $B_1 = (u_1, v_1)$, and $B_2 = (u_2, v_2)$ be four pairs of integers. We consider certain sets of pairs of two-rowed arrays, for which we introduce the notation

$$\left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}_{n,k} = \bigcup_{n_1+n_2=n} \left\{\begin{smallmatrix}(x_1,u_\circ]_{n_1}\\ [y_1,v_\circ)_{n_1+k}\end{smallmatrix}\right\} \times \left\{\begin{smallmatrix}(x_2,u_\bullet]_{n_2}\\ [y_2,v_\bullet)_{n_2-k}\end{smallmatrix}\right\}. \tag{4.1}$$

Elements of such sets are denoted by placing two two-rowed arrays side by side, namely $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$, where $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u_\circ]_{n_1}\\ [y_1,v_\circ)_{n_1+k}\end{smallmatrix}\right\}$ and $\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_2,u_\bullet]_{n_2}\\ [y_2,v_\bullet)_{n_2-k}\end{smallmatrix}\right\}$, with $n_1 + n_2 = n$. When $k = 0$, the subscript k will often be omitted.

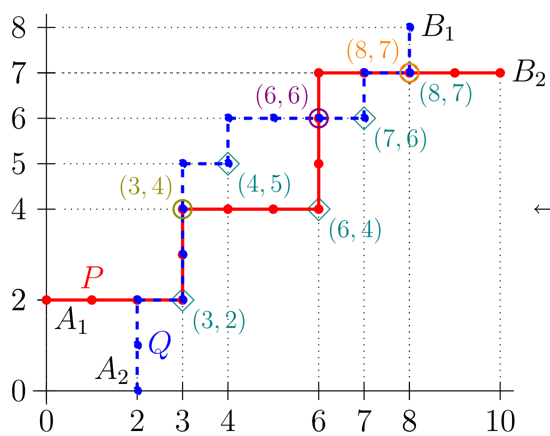
Applying the encoding (3.1) to each component of a pair of paths, we get a bijection

$$\{(P, Q) \in \mathcal{P}_{A_1\to B_\circ} \times \mathcal{P}_{A_2\to B_\bullet} : \operatorname{des}(P) + \operatorname{des}(Q) = n\} \to \left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}_{n}. \tag{4.2}$$

See Figure 8 for an example. Suppose that condition (2.15) holds, and let $z = x_1 + y_1 = x_2 + y_2$. If (P, Q) is encoded by $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$, then

$$\operatorname{maj}(P)+\operatorname{maj}(Q) = \sum_{i=1}^{n_1}(c_i+d_i-x_1-y_1)+\sum_{j=1}^{n_2}(e_j+f_j-x_2-y_2) = \|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz. \tag{4.3}$$

Next we adapt Lemma 3.1 to enumerate the sets (4.1) with respect to this statistic.



$$\longleftrightarrow \quad \begin{matrix} 0<3<6\le 10 \\ 2\le 2<4<7\end{matrix} \;\middle|\; \begin{matrix} 2<3<4<7<8\le 8 \\ 0\le 2<5<6<7<8\end{matrix} \in \left\{\begin{smallmatrix}(0,10]\\ [2,7)\end{smallmatrix}\middle|\begin{smallmatrix}(2,8]\\ [0,8)\end{smallmatrix}\right\}_6$$

Figure 8. The encoding (4.2) applied to the pair of paths from Figure 3, and the resulting pair of two-rowed arrays, where the crossings have been circled.

Lemma 4.1. *Suppose that $z = x_1 + y_1 = x_2 + y_2$. We have*

$$\sum_{n\geq 0} t^n \sum_{\substack{\mathbf{c}\\ \mathbf{d}}\left|\substack{\mathbf{e}\\ \mathbf{f}}\right.\in\left\{\substack{(x_1,u_\circ]\\ [y_1,v_\circ)}\middle|\substack{(x_2,u_\bullet]\\ [y_2,v_\bullet)}\right\}_{n,k}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz}$$
$$= q^{k(k+x_2-x_1)}\left(\sum_{n_1\geq 0} t^{n_1} q^{n_1(n_1+k)} \begin{bmatrix} u_\circ - x_1\\ n_1\end{bmatrix}_q \begin{bmatrix} v_\circ - y_1\\ n_1+k\end{bmatrix}_q\right)\times$$
$$\left(\sum_{n_2\geq 0} t^{n_2} q^{n_2(n_2-k)} \begin{bmatrix} u_\bullet - x_2\\ n_2\end{bmatrix}_q \begin{bmatrix} v_\bullet - y_2\\ n_2-k\end{bmatrix}_q\right)$$
$$= f_{k,A_1,A_2,B_\circ,B_\bullet}(t,q).$$

Proof. Using (4.1), the left-hand side expression can be factored as

$$\left(\sum_{n_1\geq 0} t^{n_1} \sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x_1,u_\circ]_{n_1}\\ [y_1,v_\circ)_{n_1+k}}\right\}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n_1z}\right)\left(\sum_{n_2\geq 0} t^{n_2} \sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x_2,u_\bullet]_{n_2}\\ [y_2,v_\bullet)_{n_2-k}}\right\}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n_2z}\right). \tag{4.4}$$

For fixed n_1, Lemma 3.1(i) gives

$$\sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x_1,u_\circ]_{n_1}\\ [y_1,v_\circ)_{n_1+k}}\right\}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n_1z} = \left(\sum_{\mathbf{c}\in(x_1,u_\circ]_{n_1}} q^{\|\mathbf{c}\|}\right)\left(\sum_{\mathbf{d}\in[y_1,v_\circ)_{n_1+k}} q^{\|\mathbf{d}\|}\right) q^{-n_1(x_1+y_1)}$$
$$= q^{n_1(n_1+k)+\binom{k}{2}+ky_1} \begin{bmatrix} u_\circ - x_1\\ n_1\end{bmatrix}_q \begin{bmatrix} v_\circ - y_1\\ n_1+k\end{bmatrix}_q, \tag{4.5}$$

where we used the simplification

$$\binom{n_1+1}{2}+\binom{n_1+k+1}{2}+n_1x_1+(n_1+k)(y_1-1)-n_1(x_1+y_1) = n_1(n_1+k)+\binom{k}{2}+ky_1.$$

Similarly,

$$\sum_{\substack{\mathbf{c}\\ \mathbf{d}}\in\left\{\substack{(x_2,u_\bullet]_{n_2}\\ [y_2,v_\bullet)_{n_2-k}}\right\}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|-n_2z} = q^{n_2(n_2-k)+\binom{k+1}{2}-ky_2} \begin{bmatrix} u_\bullet - x_2\\ n_2\end{bmatrix}_q \begin{bmatrix} v_\bullet - y_2\\ n_2-k\end{bmatrix}_q. \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) and using that $\binom{k}{2}+\binom{k+1}{2}+k(y_1-y_2) = k(k+x_2-x_1)$, we obtain the stated identity. □

4.2. **Crossings in pairs of two-rowed arrays.** Let vertex V_s be a crossing of two paths P and Q, as defined in Section 2.2. We say that V_s is an *upward* (resp. *downward*) crossing of (P,Q) if the step of P leaving V_s is an N (resp. E); equivalently, if the step of Q leaving V_s is an E (resp. N).

Crossings of a pair of paths can be read from their encoding (4.2) as a pair of two-rowed arrays. Indeed, suppose that (P,Q) is encoded by $\begin{smallmatrix}\mathbf{c}\\\mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\\mathbf{f}\end{smallmatrix}$, where $\begin{smallmatrix}\mathbf{c}\\\mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u_\circ]_{n_1}\\ [y_1,v_\circ)_{n_1}\end{smallmatrix}\right\}$ and $\begin{smallmatrix}\mathbf{e}\\\mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_2,u_\bullet]_{n_2}\\ [y_2,v_\bullet)_{n_2}\end{smallmatrix}\right\}$, and let $c_0 := x_1$, $d_0 := y_1$, $c_{n_1+1} := u_\circ$, $d_{n_1+1} := v_\circ$, $e_0 := x_2$, $f_0 := y_2$, $e_{n_2+1} := u_\bullet$, $f_{n_2+1} := v_\bullet$ by convention. For simplicity, let us assume that $A_1 \prec A_2$ or $A_1 = A_2$. Then (P,Q) has an upward crossing at (c_i,f_j), where $0 \le i \le n_1$ and $1 \le j \le n_2+1$, if all of the following hold:

(i$^\uparrow$) $e_{j-1} \le c_i < e_j$ and $d_i \le f_j < d_{i+1}$,
(ii$^\uparrow$) $(e_{j-1}, f_{j-1}, e_{j-2}, f_{j-2}, \dots, e_0, f_0) <_{\text{alt}} (c_i, d_i, c_{i-1}, d_{i-1}, \dots, c_0, d_0)$ and
$(d_i, c_{i-1}, d_{i-1}, c_{i-2}, \dots, c_0, d_0) <_{\text{alt}} (f_j, e_{j-1}, f_{j-1}, e_{j-2}, \dots, e_0, f_0)$,

where $<_{\text{alt}}$ is defined recursively by $(a_1,a_2,a_3,\dots) <_{\text{alt}} (b_1,b_2,b_3,\dots)$ if either $a_1 < b_1$, or $a_1 = b_1$ and $(b_2,b_3,\dots) <_{\text{alt}} (a_2,a_3,\dots)$. Indeed, condition (i$^\uparrow$) states that (c_i,f_j) belongs to both P and Q, and that P (resp. Q) leaves this vertex with an N (resp. E). Condition (ii$^\uparrow$) states that, if V_1 is the first vertex of the maximal sequence of consecutive common vertices ending at (c_i,f_j), then P (resp. Q) arrives at V_1 with an N (resp. E).

Similarly, (P,Q) has a downward crossing at (e_j,d_i), where $1 \le i \le n_1+1$ and $0 \le j \le n_2$, if

(i$^\downarrow$) $c_{i-1} \le e_j < c_i$ and $f_j \le d_i < f_{j+1}$,
(ii$^\downarrow$) $(c_{i-1}, d_{i-1}, c_{i-2}, d_{i-2}, \dots, c_0, d_0) <_{\text{alt}} (e_j, f_j, e_{j-1}, f_{j-1}, \dots, e_0, f_0)$ and
$(f_j, e_{j-1}, f_{j-1}, e_{j-2}, \dots, c_0, d_0) <_{\text{alt}} (d_i, c_{i-1}, d_{i-1}, c_{i-2}, \dots, e_0, f_0)$.

For example, the pair of paths in Figure 8 has a downward crossing at $(e_1,d_2) = (3,4)$. Condition (i$^\downarrow$) states that $3 \le 3 < 6$ and $2 \le 4 < 5$, and condition (ii$^\downarrow$) states that $(3,2,0,2) <_{\text{alt}} (3,2,2,0)$ and $(2,2,0) <_{\text{alt}} (4,3,2,0,2)$.

Next we generalize the definition of upward and downward crossings to pairs of two-rowed arrays $\begin{smallmatrix}\mathbf{c}\\\mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\\mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\Big|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}_{n,k}$ with $k \in \mathbb{Z}$. Suppose that $\begin{smallmatrix}\mathbf{c}\\\mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u_\circ]_{n_1}\\ [y_1,v_\circ)_{n_1+k}\end{smallmatrix}\right\}$ and $\begin{smallmatrix}\mathbf{e}\\\mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_2,u_\bullet]_{n_2}\\ [y_2,v_\bullet)_{n_2-k}\end{smallmatrix}\right\}$, where $n_1+n_2 = n$, and use the convention $c_0 := x_1$, $d_0 := y_1$, $c_{n_1+1} := u_\circ$, $d_{n_1+k+1} := v_\circ$, $e_0 := x_2$, $f_0 := y_2$, $e_{n_2+1} := u_\bullet$, $f_{n_2-k+1} := v_\bullet$. Let $m_1 = \min(n_1, n_1+k)$ and $m_2 = \min(n_2, n_2-k)$. Then $\begin{smallmatrix}\mathbf{c}\\\mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\\mathbf{f}\end{smallmatrix}$ has an upward crossing at (c_i,f_j) if $0 \le i \le m_1$ and $1 \le j \le m_2+1$, and conditions (i$^\uparrow$) and (ii$^\uparrow$) hold. Similarly, it has a downward crossing at (e_j,d_i) if $1 \le i \le m_1+1$ and $0 \le j \le m_2$, and conditions (i$^\downarrow$) and (ii$^\downarrow$) hold.

It is convenient to think of crossings of a pair of two-rowed arrays as crossings of the pair of paths obtained by truncating the arrays, similarly to what we did in Section 3.2 for single arrays. Let $T(\begin{smallmatrix}\mathbf{c}\\\mathbf{d}\end{smallmatrix})$ be the path in $\mathcal{P}_{(x_1,y_1)\to(c_{m_1+1},d_{m_1+1})}$ having valleys at positions (c_i,d_i) for $1 \le i \le m_1$, and let $T(\begin{smallmatrix}\mathbf{e}\\\mathbf{f}\end{smallmatrix})$ be the path in $\mathcal{P}_{(x_2,y_2)\to(e_{m_2+1},f_{m_2+1})}$ having valleys at positions (e_j,f_j) for $1 \le j \le m_2$. Then the upward and downward crossings of $\begin{smallmatrix}\mathbf{c}\\\mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\\mathbf{f}\end{smallmatrix}$ can be identified with the upward and downward crossings of the pair of paths $(T(\begin{smallmatrix}\mathbf{c}\\\mathbf{d}\end{smallmatrix}), T(\begin{smallmatrix}\mathbf{e}\\\mathbf{f}\end{smallmatrix}))$. See Figure 9 for an example. In particular, upward crossings are always at vertices of the form (c_i,f_j), and downward crossings are at vertices of the form (e_j,d_i), for some i,j.

$$\begin{array}{r|l} 0\leq 3 & 1<2<4\leq 4 \\ 1\leq 2<5 & 0\leq 2<4 \end{array} \in \left\{\begin{smallmatrix}(0,3]\\ [1,5)\end{smallmatrix}\middle|\begin{smallmatrix}(1,4]\\ [0,4)\end{smallmatrix}\right\}_{3,1}$$

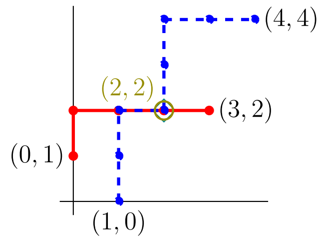


FIGURE 9. A pair of two-rowed arrays $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ with its crossing circled, and the corresponding pair of paths $(T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}), T(\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}))$.

It is clear from this description that there is a natural ordering of the crossings by increasing x-coordinate, or equivalently, by increasing y-coordinate. As in the case of single arrays, the *rth crossing* of a pair of two-rowed arrays will always refer to the rth crossing in this ordering.

For $r\geq 0$, denote by $\left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}^{\geq r}_{n,k}$ the subset of $\left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}_{n,k}$ consisting of pairs of arrays that have at least r crossings. The encoding (4.2) restricts to a bijection

$$\{\mathcal{P}^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet} : \operatorname{des}(P)+\operatorname{des}(Q)=n\} \to \left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}^{\geq r}_{n}.$$

Using Equation (4.3), it follows that, if $z=x_1+y_1=x_2+y_2$, then

$$H^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet}(t,q)=\sum_{n\geq 0} t^n \sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}\in\left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}^{\geq r}_{n}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz}. \tag{4.7}$$

To prove Theorem 2.2, we will construct bijections between $\left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}^{\geq r}_{n}$ and sets of the form $\left\{\begin{smallmatrix}(x_1,u_2]\\ [y_1,v_2)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ [y_2,v_1)\end{smallmatrix}\right\}_{n,k}$ for some $k\in\mathbb{Z}$, and then apply Lemma 4.1.

Lemma 4.2. *Let $r\geq 1$. If $A_1\prec A_2$, then the rth crossing of a pair of arrays $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}\in \left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}^{\geq r}_{n,k}$ is a downward crossing if r is odd, and an upward crossing if r is even.*

Proof. Interpreting crossings of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ as crossings of the pair of paths $(T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}), T(\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}))$, which start at A_1 and A_2, respectively, the fact that $A_1\prec A_2$ implies that downward and upward crossings must alternate, with the first crossing being downward. □

For $r\geq 1$, a symbol $\uparrow$ (resp. $\downarrow$) next to the superscript $\geq r$ denotes the subset of pairs of arrays where the rth crossing is an upward (resp. downward) crossing. For $r=0$, in the case $A_1\prec A_2$, we simplify define

$$\left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,k}=\left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n,k}=\left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}^{\geq 0}_{n,k}=\left\{\begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix}\right\}_{n,k} \tag{4.8}$$

by convention.

In the case $A_1 = A_2 = (x,y)$ and $B_1 = B_2 = (u,v)$, we define $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,k}$ (resp. $\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n,k}$) to be the set of pairs $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n,k}$ such that $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \neq \begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ and the leftmost entry in the usual zig-zag order where $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ differ is in the top row and satisfies $c_i < e_i$ (resp. $c_i > e_i$), or it is in the bottom row and satisfies $d_i > f_i$ (resp. $d_i < f_i$). Equivalently, $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ is in the first (resp. second) set if the first step where the paths $T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix})$ and $T(\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix})$ disagree is an N (resp. E) step of $T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix})$ and an E (resp. N) step of $T(\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix})$. See the examples in Figure 10.

In analogy to Lemma 3.3 for single arrays, the next lemma shows how the relative locations of the two initial points and of the two final points often force the number of crossings of a pair of two-rowed arrays to have a given parity.

Lemma 4.3. *Let $m \geq 0$.*

(a) *If $A_1 \prec A_2$, $B_1 \prec B_2$, and $s \geq 0$, then*

$$\left\{\begin{smallmatrix}(x_1,u_2]\\ [y_1,v_2)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ [y_2,v_1)\end{smallmatrix}\right\}^{\geq 2m+1}_{n,s} = \left\{\begin{smallmatrix}(x_1,u_2]\\ [y_1,v_2)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ [y_2,v_1)\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n,s} = \left\{\begin{smallmatrix}(x_1,u_2]\\ [y_1,v_2)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ [y_2,v_1)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,s} = \left\{\begin{smallmatrix}(x_1,u_2]\\ [y_1,v_2)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ [y_2,v_1)\end{smallmatrix}\right\}^{\geq 2m}_{n,s}, \tag{4.9}$$

$$\left\{\begin{smallmatrix}(x_1,u_1]\\ [y_1,v_1)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ [y_2,v_2)\end{smallmatrix}\right\}^{\geq 2m+2}_{n,-s} = \left\{\begin{smallmatrix}(x_1,u_1]\\ [y_1,v_1)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ [y_2,v_2)\end{smallmatrix}\right\}^{\geq 2m+2\uparrow}_{n,-s} = \left\{\begin{smallmatrix}(x_1,u_1]\\ [y_1,v_1)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ [y_2,v_2)\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n,-s} = \left\{\begin{smallmatrix}(x_1,u_1]\\ [y_1,v_1)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ [y_2,v_2)\end{smallmatrix}\right\}^{\geq 2m+1}_{n,-s}. \tag{4.10}$$

(b) *If $A_1 \prec A_2$ and $B_1 = B_2$, then* (4.9) *and* (4.10) *hold for $s \geq 1$.*

(c) *If $A_1 = A_2 = (x,y)$, $B_1 = B_2 = (u,v)$, and $s \geq 1$, then*

$$\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n,s} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,s} \quad \text{and} \quad \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+2\uparrow}_{n,-s} = \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n,-s}. \tag{4.11}$$

Proof. In each of Equations (4.9) and (4.10) for $A_1 \prec A_2$, the two outer equalities follow from Lemma 4.2 (and convention (4.8) in the case $m = 0$), and the left-hand side is trivially contained in the right-hand side. To prove the reverse containment, we will show that the parity of the number of crossings is forced in each case.

Let us first prove Equation (4.9) with the hypotheses of part (a). Let

$$\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u_2]\\ [y_1,v_2)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ [y_2,v_1)\end{smallmatrix}\right\}_{n,s}, \text{ so that } \begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u_2]_{n_1}\\ [y_1,v_2)_{n_1+s}\end{smallmatrix}\right\} \text{ and } \begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_2,u_1]_{n_2}\\ [y_2,v_1)_{n_2-s}\end{smallmatrix}\right\}$$

for some n_1, n_2 summing to n. Crossings of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ are crossings of the pair of paths $(T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}), T(\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix})) \in \mathcal{P}_{A_1 \to B_2'} \times \mathcal{P}_{A_2 \to B_1'}$, where $B_2' = (u_2, d_{n_1+1})$ and $B_1' = (e_{n_2-s+1}, v_1)$. Since $d_{n_1+1} \leq v_2$ and $e_{n_2-s+1} \leq u_1$, the condition $B_1 \prec B_2$ implies that $B_1' \prec B_2'$. Thus, since $A_1 \prec A_2$, the number of crossings of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ must be odd, proving Equation (4.9) in this case.

With the hypotheses of part (b), letting $B_1 = B_2 = (u,v)$ and $s \geq 1$, the same argument yields endpoints $B_2' = (u, d_{n_1+1})$ and $B_1' = (e_{n_2-s+1}, v)$ with $d_{n_1+1} < d_{n_1+s+1} = v$ and $e_{n_2-s+1} \leq u$. Let $\widetilde{T}\binom{\mathbf{e}}{\mathbf{f}}$ be the path obtained by removing the run of E steps at the end of $T\binom{\mathbf{e}}{\mathbf{f}}$, which does not affect any crossings since B_2' is strictly below these steps. Crossings of $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ are now crossings of $(T\binom{\mathbf{c}}{\mathbf{d}}, \widetilde{T}\binom{\mathbf{e}}{\mathbf{f}}) \in \mathcal{P}_{A_1 \to B_2'} \times \mathcal{P}_{A_2 \to B_1''}$, where $B_1'' = (e_{n_2-s}, v)$. Since $e_{n_2-s} < e_{n_2-s+1} \leq u$, we have $B_1'' \prec B_2'$, implying again that the number of crossings of $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ is odd, which proves Equation (4.9) also in this case.

With the hypotheses of part (c), the same argument gives a pair

$$(T\tbinom{\mathbf{c}}{\mathbf{d}}, \widetilde{T}\tbinom{\mathbf{e}}{\mathbf{f}}) \in \mathcal{P}_{A \to B_2'} \times \mathcal{P}_{A \to B_1''},$$

where $B_1'' \prec B_2'$ as before. Suppose that $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}} \in \left\{\genfrac{}{}{0pt}{}{(x,u]}{[y,v)}\middle|\genfrac{}{}{0pt}{}{(x,u]}{[y,v)}\right\}^{\geq 2m\uparrow}_{n,s}$. For $m \geq 1$, this means that the $2m$th crossing of $(T\binom{\mathbf{c}}{\mathbf{d}}, \widetilde{T}\binom{\mathbf{e}}{\mathbf{f}})$ is an upward crossing; for $m = 0$, this means that the first step where these paths disagree is an N step of $T\binom{\mathbf{c}}{\mathbf{d}}$ and an E step of $\widetilde{T}\binom{\mathbf{e}}{\mathbf{f}}$. The fact that $B_1'' \prec B_2'$ forces these paths to cross, with the $2m+1$st crossing being a downward crossing, which proves the first equality in (4.11).

The proof of Equation (4.10) is similar. Let $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}} \in \left\{\genfrac{}{}{0pt}{}{(x_1,u_1]}{[y_1,v_1)}\middle|\genfrac{}{}{0pt}{}{(x_2,u_2]}{[y_2,v_2)}\right\}_{n,-s}$, so that $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{\genfrac{}{}{0pt}{}{(x_1,u_1]_{n_1}}{[y_1,v_1)_{n_1-s}}\right\}$ and $\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}} \in \left\{\genfrac{}{}{0pt}{}{(x_2,u_2]_{n_2}}{[y_2,v_2)_{n_2+s}}\right\}$ for some n_1, n_2. Crossings of $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ are crossings of

$$(T\tbinom{\mathbf{c}}{\mathbf{d}}, T\tbinom{\mathbf{e}}{\mathbf{f}}) \in \mathcal{P}_{A_1 \to B_1'} \times \mathcal{P}_{A_2 \to B_2'}, \text{ where } B_1' = (c_{n_1-s+1}, v_1) \text{ and } B_2' = (u_2, f_{n_2+1}).$$

Since $c_{n_1-s+1} \leq u_1$ and $f_{n_2+1} \leq v_2$, the condition $B_1 \prec B_2$ implies that $B_1' \prec B_2'$. Thus, if the hypothesis of part (a) hold, the number of crossings of $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ must be even, proving Equation (4.10) in this case.

Letting now $B_1 = B_2 = (u,v)$ and $s \geq 1$, we get endpoints $B_1' = (c_{n_1-s+1}, v)$ and $B_2' = (u, f_{n_2+1})$ with $c_{n_1-s+1} \leq u$ and $f_{n_2+1} < v$. Removing the run of E steps at the end of $T\binom{\mathbf{c}}{\mathbf{d}}$, which does not affect any crossings, we obtain a pair

$$(\widetilde{T}\tbinom{\mathbf{c}}{\mathbf{d}}, T\tbinom{\mathbf{e}}{\mathbf{f}}) \in \mathcal{P}_{A_1 \to B_1''} \times \mathcal{P}_{A_2 \to B_2'}, \text{ where } B_1'' = (c_{n_1-s}, v).$$

Now $c_{n_1-s} < c_{n_1-s+1} \leq u$, and so $B_1'' \prec B_2'$, implying again that the number of crossings of $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ is even. This proves Equation (4.10) with the hypotheses of part (b).

Finally, with the hypotheses of part (c), we obtain a pair

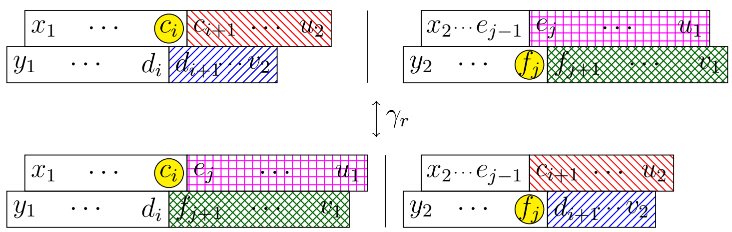
$$(\widetilde{T}\tbinom{\mathbf{c}}{\mathbf{d}}, T\tbinom{\mathbf{e}}{\mathbf{f}}) \in \mathcal{P}_{A_1 \to B_1''} \times \mathcal{P}_{A_2 \to B_2'}, \text{ where } B_1'' \prec B_2'.$$

If $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}} \in \left\{\genfrac{}{}{0pt}{}{(x,u]}{[y,v)}\middle|\genfrac{}{}{0pt}{}{(x,u]}{[y,v)}\right\}^{\geq 2m+1\downarrow}_{n,-s}$, the $2m+1$st crossing of $(\widetilde{T}\binom{\mathbf{c}}{\mathbf{d}}, T\binom{\mathbf{e}}{\mathbf{f}})$ is a downward crossing, so the paths must cross again, and the $2m+2$nd crossing must be an upward crossing. This proves the second equality in (4.11). $\square$

4.3. **The bijections γ_r and δ_r.** The bijections γ_r and δ_r play a similar role for pairs of two-rowed arrays as the bijections α_r and β_r played for single arrays. They are reminiscent of the bijection θ_r defined in [3] for pairs of paths; however, γ_r and δ_r do not restrict to bijections for pairs of paths, since they change the relative lengths of the rows of the arrays.

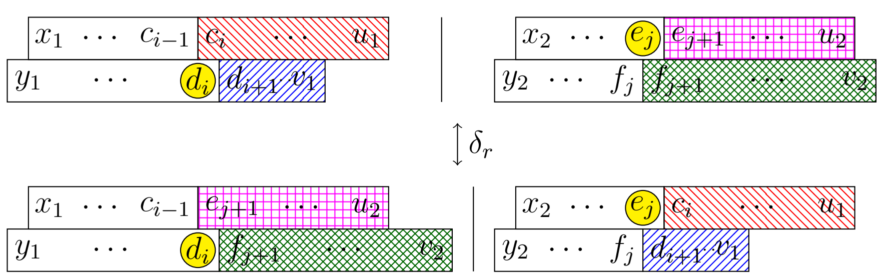
For $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{ \begin{smallmatrix}(x_1,u_\circ]\\ [y_1,v_\circ)\end{smallmatrix} \Big| \begin{smallmatrix}(x_2,u_\bullet]\\ [y_2,v_\bullet)\end{smallmatrix} \right\}_n$, we say that an upward crossing at (c_i, f_j) (resp. a downward crossing at (e_j, d_i)) is *proper* if $c_i \neq u_\circ$ and $f_j \neq v_\bullet$ (resp. $e_j \neq u_\bullet$ and $d_i \neq v_\circ$), that is, neither entry equals the upper bound for its row.

For $r \geq 1$, the map γ_r applies to pairs of two-rowed arrays $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ whose rth crossing is a proper upward crossing, say at (c_i, f_j), and it swaps the entries to the right of c_i in each row of the first array with the entries to the right of f_j in each row of the second array. Schematically, we have:



The fact that (c_i, f_j) is a proper crossing of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ guarantees that c_{i+1} and f_{j+1} exist, and that $c_i < c_{i+1}$ and $f_j < f_{j+1}$. Condition (i$^\uparrow$) in the characterization of upward crossings implies that $c_i < e_j$, $d_i < f_{j+1}$, $e_{j-1} < c_{i+1}$ and $f_j < d_{i+1}$, and so the rows of the arrays in $\gamma_r(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix})$ are increasing. The pair $\gamma_r(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix})$ still has a crossing at (c_i, f_j): condition (i$^\uparrow$) holds because $e_{j-1} \leq c_i < c_{i+1}$ and $d_i \leq f_j < f_{j+1}$, and condition (ii$^\uparrow$) holds because the relevant entries are not affected by γ_r. This crossing is clearly proper, and it is the rth crossing of $\gamma_r(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix})$ because the entries to the left of c_i and f_j, and thus the first $r-1$ crossings of the pair of arrays, are not affected by γ_r. It follows that γ_r is an involution.

Similarly, the map δ_r applies to pairs of two-rowed arrays $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ whose rth crossing is a proper downward crossing, say at (e_j, d_i), and it again swaps the entries to the right of d_i in the first array with the entries to the right of e_j in the second array. Schematically, we have:



The same argument shows that the rows of the arrays in $\delta_r(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix})$ are increasing, that $\delta_r(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix})$ still has a proper crossing at (e_j, d_i), which is its rth crossing, and that the map δ_r is an involution. In fact, if we denote by ς the involution that swaps the two two-rowed arrays in a pair, that is,

$$\varsigma(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}) = \begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}\big|\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}, \tag{4.12}$$

then the maps γ_r and δ_r are related by $\delta_r = \varsigma \circ \gamma_r \circ \varsigma$.

If $A_1 = A_2 = (x, y)$ and $B_1 = B_2 = (u, v)$, we can extend the definitions of γ_r and δ_r to the case $r = 0$ as follows. Let $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\Big|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,k}$. If the leftmost entry where $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ differ is $c_i < e_i$, then the vertex (c_i, f_i) satisfies condition (i$^\uparrow$) from the characterization of upward crossings, since $e_{i-1} < c_i < e_i$ and $d_i = f_i < d_{i+1}$, even if it fails condition (ii$^\uparrow$) since $(d_i, c_{i-1}, d_{i-1}, \dots, c_0, d_0) = (f_i, e_{i-1}, f_{i-1}, \dots, e_0, f_0)$. If $c_i \neq u$ and $f_i \neq v$, we define γ_0 by swapping the entries to the right of c_i in each row of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ with the entries to the right of f_i in each row of $\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$, just as in the usual definition of γ_r if (c_i, f_i) had been the rth crossing.

If the leftmost entry where $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$ differ is $d_i > f_i$, now it is the vertex (c_{i-1}, f_i) that satisfies condition (i$^\uparrow$), since $e_{i-1} = c_{i-1} < e_i$ and $d_{i-1} < f_i < d_i$. If $c_{i-1} \neq u$ and $f_i \neq v$, we define γ_0 by swapping the entries to the right of c_{i-1} in each row of $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}$ with the entries to the right of f_i in each row of $\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$, just as in the definition of γ_r if (c_{i-1}, f_i) had been the rth crossing. See the example in Figure 10.

The bijection δ_0 can be defined analogously, or as $\delta_0 = \varsigma \circ \gamma_0 \circ \varsigma$, but it will not be needed in the proofs.

$$\begin{array}{r|l} 0<2<4\leq 4 & 0<2\leq 4 \\ 0\leq 1<\mathbf{3} & 0\leq 1<\mathbf{2}<3\end{array} \in \left\{\begin{array}{c|c}(0,4] & (0,4]\\ [0,3) & [0,3)\end{array}\right\}^{\geq 0\uparrow}_{3,-1} \overset{\gamma_0}{\longleftrightarrow} \begin{array}{r|l} 0<2\leq 4 & 0<2<4\leq 4 \\ 0\leq 1<\mathbf{3} & 0\leq 1<\mathbf{2}<3\end{array} \in \left\{\begin{array}{c|c}(0,4] & (0,4]\\ [0,3) & [0,3)\end{array}\right\}^{\geq 0\uparrow}_{3,0}$$

FIGURE 10. An example of the bijection γ_0. For each pair of two-rowed arrays $\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\Big|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}$, the leftmost entry where they differ is $d_2 = 3 > 2 = f_2$, so $(c_1, f_2) = (2, 2)$ satisfies condition (i$^\uparrow$). The corresponding vertex in the pair of paths $(T(\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}), T(\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix}))$ has been marked with a dotted circle.

Lemma 4.4. *Fix $n \geq 0$, $k \in \mathbb{Z}$ and $r \geq 1$. Suppose that either $B_1 \prec B_2$ and $k \geq 0$, or that $B_1 = B_2$. The map γ_r restricts to a bijection*

$$\left\{\begin{array}{c|c}(x_1,u_2] & (x_2,u_1]\\ [y_1,v_2) & [y_2,v_1)\end{array}\right\}^{\geq r\uparrow}_{n,k} \overset{\gamma_r}{\longleftrightarrow} \left\{\begin{array}{c|c}(x_1,u_1] & (x_2,u_2]\\ [y_1,v_1) & [y_2,v_2)\end{array}\right\}^{\geq r\uparrow}_{n,-k-1}. \tag{4.13}$$

The map δ_r restricts to a bijection

$$\left\{\begin{array}{c|c}(x_1,u_1] & (x_2,u_2]\\ [y_1,v_1) & [y_2,v_2)\end{array}\right\}^{\geq r\downarrow}_{n,-k} \overset{\delta_r}{\longleftrightarrow} \left\{\begin{array}{c|c}(x_1,u_2] & (x_2,u_1]\\ [y_1,v_2) & [y_2,v_1)\end{array}\right\}^{\geq r\downarrow}_{n,k+1}. \tag{4.14}$$

Both γ_r and δ_r preserve the sum of the entries of the pair of arrays.

Additionally, if $A_1 = A_2$ and $B_1 = B_2$, then the above statements also hold for $r = 0$.

Proof. Suppose first that $r \geq 1$. Let us first check that pairs of arrays in the four sets above cannot have improper crossings, and so the maps γ_r and δ_r are defined. For $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}} \in \left\{\genfrac{}{}{0pt}{}{(x_1,u_\circ]}{[y_1,v_\circ)}\middle|\genfrac{}{}{0pt}{}{(x_2,u_\bullet]}{[y_2,v_\bullet)}\right\}_{n,h}$, where $h \in \mathbb{Z}$, to have an improper upward crossing at (c_i, f_j), we must have either $c_i = u_\circ$, in which case $u_\circ = c_i < e_j \leq u_\bullet$ and $h \geq 0$, or $f_j = v_\bullet$, in which case $v_\bullet = f_j < d_{i+1} \leq v_\circ$ and $h \geq 0$. Similarly, for $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ to have an improper downward crossing at (e_j, d_i), we must have either $e_j = u_\bullet$, in which case $u_\bullet = e_j < c_i \leq u_\circ$ and $h \leq 0$, or $d_i = v_\circ$, in which case $v_\circ = d_i < f_{j+1} \leq v_\bullet$ and $h \leq 0$.

If $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ is in the left-hand side of (4.13) (resp. (4.14)) and has an improper upward (resp. downward) crossing, the previous paragraph forces $u_2 < u_1$ or $v_2 < v_1$, contradicting the hypothesis that $B_1 \prec B_2$ or $B_1 = B_2$. If $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ is in the right-hand side instead, the forced inequalities $u_1 < u_2$ or $v_1 < v_2$ hold when $B_1 \prec B_2$, but the requirement on h states that $k + 1 \leq 0$, contradicting the hypothesis that $k \geq 0$ in this case.

Having already seen that γ_r and δ_r are involutions preserving the first r crossings and preserving the sum of the entries, it remains to describe their images when restricted to the above sets. Let $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ be in one of the sets in (4.13), so that one has

$$\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}} \in \left\{\genfrac{}{}{0pt}{}{(x_1,u_\circ]_{n_1}}{[y_1,v_\circ)_{n_1+h}}\right\} \qquad \text{and} \qquad \genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}} \in \left\{\genfrac{}{}{0pt}{}{(x_2,u_\bullet]_{n_2}}{[y_2,v_\bullet)_{n_2-h}}\right\}$$

for some n_1, n_2 summing to n, and $h \in \mathbb{Z}$. If the rth crossing of $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ is an upward crossing at (c_i, f_j), then

$$\gamma_r\Big(\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}\Big) \in \left\{\genfrac{}{}{0pt}{}{(x_1,u_\bullet]_{n_2+i-j+1}}{[y_1,v_\bullet)_{n_2-h+i-j}}\right\} \times \left\{\genfrac{}{}{0pt}{}{(x_2,u_\circ]_{n_1-i+j-1}}{[y_2,v_\circ)_{n_1+h-i+j}}\right\} \subseteq \left\{\genfrac{}{}{0pt}{}{(x_1,u_\bullet]}{[y_1,v_\bullet)}\middle|\genfrac{}{}{0pt}{}{(x_2,u_\circ]}{[y_2,v_\circ)}\right\}_{n,-h-1}.$$

When $h \in \{k, -k-1\}$, then $-h-1$ equals the other element in the set, so this argument works in both directions.

Similarly, if $\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}$ is in one of the sets in (4.14) and its rth crossing is a downward crossing at (e_j, d_i), then

$$\delta_r\Big(\genfrac{}{}{0pt}{}{\mathbf{c}}{\mathbf{d}}\Big|\genfrac{}{}{0pt}{}{\mathbf{e}}{\mathbf{f}}\Big) \in \left\{\genfrac{}{}{0pt}{}{(x_1,u_\bullet]_{n_2+i-j-1}}{[y_1,v_\bullet)_{n_2-h+i-j}}\right\} \times \left\{\genfrac{}{}{0pt}{}{(x_2,u_\circ]_{n_1-i+j+1}}{[y_2,v_\circ)_{n_1+h-i+j}}\right\} \subseteq \left\{\genfrac{}{}{0pt}{}{(x_1,u_\bullet]}{[y_1,v_\bullet)}\middle|\genfrac{}{}{0pt}{}{(x_2,u_\circ]}{[y_2,v_\circ)}\right\}_{n,-h+1}.$$

When $h \in \{-k, k+1\}$, then $-h+1$ equals the other element in the set.

Finally, in the case that $A_1 = A_2$ and $B_1 = B_2$, a similar argument shows that the maps γ_0 and δ_0 are defined and they are bijections between the stated sets. $\square$

4.4. **Proof of Theorem 2.2.** The proof is divided into four cases according to which endpoints of the paths coincide. In each case, we determine $H^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet}(t,q)$ by first using Equation (4.7) to write it as a sum over pairs of two-rowed arrays. Then we repeatedly apply the maps from Lemma 4.4 to construct bijections between $\left\{\genfrac{}{}{0pt}{}{(x_1,u_\circ]}{[y_1,v_\circ)}\middle|\genfrac{}{}{0pt}{}{(x_2,u_\bullet]}{[y_2,v_\bullet)}\right\}^{\geq r}_{n}$ and certain sets of pairs of two-rowed arrays with no requirement on the number of crossings, and finally we use Lemma 4.1 to obtain the desired expressions. Again, the cases are labeled as in [3] for consistency.

Case 1: endpoints $A_1 \prec A_2$ and $B_1 \prec B_2$. If $P \in \mathcal{P}_{A_1 \to B_2}$ and $Q \in \mathcal{P}_{A_2 \to B_1}$, the relative position of the endpoints forces $\chi(P,Q)$ to be odd, which proves the first equality in Equation (2.16). Using Lemmas 4.3 and 4.4, we construct a sequence of bijections $\delta_1 \circ \gamma_2 \circ \cdots \circ \delta_{2m-1} \circ \gamma_{2m}$:

$$\begin{aligned}
&\left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}^{\geq 2m}_{n} = \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,0} \xrightarrow{\gamma_{2m}} \left\{\begin{smallmatrix}(x_1,u_1]\\ {[y_1,v_1)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ {[y_2,v_2)}\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,-1} = \left\{\begin{smallmatrix}(x_1,u_1]\\ {[y_1,v_1)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ {[y_2,v_2)}\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_{n,-1}\\
&\qquad \xrightarrow{\delta_{2m-1}} \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_{n,2} = \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}^{\geq 2m-2\uparrow}_{n,2}\\
&\qquad \xrightarrow{\gamma_{2m-2}} \cdots \xrightarrow{\delta_1} \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}^{\geq 1\downarrow}_{n,2m} = \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,2m} = \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}_{n,2m},
\end{aligned} \tag{4.15}$$

where the last equality comes from (4.8). See Figure 11 for an example. Since these bijections preserve the sum of the entries of the arrays, Equation (4.7) and Lemma 4.1 give

$$H^{\geq 2m}_{A_1\to B_2, A_2\to B_1}(t,q) = \sum_{n\geq 0} t^n \sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\middle|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}_{n,2m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz} = f_{2m,A_1,A_2,B_2,B_1}(t,q),$$

proving Equation (2.16).

$$\left\{\begin{smallmatrix}(0,10]\\ {[2,7)}\end{smallmatrix}\middle|\begin{smallmatrix}(2,8]\\ {[0,8)}\end{smallmatrix}\right\}^{\geq 2}_{6} = \left\{\begin{smallmatrix}(0,10]\\ {[2,7)}\end{smallmatrix}\middle|\begin{smallmatrix}(2,8]\\ {[0,8)}\end{smallmatrix}\right\}^{\geq 2\uparrow}_{6,0} \xrightarrow{\gamma_2} \left\{\begin{smallmatrix}(0,8]\\ {[2,8)}\end{smallmatrix}\middle|\begin{smallmatrix}(2,10]\\ {[0,7)}\end{smallmatrix}\right\}^{\geq 2\uparrow}_{6,-1} = \left\{\begin{smallmatrix}(0,8]\\ {[2,8)}\end{smallmatrix}\middle|\begin{smallmatrix}(2,10]\\ {[0,7)}\end{smallmatrix}\right\}^{\geq 1\downarrow}_{6,-1}$$

$$\begin{array}{r|l} 0<3<6\leq 10 & 2<3<4<7<8\leq 8\\ 2\leq 2<4<7 & 0\leq 2<5<6<7<8 \end{array} \qquad \begin{array}{r|l} 0<3<6<7<8\leq 8 & 2<3<4\leq 10\\ 2\leq 2<4<7<8 & 0\leq 2<5<6<7 \end{array}$$

$$\xrightarrow{\delta_1} \left\{\begin{smallmatrix}(0,10]\\ {[2,7)}\end{smallmatrix}\middle|\begin{smallmatrix}(2,8]\\ {[0,8)}\end{smallmatrix}\right\}^{\geq 1\downarrow}_{6,2} = \left\{\begin{smallmatrix}(0,10]\\ {[2,7)}\end{smallmatrix}\middle|\begin{smallmatrix}(2,8]\\ {[0,8)}\end{smallmatrix}\right\}_{6,2}$$

$$\begin{array}{r|l} 0<3<4\leq 10 & 2<3<6<7<8\leq 8\\ 2\leq 2<4<5<6<7 & 0\leq 2<7<8 \end{array}$$

FIGURE 11. An example of the bijection (4.15), where $m=1$ and $n=6$.

Similarly, if $P \in \mathcal{P}_{A_1\to B_1}$ and $Q \in \mathcal{P}_{A_2\to B_2}$, then $\chi(P,Q)$ must be even, which proves the first equality in Equation (2.17). In this case, we construct a sequence of bijections $\delta_1 \circ \gamma_2 \circ \cdots \circ \delta_{2m+1}$:

$$\begin{aligned}
&\left\{\begin{smallmatrix}(x_1,u_1]\\ {[y_1,v_1)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ {[y_2,v_2)}\end{smallmatrix}\right\}^{\geq 2m+1}_{n} = \left\{\begin{smallmatrix}(x_1,u_1]\\ {[y_1,v_1)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ {[y_2,v_2)}\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n,0} \xrightarrow{\delta_{2m+1}} \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n,1} = \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,1}\\
&\qquad \xrightarrow{\gamma_{2m}} \left\{\begin{smallmatrix}(x_1,u_1]\\ {[y_1,v_1)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ {[y_2,v_2)}\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,-2} = \left\{\begin{smallmatrix}(x_1,u_1]\\ {[y_1,v_1)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_2]\\ {[y_2,v_2)}\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_{n,-2}\\
&\qquad \xrightarrow{\delta_{2m-1}} \cdots \xrightarrow{\delta_1} \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}^{\geq 1\downarrow}_{n,2m+1} = \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}_{n,2m+1}.
\end{aligned}$$

Equation (4.7) and Lemma 4.1 now give

$$H^{\geq 2m+1}_{A_1\to B_1, A_2\to B_2}(t,q) = \sum_{n\geq 0} t^n \sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\middle|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u_2]\\ {[y_1,v_2)}\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u_1]\\ {[y_2,v_1)}\end{smallmatrix}\right\}_{n,2m+1}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz} = f_{2m+1,A_1,A_2,B_2,B_1}(t,q),$$

proving Equation (2.17).

We now handle Case 3, followed by Cases 2 and 4.

Case 3: endpoints $A_1 \prec A_2$ and B. If $P \in \mathcal{P}_{A_1\to B}$ and $Q \in \mathcal{P}_{A_2\to B}$, the parity of $\chi(P,Q)$ is no longer forced by the endpoints, so we consider two cases. When $r = 2m$ for some $m \geq 1$, the rth crossing is an upward crossing by Lemma 4.2, and Lemmas 4.3 and 4.4 give a sequence of bijections $\delta_1 \circ \gamma_2 \circ \cdots \circ \delta_{2m-1} \circ \gamma_{2m}$:

$$\begin{aligned}
\left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m}_{n} &= \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,0} \xrightarrow{\gamma_{2m}} \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,-1} = \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_{n,-1} \\
&\xrightarrow{\delta_{2m-1}} \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_{n,2} = \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m-2\uparrow}_{n,2} \\
&\xrightarrow{\gamma_{2m-2}} \cdots \xrightarrow{\delta_1} \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 1\downarrow}_{n,2m} = \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,2m} = \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}_{n,2m},
\end{aligned} \tag{4.16}$$

using again (4.8). Equation (4.7) and Lemma 4.1 give

$$H^{\geq 2m}_{A_1\to B, A_2\to B}(t,q) = \sum_{n\geq 0} t^n \sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\middle|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}_{n,2m}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz} = f_{2m,A_1,A_2,B,B}(t,q),$$

proving Equation (2.19) for even r.

When $r = 2m+1$ for some $m \geq 0$, the rth crossing is a downward crossing by Lemma 4.2, and we get a sequence of bijections $\delta_1 \circ \gamma_2 \circ \cdots \circ \delta_{2m+1}$:

$$\begin{aligned}
\left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m+1}_{n} &= \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n,0} \xrightarrow{\delta_{2m+1}} \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n,1} = \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,1} \\
&\xrightarrow{\gamma_{2m}} \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,-2} = \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_{n,-2} \\
&\xrightarrow{\delta_{2m-1}} \cdots \xrightarrow{\delta_1} \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}^{\geq 1\downarrow}_{n,2m+1} = \left\{\begin{smallmatrix}(x_1,u]\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}_{n,2m+1},
\end{aligned} \tag{4.17}$$

from where

$$H^{\geq 2m+1}_{A_1\to B_1, A_2\to B_2}(t,q) = \sum_{n\geq 0} t^n \sum_{\begin{smallmatrix}\mathbf{c}\\ \mathbf{d}\end{smallmatrix}\middle|\begin{smallmatrix}\mathbf{e}\\ \mathbf{f}\end{smallmatrix} \in \left\{\begin{smallmatrix}(x_1,u\\ [y_1,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x_2,u]\\ [y_2,v)\end{smallmatrix}\right\}_{n,2m+1}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz} = f_{2m+1,A_1,A_2,B,B}(t,q),$$

proving Equation (2.19) for odd r.

Case 2: endpoints A and $B_1 \prec B_2$. We will reduce this case to Case 3 by applying the involution ν, defined above Equation (3.27), componentwise to each of the two-rowed arrays in a pair. With some abuse of notation, we also denote this map on pairs of two-rowed arrays by ν. It restricts to a bijection

$$\left\{\begin{smallmatrix}(x,u_1]\\ [y,v_1)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u_2]\\ [y,v_2)\end{smallmatrix}\right\}_{n,k} \overset{\nu}{\longleftrightarrow} \left\{\begin{smallmatrix}(-v_1,-y]\\ [-u_1,-x)\end{smallmatrix}\middle|\begin{smallmatrix}(-v_2,-y]\\ [-u_2,-x)\end{smallmatrix}\right\}_{n,-k}$$

for any $k \in \mathbb{Z}$. In the case $k = 0$, translating ν into a map on pairs of paths via the encoding (4.2) yields the involution that reflects each path along the line $x + y = 0$. In particular, it preserves the number of crossings, so it restricts to a bijection

$$\left\{\begin{smallmatrix}(x,u_1]\\ [y,v_1)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u_2]\\ [y,v_2)\end{smallmatrix}\right\}^{\geq r}_{n} \overset{\nu}{\longleftrightarrow} \left\{\begin{smallmatrix}(-v_1,-y]\\ [-u_1,-x)\end{smallmatrix}\middle|\begin{smallmatrix}(-v_2,-y]\\ [-u_2,-x)\end{smallmatrix}\right\}^{\geq r}_{n}.$$

The hypothesis $(u_1, v_1) \prec (u_2, v_2)$ implies that the initial points of the reflected paths satisfy $(-v_1, -u_1) \prec (-v_2, -u_2)$, whereas the final point is the same for both paths, namely $(-y, -x)$. This allows us to apply Case 3.

When $r = 2m$, Equation (4.16) gives a bijection

$$\delta_1 \circ \gamma_2 \circ \cdots \circ \delta_{2m-1} \circ \gamma_{2m} : \left\{ \begin{smallmatrix} (-v_1,-y] \\ [-u_1,-x) \end{smallmatrix} \middle| \begin{smallmatrix} (-v_2,-y] \\ [-u_2,-x) \end{smallmatrix} \right\}_n^{\geq 2m} \longrightarrow \left\{ \begin{smallmatrix} (-v_1,-y] \\ [-u_1,-x) \end{smallmatrix} \middle| \begin{smallmatrix} (-v_2,-y] \\ [-u_2,-x) \end{smallmatrix} \right\}_{n,2m}.$$

Conjugating by ν and composing with the map ς from Equation (4.12) yields a bijection

$$\varsigma \circ \nu \circ \delta_1 \circ \gamma_2 \circ \cdots \circ \delta_{2m-1} \circ \gamma_{2m} \circ \nu : \left\{ \begin{smallmatrix} (x,u_1] \\ [y,v_1) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u_2] \\ [y,v_2) \end{smallmatrix} \right\}_n^{\geq 2m} \longrightarrow \left\{ \begin{smallmatrix} (x,u_2] \\ [y,v_2) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u_1] \\ [y,v_1) \end{smallmatrix} \right\}_{n,2m}$$

that preserves the sum of the entries. Similarly, when $r = 2m+1$, conjugating the bijection (4.17) with ν and composing with ς produces a bijection

$$\varsigma \circ \nu \circ \delta_1 \circ \gamma_2 \circ \cdots \circ \delta_{2m+1} \circ \nu : \left\{ \begin{smallmatrix} (x,u_1] \\ [y,v_1) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u_2] \\ [y,v_2) \end{smallmatrix} \right\}_n^{\geq 2m+1} \longrightarrow \left\{ \begin{smallmatrix} (x,u_2] \\ [y,v_2) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u_1] \\ [y,v_1) \end{smallmatrix} \right\}_{n,2m+1}.$$

In both cases, using Equation (4.7) and Lemma 4.1, we get

$$H^{\geq r}_{A\to B_1, A\to B_2}(t,q) = \sum_{n\geq 0} t^n \sum_{\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix} \middle| \begin{smallmatrix}\mathbf{e} \\ \mathbf{f}\end{smallmatrix} \in \left\{ \begin{smallmatrix} (x,u_1] \\ [y,v_1) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u_2] \\ [y,v_2) \end{smallmatrix} \right\}_{n,r}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz} = f_{r,A,A,B_2,B_1}(t,q),$$

proving Equation (2.18).

Case 4: endpoints A and B. The map ς from Equation (4.12) restricts to a bijection

$$\left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_{n,k}^{\geq r\uparrow} \overset{\varsigma}{\longleftrightarrow} \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_{n,-k}^{\geq r\downarrow}$$

for any $r \geq 0$ and $k \in \mathbb{Z}$. For $r \geq 1$, we also have

$$\left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_n^{\geq r} = \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_n^{\geq r\uparrow} \sqcup \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_n^{\geq r\downarrow},$$

and so Equation (4.7) gives

$$H^{\geq r}_{A\to B, A\to B}(t,q) = 2\sum_{n\geq 0} t^n \sum_{\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix} \middle| \begin{smallmatrix}\mathbf{e} \\ \mathbf{f}\end{smallmatrix} \in \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_n^{\geq r\uparrow}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz} \tag{4.18}$$

$$= 2\sum_{n\geq 0} t^n \sum_{\begin{smallmatrix}\mathbf{c} \\ \mathbf{d}\end{smallmatrix} \middle| \begin{smallmatrix}\mathbf{e} \\ \mathbf{f}\end{smallmatrix} \in \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_n^{\geq r\downarrow}} q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz}. \tag{4.19}$$

Our next goal is to prove that

$$H^{\geq r}_{A\to B, A\to B}(t,q) + H^{\geq r+1}_{A\to B, A\to B}(t,q) = 2 f_{r+1,A,A,B,B}(t,q) \tag{4.20}$$

for all $r \geq 1$.

For arrays with at least $r = 2m$ crossings, Lemmas 4.3 and 4.4 give bijections $\delta_1 \circ \gamma_2 \circ \cdots \circ \delta_{2m-1} \circ \gamma_{2m}$:

$$\begin{aligned} \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_n^{\geq 2m\uparrow} \xrightarrow{\gamma_{2m}} \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_{n,-1}^{\geq 2m\uparrow} &= \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_{n,-1}^{\geq 2m-1\downarrow} \\ \xrightarrow{\delta_{2m-1}} \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_{n,2}^{\geq 2m-1\downarrow} &= \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_{n,2}^{\geq 2m-2\uparrow} \\ \xrightarrow{\gamma_{2m-2}} \cdots \xrightarrow{\delta_1} \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_{n,2m}^{\geq 1\downarrow} &= \left\{ \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \middle| \begin{smallmatrix} (x,u] \\ [y,v) \end{smallmatrix} \right\}_{n,2m}^{\geq 0\uparrow}. \end{aligned} \tag{4.21}$$

Similarly, for arrays with at least $r=2m+1$ crossings, we get bijections $\delta_1\circ\gamma_2\circ\cdots\circ\delta_{2m-1}\circ\gamma_{2m}\circ\delta_{2m+1}$:

$$\begin{aligned}\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n}\xrightarrow{\delta_{2m+1}}\ &\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m+1\downarrow}_{n,1}=\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,1}\\ \xrightarrow{\gamma_{2m}}\ &\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m\uparrow}_{n,-2}=\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 2m-1\downarrow}_{n,-2}\\ \xrightarrow{\delta_{2m-1}}\cdots\xrightarrow{\delta_1}\ &\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 1\downarrow}_{n,2m+1}=\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,2m+1}.\end{aligned}\tag{4.22}$$

In both cases, we can compose these bijections with $\varsigma\circ\gamma_0$:

$$\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,r}\xrightarrow{\gamma_0}\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,-r-1}\xrightarrow{\varsigma}\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n,r+1}.\tag{4.23}$$

Composing (4.21) with (4.23), where $r=2m$, and using Equation (4.18), we get

$$H^{\geq 2m}_{A\to B,A\to B}(t,q)=2\sum_{n\geq 0}t^n\sum_{\begin{array}{c|c}\mathbf{c}&\mathbf{e}\\ \mathbf{d}&\mathbf{f}\end{array}\in\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n,2m+1}}q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz}$$

for $m\geq 1$. Similarly, the bijection (4.22) and Equation (4.19), where $r=2m+1$, give

$$H^{\geq 2m+1}_{A\to B,A\to B}(t,q)=2\sum_{n\geq 0}t^n\sum_{\begin{array}{c|c}\mathbf{c}&\mathbf{e}\\ \mathbf{d}&\mathbf{f}\end{array}\in\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,2m+1}}q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz}.$$

Adding the last two equations, using the fact that

$$\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,k}\sqcup\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n,k}=\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}_{n,k}$$

for all $k\neq 0$, and applying Lemma 4.1, we obtain a proof of Equation (4.20) for $r=2m$.

On the other hand, composing (4.22) (with $m-1$ playing the role of m) with (4.23) (with $r=2m-1$) and using Equation (4.19), we get

$$H^{\geq 2m-1}_{A\to B,A\to B}(t,q)=2\sum_{n\geq 0}t^n\sum_{\begin{array}{c|c}\mathbf{c}&\mathbf{e}\\ \mathbf{d}&\mathbf{f}\end{array}\in\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\downarrow}_{n,2m}}q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz}$$

for $m\geq 1$. Similarly, the bijection (4.21) and Equation (4.18), where $r=2m$, give

$$H^{\geq 2m}_{A\to B,A\to B}(t,q)=2\sum_{n\geq 0}t^n\sum_{\begin{array}{c|c}\mathbf{c}&\mathbf{e}\\ \mathbf{d}&\mathbf{f}\end{array}\in\left\{\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\middle|\begin{smallmatrix}(x,u]\\ [y,v)\end{smallmatrix}\right\}^{\geq 0\uparrow}_{n,2m}}q^{\|\mathbf{c}\|+\|\mathbf{d}\|+\|\mathbf{e}\|+\|\mathbf{f}\|-nz}.$$

Adding the last two equations and applying Lemma 4.1, we obtain a proof of Equation (4.20) for $r=2m-1$.

Solving Equation (4.20) for $H^{\geq r}_{A\to B,A\to B}(t,q)$ and iterating, we obtain

$$H^{\geq r}_{A\to B,A\to B}(t,q)=2\left(f_{r+1,A,A,B,B}(t,q)-f_{r+2,A,A,B,B}(t,q)+f_{r+3,A,A,B,B}(t,q)-\cdots\right)$$

which proves Equation (2.20) for $r\geq 1$. The case $r=0$ follows immediately from Equation (4.7) and Lemma 4.1.

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THREE FAMILIES OF q-LOMMEL POLYNOMIALS

JANG SOO KIM[1] AND DENNIS STANTON[2]

[1]Department of Mathematics, Sungkyunkwan University (SKKU), Suwon, Gyeonggi-do 16419, South Korea; https://jangsookim.github.io/

[2]School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA; https://www-users.cse.umn.edu/~stant001/

Abstract. Three q-versions of Lommel polynomials are studied. Included are explicit representations, recurrences, continued fractions, and connections to associated Askey–Wilson polynomials. Combinatorial results are emphasized, including a general theorem when R_I moments coincide with orthogonal polynomial moments. The combinatorial results use weighted Motzkin paths, Schröder paths, and parallelogram polyominoes.

Keywords: Lommel polynomial, Bessel function, orthogonal polynomial.

1. Introduction

Lehmer [27] used the following Bessel function identity to study zeros of Bessel functions

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} = 2\sum_{n=1}^{\infty} \sigma_{2n}(\nu) x^{2n-1}, \tag{1.1}$$

where $\sigma_{2n}(\nu)$ is the $2n^{\text{th}}$ power sum of the inverses of the positive zeros $j_{\nu,k}$ of $J_\nu(x)$:

$$\sigma_{2n}(\nu) = \sum_{k=1}^{\infty} j_{\nu,k}^{-2n}. \tag{1.2}$$

Lehmer noted that $\sigma_{2n}(\nu)$ is a rational function of ν, with a predictable denominator, and a numerator with nonnegative coefficients. Kishore [19] proved Lehmer's positivity conjecture. Lalanne ([25, Proposition 3.6], [26, Theorem 4.7]) proved q-versions of Kishore's result using weighted binary trees and also weighted Dyck paths.

The above series is related to the Lommel polynomials $L_{n,\nu}$, which are orthogonal polynomials with respect to the linear functional

$$\mathcal{L}(P(x)) := 2(\nu+1)\sum_{k=1}^{\infty}\left(P\left(j_{\nu,k}^{-1}\right) + P\left(-j_{\nu,k}^{-1}\right)\right) j_{\nu,k}^{-2},$$

i.e., $\mathcal{L}(L_{n,\nu}(x)L_{m,\nu}(x)) = 0$ if $n \neq m$ and $\mathcal{L}(1) = 1$; see [15, Eq. (6.5.17)]. Thus $\sigma_{2n}(\nu)$ in (1.2) is the $(2n-2)^{\text{th}}$ moment for the Lommel polynomials, while (1.1) is the Lommel moment generating function.

The purpose of this paper is to study three sets of q-Lommel polynomials, whose moment generating functions are quotients of q-Bessel functions. These polynomials were analytically studied by Ismail [15], Koelink and Van Assche [23], and Koelink [21]. In this paper we concentrate on the combinatorial aspect of these three sets of q-Lommel polynomials.

The literature already contains some combinatorial results on the quotient of Bessel functions and the quotient of q-Bessel functions. Delest and Fédou [9] showed that a generating function for parallelogram polyominoes can be written as a ratio of Jackson's third q-Bessel functions. Bousquet-Mélou and Viennot [4] generalized their result by adding one more parameter. A recounting of the history of the combinatorics of the q-analogue of the quotient of Bessel functions may be found in [3, Section 1] (see also [26, Section 4]). It includes results by Klarner and Rivest [20, Eq. (19)], Fédou [12], Lalanne [25, 26], Brak and Guttmann [5], and Barcucci et al. [1, Corollary 3.5], [2, Theorems 4.3 and 5.3].

In this paper we put these results in perspective by relating them to q-Lommel polynomials. The moment generating function has a continued fraction expansion. Using the general theory of orthogonal and type R_I polynomials we give finite versions of the infinite continued fractions. We show that a generating function for bounded diagonal parallelogram polyominoes is given by a ratio of q-Lommel polynomials, which is a finite version of the result of Bousquet-Mélou and Viennot [4].

Even though the Lommel polynomials have a hypergeometric representation as a ${}_2F_3$, they do not appear in the Askey scheme. In this paper we rectify this, by realizing two sets of q-Lommel polynomials as limiting cases of associated Askey–Wilson polynomials. One may ask for an associated Askey scheme which contains this limiting case (see Problem 8.7).

The paper is organized in the following way. In Section 2 we define the three sets of q-Lommel polynomials using three-term recurrence relations. The classical connection between these polynomials and q-Bessel functions is given in Section 3. The associated Askey–Wilson polynomials are reviewed in Section 4, along with explicit limiting cases to the q-Lommel polynomials; see Theorems 4.7 and 4.8. In Section 5 we independently prove the continued fraction expansions for the moment generating functions, and give two surprising equalities of continued fractions in Corollary 5.6 and Theorem 5.12. Combinatorial interpretations of these continued fractions are given in Section 6; see Theorem 6.9 and Corollary 6.11. A general combinatorial result for the concurrence of type R_I moments and orthogonal polynomials moments is given in Section 7; see Theorem 7.2. In Section 8 we propose some open problems.

We use the standard notations ${}_pF_q$ for hypergeometric series and ${}_p\phi_q$ for basic hypergeometric series (also sometimes called q-hypergeometric series) [14].

2. q-Lommel polynomials

In this section we give the defining recurrence relations for the Lommel, the classical q-Lommel, the even-odd q-Lommel, and the type R_I q-Lommel polynomials.

Definition 2.1. The *monic Lommel polynomials* $h_n(x;c)$ are defined by

$$h_{n+1}(x;c) = xh_n(x;c) - \frac{1}{(c+n)(c+n-1)}h_{n-1}(x;c), n \geq 0, \quad h_{-1}(x;c) = 0, h_0(x;c) = 1.$$

We consider three versions of q-Lommel polynomials.

Definition 2.2 ([15, § 14.4])**.** The *classical q-Lommel polynomials* are defined by

$$h_{n+1}(x;c,q) = xh_n(x;c,q) - \lambda_n h_{n-1}(x;c,q), \quad n \geq 0, \quad h_{-1}(x;c,q) = 0, \quad h_0(x;c,q) = 1,$$

$$\text{where} \quad \lambda_n = \frac{cq^{n-1}}{(1-cq^{n-1})(1-cq^n)}.$$

Definition 2.3. The *even-odd q-Lommel polynomials* are defined by

$$p_{n+1}(x;c,q) = xp_n(x;c,q) - \lambda_n p_{n-1}(x;c,q), \quad n \geq 0, \quad p_{-1}(x;c,q) = 0, \quad p_0(x;c,q) = 1,$$

$$\text{where} \quad \lambda_{2n} = \frac{cq^{3n-1}}{(1-cq^{2n-1})(1-cq^{2n})}, \qquad \lambda_{2n+1} = \frac{q^n}{(1-cq^{2n})(1-cq^{2n+1})}.$$

Note that each polynomial h_n and p_n may be considered as a q-analogue of the classical Lommel polynomials since

$$\lim_{q\to 1}(1-q)^n h_n(x/(1-q);q^c,q) = h_n(x;c), \quad \lim_{q\to 1}(1-q)^n p_n(x/(1-q);q^c,q) = h_n(x;c).$$

Definition 2.4. The *type R_I q-Lommel polynomials* are defined by

$$r_{n+1}(x;c,q) = (x-b_n)r_n(x;c,q) - xa_n r_{n-1}(x;c,q), \quad r_{-1}(x;c,q) = 0, \quad r_0(x;c,q) = 1, \tag{2.1}$$

$$\text{where} \quad b_n = \frac{q^n}{1-cq^n}, \qquad a_n = \frac{cq^{2n-1}}{(1-cq^{n-1})(1-cq^n)}.$$

Note that if

$$\hat{r}_n(x;c) = \lim_{q\to 1}(1-q)^{2n} r_n(x/(1-q)^2;q^c,q),$$

then

$$\hat{r}_{n+1}(x;c) = x\hat{r}_n(x;c) - \frac{x}{(c+n-1)(c+n)}\hat{r}_{n-1}(x;c). \tag{2.2}$$

The polynomials $\hat{r}_n(x;c)$ in (2.2) are closely related to the monic Lommel polynomials. For example, it is known that their moments are the same; see (7.2).

Koelink and Van Assche study the even-odd and the type R_I q-Lommel polynomials[1] in [23, Section 4], and Koelink continues this analytic study in [21].

Orthogonality relations for the classical q-Lommel are in [15, Theorem 14.4.3], while those for the even-odd q-Lommel and the type R_I q-Lommel are in [23, Theorem 4.2] and [23, Theorem 3.4].

3. q-Bessel functions and q-Lommel polynomials

In this section we give the recurrence relation which connects q-Bessel functions to the classical q-Lommel polynomials and the type R_I q-Lommel polynomials.

Definition 3.1. The *Bessel functions* $J_\nu(x)$ are defined by

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \sum_{n\geq 0} \frac{(-z^2/4)^n}{n!(\nu+1)_n}.$$

Definition 3.2 ([6, p. 188, (6.2)])**.** The *classical Lommel polynomials* $L_{n,\nu}(z)$ are (non-monic) polynomials in z^{-1} defined by $L_{0,\nu}(z) = 1$, $L_{1,\nu}(z) = 2\nu/z$, and

$$L_{n+1,\nu}(z) = \frac{2(n+\nu)}{z} L_{n,\nu}(z) - L_{n-1,\nu}(z), \qquad n \geq 1.$$

Equivalently,

$$h_n(x;c) = L_{n,c}(2/x)/(c)_n.$$

[1] Continued fractions of type R and the corresponding orthogonal polynomials of type R_I or R_{II} were introduced by Ismail and Masson in [16], the notation being a mnemonic for rational interpolation.

The Bessel function satisfy $J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) - J_{\nu-1}(z)$. Iterating this recurrence offers the following connection with Lommel polynomials.

Proposition 3.3 ([6, p. 187]). *The Bessel functions and the classical Lommel polynomials are related by the recurrence*

$$J_{\nu+n}(z) = L_{n,\nu}(z) J_\nu(z) - L_{n-1,\nu+1}(z) J_{\nu-1}(z).$$

Definition 3.4. *Jackson's first q-Bessel function* $J_\nu^{(1)}(z;q)$ and *second q-Bessel function* $J_\nu^{(2)}(z;q)$ are defined by

$$J_\nu^{(1)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} (z/2)^\nu {}_2\phi_1\left(0,0;q^{\nu+1};q,-z^2/4\right),$$
$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} (z/2)^\nu {}_0\phi_1\left(-;q^{\nu+1};q,-q^{\nu+1}z^2/4\right).$$

In this paper we consider only the first and third q-Bessel function, as the second q-Bessel can be obtained from the first by changing q to q^{-1}. Recall that we consider formal power series in z, and have no restriction on q.

Proposition 3.5 ([15, Eq. (14.4.1)]). *The first q-Bessel functions satisfy*

$$q^{n\nu+\binom{n}{2}} J_{\nu+n}^{(1)}(x;q) = L_{n,\nu}^{(1)}(x;q) J_\nu^{(1)}(x;q) - L_{n-1,\nu+1}^{(1)}(x;q) J_{\nu-1}^{(1)}(x;q). \tag{3.1}$$

where $L_{0,\nu}^{(1)}(x;q) = 1$, $L_{1,\nu}^{(1)}(x;q) = 2(1-q^{n+\nu})/x$, *and*

$$\frac{2}{x}(1-q^{n+\nu}) L_{n,\nu}^{(1)}(x;q) = L_{n+1,\nu}^{(1)}(x;q) + q^{n+\nu-1} L_{n-1,\nu}^{(1)}(x;q), \qquad n \geq 1.$$

Again, we need a rescaling to obtain the classical q-Lommel polynomials,

$$h_n(x;c,q) = L_{n,\nu}^{(1)}(2/x;q)/(q^\nu;q)_n, \quad c = q^\nu.$$

Definition 3.6. The *Jackson's third q-Bessel functions* $J_\nu^{(3)}(z;q)$ are defined by

$$J_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty z^\nu}{(q;q)_\infty} {}_1\phi_1\left(0;q^{\nu+1};q,qz^2\right).$$

Define the Laurent polynomials $L_{m,\nu}^{(3)}(z;q)$ by

$$L_{m+1,\nu}^{(3)}(z;q) = \left(z + z^{-1}(1-q^{\nu+m})\right) L_{m,\nu}^{(3)}(z;q) - L_{m-1,\nu}^{(3)}(z;q).$$

We rescale these Laurent polynomials to obtain polynomials

$$r_n^{(3)}(x;c,q) := \frac{x^{n/2}}{(q^{-\nu};q^{-1})_n} L_{n,\nu}^{(3)}(x^{-1/2};q^{-1}), \quad c = q^\nu. \tag{3.2}$$

Then $r_n^{(3)}(x;c,q)$ are the type R_I polynomials defined by $r_{-1}^{(3)}(x;c,q) = 0$, $r_0^{(3)}(x;c,q) = 1$, and

$$r_{n+1}^{(3)}(x;c,q) = (x - \hat{b}_n) r_n^{(3)}(x;c,q) - x\hat{a}_n r_{n-1}^{(3)}(x;c,q), \qquad n \geq 0,$$
$$\text{where } \hat{b}_n = \frac{cq^n}{1-cq^n}, \qquad \hat{a}_n = \frac{c^2 q^{2n-1}}{(1-cq^{n-1})(1-cq^n)}.$$

Using the recurrences one can easily check that

$$r_n(x;c,q) = \frac{r_n^{(3)}(cx;c,q)}{c^n},$$

where $r_n(x;c,q)$ are the type R_I q-Lommel polynomials $r_n(x;c,q)$ in Definition 2.4.

Koelink and Swarttouw [22, Eq. (4.12)] showed that the third q-Bessel functions satisfy the following property analogous to (3.3) and (3.1).

Proposition 3.7. *The third q-Bessel functions satisfy*

$$J^{(3)}_{\nu+m}(z;q) = L^{(3)}_{m,\nu}(z;q)J^{(3)}_{\nu}(z;q) - L^{(3)}_{m-1,\nu+1}(z;q)J^{(3)}_{\nu-1}(z;q).$$

Koelink and Swarttouw [22, Eq. (4.24)] also showed that

$$\lim_{m\to\infty} z^m R^{(3)}_{m,\nu}(z;q) = \frac{(q;q)_\infty z^{1-\nu}}{(z^2;q)_\infty} J^{(3)}_{\nu-1}(z;q),$$

which implies

$$\lim_{m\to\infty} \frac{R^{(3)}_{m,\nu+2}(z;q)}{R^{(3)}_{m+1,\nu+1}(z;q)} = \frac{J^{(3)}_{\nu+1}(z;q)}{J^{(3)}_{\nu}(z;q)}. \tag{3.3}$$

By (3.2) and (3.3) we have

$$\frac{J^{(3)}_{\nu+1}(x^{1/2};q^{-1})}{J^{(3)}_{\nu}(x^{1/2};q^{-1})} = \lim_{n\to\infty} \frac{-q^{\nu+1} r^{(3)}_n(x^{-1};q^{\nu+2},q)}{x^{1/2}(1-q^{\nu+1}) r^{(3)}_{n+1}(x^{-1};q^{\nu+1},q)}. \tag{3.4}$$

The q-Bessel function relation for the even-odd q-Lommel polynomials which corresponds to Proposition 3.5 is given in [23, Proposition 4.1].

4. q-Lommel polynomials and the Askey scheme

The q-Lommel polynomials do not appear in the Askey scheme. In this section we realize both the classical q-Lommel and the even-odd q-Lommel polynomials as limiting cases of the associated Askey–Wilson polynomials; see Theorems 4.7 and 4.8. We then use results of Ismail and Masson [16] to give explicit formulas for each polynomial. Finally, we prove that the moments for even-odd q-Lommel and the type R_I q-Lommel agree; see Theorem 4.14.

An explicit formula for the Lommel polynomial $h_n(x;c)$ is

$$h_n(x;c) = x^n {}_2F_3(-n/2,(1-n)/2;c,1-c-n,-n;-4/x^2).$$

In this section we give explicit formulas for our three families of q-Lommel polynomials. The classical q-Lommel polynomials have a corresponding single sum formula [15, Theorem 14.4.1]:

$$h_n(x;c,q) = \frac{1}{(c;q)_n} \sum_{j=0}^{\lfloor n/2\rfloor} \frac{(-1)^j\,(c,q;q)_{n-j}}{(q,c;q)_j\,(q;q)_{n-2j}} x^{n-2j} c^j q^{j(j-1)}.$$

Here are the main results for the even-odd q-Lommel polynomials.

Theorem 4.1. *The even even-odd q-Lommel polynomials have the explicit formula*

$$p_{2n}(x;c,q) = (-1)^n \frac{q^{\binom{n}{2}}}{(c;q)_{2n}} \sum_{k=0}^{n} \frac{(q^{-n},cq^n,c;q)_k}{(q;q)_k} q^k x^{2k}$$
$$\times \sum_{s=0}^{n-k} \frac{(cq^{k-1};q)_s}{(q;q)_s} \frac{1-cq^{k-1+2s}}{1-cq^{k-1}} \frac{(cq^{n+k},q^{k-n},q^k;q)_s}{(q^{-n},cq^n,c;q)_s} c^s q^{-sk+s(s-1)}.$$

Theorem 4.2. *The odd even-odd q-Lommel polynomials have the explicit formula*

$$p_{2n+1}(x;c,q) = (-c)^n \frac{q^{n^2+\binom{n+1}{2}}}{(cq;q)_{2n}} \sum_{k=0}^{n} \frac{(q^{-n}, cq^{n+1}, cq;q)_k}{(q;q)_k} c^{-k} q^{-k^2} x^{2k+1}$$
$$\times \sum_{s=0}^{n-k} \frac{(cq^k;q)_s}{(q;q)_s} \frac{1-cq^{k+2s}}{1-cq^k} \frac{(cq^{n+k+1}, q^{k-n}, q^{k+1};q)_s}{(q^{-n}, cq^{n+1}, c;q)_s} c^s q^{-(3k+2)s-s(s-1)}.$$

Note that the inner sums in Theorems 4.1 and 4.2 are basic hypergeometric series. The coefficient of x^{2k} in Theorem 4.1 and x^{2k+1} in Theorem 4.2 are terminating basic hypergeometric series.

In order to prove Theorems 4.1 and 4.2, we first write the even even-odd polynomials as orthogonal polynomials in x^2 using the odd-even trick. Then we realize the new polynomials as limiting cases of *associated Askey–Wilson polynomials*, for which explicit formulas are known. The same method will work for the odd even-odd polynomials.

We begin with the associated Askey–Wilson polynomials. First, recall that the *monic Askey–Wilson polynomials* satisfy the following three-term recurrence (which, as claimed by Favard's theorem, is characterizing orthogonal polynomials):

$$p_{n+1}(x) = (x-b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad n \geq 1, \tag{4.1}$$

where

$$b_n = \frac{1}{2}(a + a^{-1} - A_n - C_n), \qquad \lambda_n = \frac{1}{4}A_{n-1}C_n,$$
$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$
$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

Then, the *associated Askey–Wilson polynomials* are defined by the same three-term recurrence relation (4.1), with q^n replaced by αq^n.

Definition 4.3. The *associated Askey–Wilson polynomials* $p_n^{(\alpha)}(x)$ are defined as a solution to

$$p_{n+1}^{(\alpha)}(x) = (x - b_n(\alpha))p_n^{(\alpha)}(x) - \lambda_n(\alpha)p_{n-1}^{(\alpha)}(x), \quad n \geq 1, \tag{4.2}$$

$$b_n(\alpha) = \frac{1}{2}(a + a^{-1} - A_n(\alpha) - C_n(\alpha)), \qquad \lambda_n(\alpha) = \frac{1}{4}A_{n-1}(\alpha)C_n(\alpha),$$

$$A_n(\alpha,q) = \frac{(1-ab\alpha q^n)(1-ac\alpha q^n)(1-ad\alpha q^n)(1-abcd\alpha q^{n-1})}{a(1-abcd\alpha^2 q^{2n-1})(1-abcd\alpha^2 q^{2n})},$$
$$C_n(\alpha,q) = \frac{a(1-\alpha q^n)(1-bc\alpha q^{n-1})(1-bd\alpha q^{n-1})(1-cd\alpha q^{n-1})}{(1-abcd\alpha^2 q^{2n-2})(1-abcd\alpha^2 q^{2n-1})}.$$

There are two linearly independent solutions to (4.2), depending on the initial conditions. Ismail and Rahman [15, Eq. (4.15), Eq. (8.9)] gave these two independent solutions as double sums, the inner sum being a very-well-poised ${}_{10}W_9$.

Theorem 4.4. *Two linearly independent solutions $\psi_n^{(\alpha,\epsilon)}(x,q), \epsilon = 1, 2$ to (4.2) are given by*

$$\psi_n^{(\alpha,\epsilon)}(x;q) = K_n \sum_{k=0}^{n} \frac{(q^{-n}, abcd\alpha^2 q^{n-1}, abcd\alpha^2/q, az, a/z; q)_k}{(q, ab\alpha, ac\alpha, ad\alpha, abcd\alpha/q; q)_k} q^k$$
$$\times {}_{10}W_9\left(abcd\alpha^2 q^{k-2}; \alpha, bc\alpha/q, bd\alpha/q, cd\alpha/q, S, abcd\alpha^2 q^{n+k-1}, q^{k-n}; q; T\right)$$

where

$$K_n = (2a)^{-n} \frac{(ab\alpha, ac\alpha, ad\alpha, abcd\alpha/q; q)_n}{(abcd\alpha^2 q^{n-1}, abcd\alpha^2/q; q)_n}$$

and the two choices for ϵ correspond to

$$(S,T) = (q^{k+1}, a^2), \text{ for } \epsilon = 1, \ (S,T) = (q^k, qa^2), \text{ for } \epsilon = 2.$$

We next explain how Theorem 4.1 follows from Theorem 4.4. First we rewrite the recurrence relation [6] in terms of polynomials in x^2. Let $t_n(x)$ and $s_n(x)$ be the polynomials satisfying

$$p_{2n}(x;c,q) = t_n(x^2),$$
$$p_{2n+1}(x;c,q) = x s_n(x^2).$$

Proposition 4.5. *We have*

$$t_{n+1}(x) = (x - B_n)t_n(x) - \Lambda_n t_{n-1}(x), \quad t_{-1} = 0, \quad t_0(x) = 1.$$

where

$$B_0 = \frac{1}{(1-c)(1-cq)},$$
$$B_n = \lambda_{2n} + \lambda_{2n+1}, \quad n \geq 1,$$
$$\Lambda_n = \lambda_{2n-1}\lambda_{2n}, \quad n \geq 1.$$

Proposition 4.6. *We have*

$$s_{n+1}(x) = (x - B_n)s_n(x) - \Lambda_n s_{n-1}(x), \quad s_{-1} = 0, \quad s_0(x) = 1.$$

where

$$B_n = \lambda_{2n+2} + \lambda_{2n+1}, \quad n \geq 1,$$
$$\Lambda_n = \lambda_{2n+1}\lambda_{2n}, \quad n \geq 1.$$

We shall obtain the recurrence relations in Propositions 4.5 and 4.6 by an appropriate limiting case of Theorem 4.4. Our goal is to obtain $(A_n, C_n) = (\lambda_{2n+1}, \lambda_{2n})$ for $t_n(x)$ and $(A_n, C_n) = (\lambda_{2n+2}, \lambda_{2n+1})$ for $s_n(x)$. Then we match the initial conditions to find the correct linear combination of the two solutions.

First choosing $a = c^{-1}q^{-1}\alpha$, $b = c = d = 1/\alpha$, we obtain

$$A_n(\alpha, 1/q) = \frac{\alpha(1 - cq^{n+1}/\alpha)^3(1 - \alpha c q^n)}{cq(1 - cq^{2n})(1 - cq^{2n+1})},$$
$$C_n(\alpha, 1/q) = \frac{cq(1 - q^n/\alpha)(1 - \alpha q^{n-1})^3}{\alpha(1 - cq^{2n-1})(1 - cq^{2n})}.$$

By rescaling x by $B\alpha^2x/2$, i.e., $\hat{p}_n(x) = 2^n\alpha^{-2n}B^{-n}p_n^{(\alpha)}(B\alpha^2x/2)$, we have

$$\hat{p}_{n+1}(x) = (x - \hat{b}_n(\alpha))\hat{p}_n(x) - \hat{\lambda}_n(\alpha)\hat{p}_{n-1}(x),$$

$$\hat{b}_n(\alpha) = \frac{1}{B\alpha^2}\left(\frac{cq}{\alpha} + \frac{\alpha}{cq} - A_n(\alpha,1/q) - C_n(\alpha,1/q)\right),$$
$$\hat{\lambda}_n(\alpha) = \frac{1}{B^2\alpha^4}A_n(\alpha,1/q)C_n(\alpha,1/q).$$

If $\alpha \to \infty$, the first two terms in $\hat{b}_n(\alpha)$ vanish. Choosing $B = 1/q$, we obtain the desired values for Proposition 4.5

$$\lim_{\alpha\to\infty}\frac{-1}{B\alpha^2}A_n(\alpha,1/q) = \frac{q^n}{(1-cq^{2n})(1-cq^{2n+1})} = \lambda_{2n+1},$$
$$\lim_{\alpha\to\infty}\frac{-1}{B\alpha^2}C_n(\alpha,1/q) = \frac{cq^{3n-1}}{(1-cq^{2n-1})(1-cq^{2n})} = \lambda_{2n}.$$

The first degree limiting polynomial matches the second Ismail–Rahman solution in Theorem 4.4 with $(a,b,c,d) = (\alpha/cq, 1/\alpha, 1/\alpha, 1/\alpha)$,

$$x - \frac{1}{(1-c)(1-cq)}$$

so that

$$\lim_{\alpha\to\infty}\hat{p}_n(x) = \lim_{\alpha\to\infty}\psi_n^{(\alpha,2)}(x;1/q),$$

which is the stated explicit formula in Theorem 4.1.

For the odd even-odd polynomials in Proposition 4.6, we choose $(a,b,c,d) = (cq^2\alpha, 1/\alpha, 1/\alpha, 1/\alpha)$,

$$A_n(\alpha,q) = \frac{(1-c\alpha q^{n+2})^3(1-cq^{n+1}/\alpha)}{\alpha cq^2(1-cq^{2n+1})(1-cq^{2n+2})},$$
$$C_n(\alpha,q) = \frac{\alpha cq^2(1-\alpha q^n)(1-q^{n-1}/\alpha)^3}{(1-cq^{2n})(1-cq^{2n+1})}.$$

As before choosing $\hat{p}_n(x) = 2^n\alpha^{-2n}B^{-n}p_n^{(\alpha)}(B\alpha^2x/2)$ and $B = -cq^2$ we find

$$\lim_{\alpha\to\infty}\frac{1}{B\alpha^2}A_n(\alpha,q) = \frac{cq^{3n+2}}{(1-cq^{2n+1})(1-cq^{2n+2})} = \lambda_{2n+2},$$
$$\lim_{\alpha\to\infty}\frac{1}{B\alpha^2}C_n(\alpha,q) = \frac{q^n}{(1-cq^{2n})(1-cq^{2n+1})} = \lambda_{2n+1}.$$

The first degree limiting polynomial matches the first Ismail–Rahman solution in Theorem 4.4 with $(a,b,c,d) = (cq^2\alpha, 1/\alpha, 1/\alpha, 1/\alpha)$,

$$x - \frac{1+cq}{(1-c)(1-cq^2)}$$

so that

$$\lim_{\alpha\to\infty}\hat{p}_n(x) = \lim_{\alpha\to\infty}\psi_n^{(\alpha,1)}(x;q),$$

which is the stated explicit formula in Theorem 4.2.

We summarize these limits for the even-odd q-Lommel polynomials.

Theorem 4.7. *The even-odd q-Lommel polynomials are the following limits of associated Askey–Wilson polynomials*

$$p_{2n}(x;c,q) = \lim_{\alpha\to\infty} \frac{(2q)^n}{\alpha^{2n}} \psi_n^{(\alpha,2)}(\alpha^2x^2/2q;1/q), \quad (a,b,c,d) = (\alpha/cq, 1/\alpha, 1/\alpha, 1/\alpha),$$

$$p_{2n+1}(x;c,q) = x \lim_{\alpha\to\infty} \frac{(-2/cq^2)^n}{\alpha^{2n}} \psi_n^{(\alpha,1)}(-cq^2\alpha^2x^2/2;q), \ (a,b,c,d) = (cq^2\alpha, 1/\alpha, 1/\alpha, 1/\alpha).$$

For the classical q-Lommel polynomials $h_n(x;c;q)$, for the even polynomials choose

$$(a,b,c,d) = (1, q/\alpha^2, c, 1)$$

and for the odd polynomials choose

$$(a,b,c,d) = (1, q^2/\alpha^2, c, 1).$$

Similar calculations to the proof of Theorem 4.7 show the next result.

Theorem 4.8. *The classical q-Lommel polynomials are the following limits of associated Askey–Wilson polynomials*

$$h_{2n}(x;c,q) = \lim_{\alpha\to\infty} \frac{(-2)^n}{\alpha^{2n}} \psi_n^{(\alpha,2)}(-\alpha^2x^2/2;q), \quad (a,b,c,d) = (1, q/\alpha^2, c, 1),$$

$$h_{2n+1}(x;c,q) = x \lim_{\alpha\to\infty} \frac{(-2q)^n}{\alpha^{2n}} \psi_n^{(\alpha,1)}(-\alpha^2x^2/2q;q), \quad (a,b,c,d) = (1, q^2/\alpha^2, c, 1).$$

Theorem 4.9 is [15, Theorem 14.4.1].

Theorem 4.9. *The classical q-Lommel polynomials are*

$$h_n(x;c,q) = \sum_{k=0}^{n/2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{(-c)^k q^{k^2-k}}{(c;q)_k (cq^{n-1};q^{-1})_k} x^{n-2k}.$$

Proof. We consider the even case, the proof for the odd case is similar. The inner sum becomes an evaluable very-well-poised ${}_6W_5$

$${}_6W_5\left(cq^{k-1}; q^k, cq^{n+k}, q^{k-n} \;\middle|\; q; q^{-2k}\right) = \frac{(cq^{n-1};q^{-1})_k (q^{n+1};q)_k}{(c;q)_k (q^{k+1};q)_k} q^{-k(n-k)}.$$

By considering the coefficient of x^{2n-2k}, we arrive at Theorem 4.9 with n replaced by $2n$. The odd case actually gives the same result. □

For the type R_I q-Lommel polynomials there is a simple generating function which gives an explicit expression.

Proposition 4.10. *The type R_I q-Lommel polynomials have the generating function*

$$\sum_{n=0}^{\infty} (c^{-1};q^{-1})_n \ r_n(x;c,q) t^n = \sum_{k=0}^{\infty} \frac{(-xt/c)^k q^{-\binom{k}{2}}}{(t/c, tx; q^{-1})_{k+1}}.$$

Proof. If $G(x,t)$ is the generating function on the left side, then Definition 2.4 implies

$$G(x,t) - 1 = (x+1/c)tG(x,t) - xt/c\, G(x,tq^{-1}) - xt^2/c\, G(x,t),$$

$$G(x,t) = \frac{1}{(1-xt)(1-t/c)} - \frac{xt/c}{(1-xt)(1-t/c)} G(x,tq^{-1})$$

whose iterate is the result. □

Theorem 4.11. *The type R_I q-Lommel polynomials have the explicit formula*

$$r_n(x;c,q) = \frac{1}{(c^{-1};q^{-1})_n} \sum_{k=0}^{n} \sum_{a=0}^{n-k} (-x/c)^k q^{-\binom{k}{2}} \begin{bmatrix} k+a \\ a \end{bmatrix}_{q^{-1}} c^{-a} \begin{bmatrix} n-a \\ k \end{bmatrix}_{q^{-1}} x^{n-k-a}.$$

Proof. Apply the q^{-1}-binomial theorem to Proposition 4.10 to find the resulting coefficient of t^n. □

Proposition 4.12. *We have the connection coefficient relation*

$$r_n(x^2;c,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{c^k q^{n^2-(n-k)^2}}{(cq^{n-1}, cq^{2n-k};q^{-1})_k} p_{2n-2k}(x;c,q).$$

Proof. Induction on n using the three-term relations. □

Proposition 4.13. *If $\mathcal{L}_p$ is the linear functional for the even-odd polynomials $p_n(x;c,q)$, then*

$$\mathcal{L}_p(r_n(x^2;c,q)) = \frac{c^n q^{n^2}}{(c,cq;q)_n}.$$

Proof. Apply $\mathcal{L}_p$ to both sides of Proposition 4.12. By orthogonality, $\mathcal{L}_p(p_j(x)) = 0$ for $j > 0$, so only the $k = n$ term survives. □

Theorem 4.14. *The moments of the type R_I q-Lommel polynomials are equal to the even moments of the even-odd q-Lommel polynomials,*

$$\mathcal{L}_r(x^m) = \mathcal{L}_p(x^{2m}), \quad m \geq 0.$$

Proof. Using the three-term recurrence (2.1), the type R_I moments $\mathcal{L}_r(x^m)$ are recursively determined by [18, Corollary 3.15]

$$\mathcal{L}_r(r_n(x;c,q)) = a_1 a_2 \cdots a_n = \frac{c^n q^{n^2}}{(c,cq;q)_n}, \quad n \geq 0.$$

By Proposition 4.13 the moments $\mathcal{L}_p(x^{2m})$ satisfy the same recurrence. □

For completeness, we give the inverse relation to Proposition 4.12.

Proposition 4.15. *We have the connection coefficient relation*

$$p_{2n}(x;c,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-c)^k q^{2nk-\binom{k+1}{2}}}{(cq^{n-1}, cq^{2n-1};q^{-1})_k} r_{n-k}(x^2;c,q).$$

Proposition 4.16. *The even-odd q-Lommel polynomials have the explicit expressions*

$$p_{2n}(x;c,q) = \frac{1}{(c;q)_{2n}} \sum_{k=0}^{n} (-1)^k x^{2n-2k} (cq^k;q)_{2n-2k} \sum_{j=0}^{k} \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q \begin{bmatrix} n-k+j-1 \\ j \end{bmatrix}_q c^j q^{jn+\binom{k}{2}},$$

$$p_{2n+1}(x;c,q) = \frac{1}{(c;q)_{2n+1}} \sum_{k=0}^{n} (-1)^k x^{2n-2k+1} (cq^k;q)_{2n-2k+1} \sum_{j=0}^{k} \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q \begin{bmatrix} n-k+j \\ j \end{bmatrix}_q c^j q^{jn+\binom{k}{2}}.$$

Proof. This follows from Definition 2.3, by considering the coefficients of $x^{2n-2k-1}$. □

5. Moments and continued fractions

In this section we review the known facts which connect continued fractions to moment generating functions. We independently prove the continued fractions for the moment generating functions of each of the three q-Lommel polynomials.

Definition 5.1. Take a sequence of orthogonal polynomials $p_n(x)$ which satisfy $p_{-1}(x) = 0$, $p_0(x) = 1$, and

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad n \geq 0,$$

and whose linear functional for orthogonality is $\mathcal{L}_p$. Define

$$\mu_n(\{b_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 0}) = \mathcal{L}_p(x^n).$$

The moment generating function for $\mathcal{L}_p$ is

$$\sum_{n=0}^{\infty} \mathcal{L}_p(x^n)t^n = \sum_{n=0}^{\infty} \mu_n(\{b_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 0})t^n.$$

A Jacobi continued fraction also exists, converging as formal power series in t:

$$\sum_{n=0}^{\infty} \mathcal{L}_p(x^n)t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{1 - \ddots}}}. \tag{5.1}$$

Definition 5.2 ([18])**.** For general type R_I orthogonal polynomials

$$r_{n+1}(x) = (x - b_n)r_n(x) - (a_n x + \lambda_n)r_{n-1}(x), \quad n \geq 0,$$

with linear functional $\mathcal{L}_r$, define

$$\mu_n(\{b_k\}_{k\geq 0}, \{a_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 0}) = \mathcal{L}_r(x^n). \tag{5.2}$$

The corresponding continued fraction for the type R_I moment generating function is [18, Corollary 3.7]

$$\sum_{n=0}^{\infty} \mathcal{L}_r(x^n)t^n = \cfrac{1}{1 - b_0 t - \cfrac{a_1 t + \lambda_1 t^2}{1 - b_1 t - \cfrac{a_2 t + \lambda_2 t^2}{1 - \ddots}}}. \tag{5.3}$$

Note that both continued fractions in (5.1) and (5.3) are explicitly given in terms of the three-term recurrence coefficients. We shall evaluate the continued fractions as quotients of basic hypergeometric series, namely q-Bessel functions, using contiguous relations.

For the Lommel polynomials $h_n(x; c)$, it is known that the moment generating function is a quotient of Bessel functions, with $\lambda_n = 1/(c + n - 1)(c + n)$,

$$\sum_{n=0}^{\infty} \mathcal{L}_h(x^n)t^n = \frac{{}_0F_1(c + 1; -t^2)}{{}_0F_1(c; -t^2)} = \cfrac{1}{1 - \cfrac{\lambda_1 t^2}{1 - \cfrac{\lambda_2 t^2}{1 - \ddots}}}.$$

The moment generating function for the classical q-Lommel polynomials is a quotient of q-Bessel functions. In this section we shall see that a corresponding result holds for our other two q-Lommel polynomials, and in fact their moment generating functions are equal.

Theorem 5.3 ([15, Theorem 14.4.3])**.** *The moment generating function for the classical q-Lommel polynomials $h_n(x;c,q)$ is a quotient of Jackson's first q-Bessel functions*

$$\sum_{n=0}^{\infty} \mathcal{L}_h(x^n)t^n = \frac{{}_2\phi_1\left(0,0;cq;q,-t^2\right)}{{}_2\phi_1\left(0,0;c;q;-t^2\right)} = \cfrac{1}{1-\cfrac{\lambda_1 t^2}{1-\cfrac{\lambda_2 t^2}{1-\ddots}}},$$

where, as in Definition 2.2, $\lambda_n = cq^{n-1}/((1-cq^{n-1})(1-cq^n))$.

Theorem 5.4. *The moment generating function for the even-odd q-Lommel polynomials $p_n(x;c,q)$ is a quotient of Jackson's third q-Bessel functions*

$$\sum_{n=0}^{\infty} \mathcal{L}_p(x^n)t^n = \frac{{}_1\phi_1\left(0;cq;q;qt^2\right)}{{}_1\phi_1\left(0;c;q;t^2\right)} = \cfrac{1}{1-\cfrac{\lambda_1 t^2}{1-\cfrac{\lambda_2 t^2}{1-\ddots}}},$$

where, as in Definition 2.3,

$$\lambda_{2n} = \frac{cq^{3n-1}}{(1-cq^{2n-1})(1-cq^{2n})}, \qquad \lambda_{2n+1} = \frac{q^n}{(1-cq^{2n})(1-cq^{2n+1})}.$$

Theorem 5.5. *The moment generating function for the type R_I q-Lommel polynomials $r_n(x;c,q)$ is a quotient of Jackson's third q-Bessel functions*

$$\sum_{n=0}^{\infty} \mathcal{L}_r(x^n)z^n = \frac{{}_1\phi_1\left(0;cq;q;qz\right)}{{}_1\phi_1\left(0;c;q;z\right)} = \cfrac{1}{1-b_0z-\cfrac{a_1 z}{1-b_1z-\cfrac{a_2 z}{1-b_2z-\ddots}}},$$

where, as in Definition 2.4,

$$a_n = \frac{cq^{2n-1}}{(1-cq^{n-1})(1-cq^n)}, \qquad b_n = \frac{q^n}{1-cq^n}.$$

Theorem 4.14 implies that the two continued fractions in Theorems 5.4 and 5.5 with $z=t^2$ are equal.

Corollary 5.6. *We have the equality of continued fractions*

$$\cfrac{1}{1-b_0z-\cfrac{a_1 z}{1-b_1z-\cfrac{a_2 z}{1-b_2z-\ddots}}} = \cfrac{1}{1-\cfrac{\lambda_1 z}{1-\cfrac{\lambda_2 z}{1-\ddots}}},$$

where a_n, b_n, and λ_n are defined as in Theorems 5.4 and 5.5.

Theorems 5.3, 5.4, and 5.5 may all be proven using contiguous relations for hypergeometric and basic hypergeometric series. To prove Theorems 5.3 and 5.4 we use Heine's contiguous relation [10, Eq. 17.6.19] which is

$$ {}_2\phi_1\left(aq,b;cq;q,z\right) - {}_2\phi_1\left(a,b;c;q,z\right) = \frac{(1-b)(a-c)z}{(1-c)(1-cq)} {}_2\phi_1\left(aq,bq;cq^2;q,z\right). $$

Equivalently,

$$ \frac{{}_2\phi_1\left(aq,b;cq;q,z\right)}{{}_2\phi_1\left(a,b;c;q,z\right)} = \frac{1}{1-\dfrac{(1-b)(a-c)z}{(1-c)(1-cq)}\cdot\dfrac{{}_2\phi_1\left(bq,aq;cq^2;q,z\right)}{{}_2\phi_1\left(b,aq;cq;q,z\right)}}. \tag{5.4} $$

Applying (5.4) iteratively, we obtain Heine's continued fraction, which is a q-analogue of Gauss's continued fraction.

Lemma 5.7 (Heine's fraction)**.** *We have*

$$ \frac{{}_2\phi_1\left(aq,b;cq;q,z\right)}{{}_2\phi_1\left(a,b;c;q,z\right)} = \cfrac{1}{1-\cfrac{\beta_1 z}{1-\cfrac{\beta_2 z}{1-\ddots}}}, $$

where

$$ \beta_{2n+1} = \frac{(1-bq^n)(a-cq^n)q^n}{(1-cq^{2n})(1-cq^{2n+1})}, \qquad \beta_{2n} = \frac{(1-aq^n)(b-cq^n)q^{n-1}}{(1-cq^{2n-1})(1-cq^{2n})}. $$

Theorem 5.3 is the special case $a=b=0$ and $z=-t^2$ of Lemma 5.7. Theorem 5.4 is also the limiting case $z=-t^2/a$, $b=0$, $a\to\infty$ of Lemma 5.7.

For Theorem 5.5 we need the q-Nörlund fraction [8, Eq. (19.2.7)]. However, to simplify the expressions we need some notation for continued fractions.

Definition 5.8. For sequences a_i and b_i, let

$$ \mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{a_i}{b_i}\right) = \cfrac{a_0}{b_0+\cfrac{a_1}{b_1+\ddots+\cfrac{a_m}{b_m}}}, \qquad \mathop{\mathbf{K}}_{i=0}^{\infty}\left(\frac{a_i}{b_i}\right) = \cfrac{a_0}{b_0+\cfrac{a_1}{b_1+\ddots}}. $$

The following lemma will be used later.

Lemma 5.9. *For any sequences $\{a_i : 0\le i\le m\}$, $\{b_i : 0\le i\le m\}$, and $\{c_i : -1\le i\le m\}$, we have*

$$ \mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{a_i}{b_i}\right) = \frac{1}{c_{-1}}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{a_i c_{i-1} c_i}{b_i c_i}\right). $$

Proof. By multiplying the numerator and denominator of the i^{th} fraction by c_i, we obtain

$$ \cfrac{a_0}{b_0+\cfrac{a_1}{b_1+\ddots+\cfrac{a_m}{b_m}}} = \cfrac{a_0c_0}{b_0c_0+\cfrac{a_1c_0c_1}{b_1c_1+\ddots+\cfrac{a_mc_{m-1}c_m}{b_mc_m}}}, $$

which is equivalent to the equation in the lemma. $\square$

Lemma 5.10 (q-Nörlund fraction)**.** *We have*

$$\frac{{}_2\phi_1(a,b;c;q,z)}{{}_2\phi_1(aq,bq;cq;q,z)} = \frac{1-c-(a+b-ab-abq)z}{1-c} + \frac{1}{1-c}\mathop{\mathbf{K}}_{m=1}^{\infty}\left(\frac{c_m(z)}{e_m+d_m z}\right),$$

where

$$\begin{aligned} c_m(z) &= (1-aq^m)(1-bq^m)(cz-abq^m z^2)q^{m-1}, \\ e_m &= 1-cq^m, \\ d_m &= -(a+b-abq^m-abq^{m+1})q^m. \end{aligned}$$

The q-Nörlund fraction can be restated in the form of a continued fraction for type R_I orthogonal polynomials.

Proposition 5.11 (q-Nörlund fraction restated)**.** *We have*

$$\frac{{}_2\phi_1(aq,bq;cq;q,z)}{{}_2\phi_1(a,b;c;q,z)} = \cfrac{1}{1-b_0z-\cfrac{a_1z+\lambda_1z^2}{1-b_1z-\cfrac{a_2z+\lambda_2z^2}{1-b_2z-\ddots}}},$$

where

$$\begin{aligned} b_m &= \frac{(a+b-abq^m-abq^{m+1})q^m}{1-cq^m}, \\ a_m &= -\frac{(1-aq^m)(1-bq^m)cq^{m-1}}{(1-cq^{m-1})(1-cq^m)}, \\ \lambda_m &= \frac{(1-aq^m)(1-bq^m)abq^{2m-1}}{(1-cq^{m-1})(1-cq^m)}. \end{aligned}$$

Proof. By taking the inverse on each side of the equation in Lemma 5.10, we obtain

$$\frac{{}_2\phi_1(aq,bq;cq;q,z)}{{}_2\phi_1(a,b;c;q,z)} = \frac{1-c}{c_0(z)}\mathop{\mathbf{K}}_{m=0}^{\infty}\left(\frac{c_m(z)}{e_m+d_m z}\right).$$

Applying Lemma 5.9 with $c_i = 1/(1-cq^i)$ and $m\to\infty$ yields

$$\frac{{}_2\phi_1(aq,bq;cq;q,z)}{{}_2\phi_1(a,b;c;q,z)} = \frac{(1-cq^{-1})(1-c)}{c_0(z)}\mathop{\mathbf{K}}_{m=0}^{\infty}\left(\frac{c_m(z)/(1-cq^{m-1})(1-cq^m)}{e_m/(1-cq^m)+d_m z/(1-cq^m)}\right),$$

which is the same as the desired identity. □

Proof of Theorem 5.5. Replace z by z/b, put $a=0$, and let $b\to\infty$ in Proposition 5.11. The result is Theorem 5.5. □

Note that when $b=0$ both Lemma 5.7 and Proposition 5.11 give a continued fraction expression for

$$\frac{{}_2\phi_1(a,0;c;q,z)}{{}_2\phi_1(aq,0;cq;q,z)}.$$

Therefore we obtain the following theorem.

Theorem 5.12. *We have the equality of continued fractions*

$$\cfrac{1}{1-b_0z-\cfrac{a_1z}{1-b_1z-\cfrac{a_2z}{1-b_2z-\ddots}}}=\cfrac{1}{1-\cfrac{\lambda_1z}{1-\cfrac{\lambda_2z}{1-\ddots}}},$$

where

$$a_n=\frac{(aq^n-1)cq^{n-1}}{(1-cq^{n-1})(1-cq^n)},\qquad b_n=\frac{aq^n}{1-cq^n},$$

$$\lambda_{2n}=\frac{-cq^{2n-1}(1-aq^n)}{(1-cq^{2n-1})(1-cq^{2n})},\qquad \lambda_{2n+1}=\frac{(a-cq^n)q^n}{(1-cq^{2n})(1-cq^{2n+1})}.$$

When Theorem 5.12 is interpreted as an equality for moment generating functions, we find the following generalization of Theorem 4.14 which holds for q-Lommel polynomials.

Corollary 5.13. *Let λ_n, a_n and b_n be given by Theorem 5.12. The $2n^{\text{th}}$ moment of the orthogonal polynomials defined by $p_{n+1}(x)=xp_n(x)-\lambda_n p_{n-1}(x)$ is equal to the n^{th} moment of the type R_I polynomials defined by $r_{n+1}(x)=(x-b_n)r_n(x)-a_nxr_{n-1}(x)$.*

6. Combinatorics of moments of type R_I q-Lommel polynomials

The moment generating function for type R_I polynomials is given by the continued fraction in (5.3). For type R_I q-Lommel polynomials we give in this section a general combinatorial interpretation for this infinite continued fraction in terms of parallelogram polyominoes. We also interpret the finite continued fraction and give an explicit rational expression using q-Lommel polynomials. To be specific we give a combinatorial interpretation for the ratio

$$r_n^{(3)}(x^{-1};q^{\nu+2},q)/r_{n+1}^{(3)}(x^{-1};q^{\nu+1},q)$$

of (rescaled) type R_I q-Lommel polynomials, Theorem 6.9. This is a finite version of the result of Bousquet-Mélou and Viennot [4]. The $n\to\infty$ limit of Theorem 6.9 yields a quotient of q-Bessel functions,

$$J_{\nu+1}^{(3)}(x^{1/2};q^{-1})/J_{\nu}^{(3)}(x^{1/2};q^{-1})$$

which is the moment generating function for the type R_I q-Lommel polynomials. This material appears in our unpublished manuscript [17, Section 5].

We shall need several definitions related to parallelogram polyominoes and Motzkin paths.

Definition 6.1. An *NE-path* is a lattice path from $(0,0)$ to (a,b) for some positive integers a,b consisting of north steps $(0,1)$ and east steps $(1,0)$. A *parallelogram polyomino* is a set of unit squares enclosed by two NE-paths with the same ending points that do not intersect except the starting and ending points. Denote by $\mathcal{P}$ the set of parallelogram polyominoes.

For a parallelogram polyomino $\alpha\in\mathcal{P}$ let $U(\alpha)$ be the upper boundary path and $D(\alpha)$ the lower boundary path; see Figure 1. A *diagonal* of α is the set of squares in α whose centers are on the line $x+y=i$ for some integer i. The *size* of a diagonal is the number of squares in it. See Figure 2.

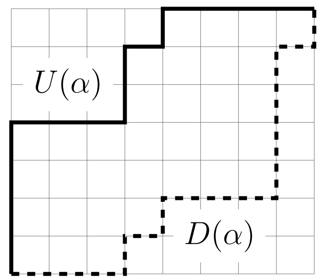

FIGURE 1. The boundary paths $U(\alpha)$ and $D(\alpha)$ for a parallelogram polyomino.

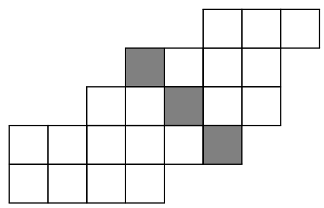

FIGURE 2. A diagonal with size 3 in a parallelogram polyomino.

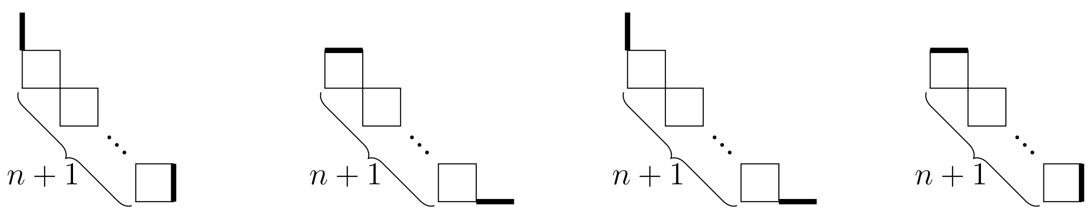

FIGURE 3. From left to right are shown an NN-diagonal, EE-diagonal, NE-diagonal, and EN-diagonal of size $n+1$ whose weights are, respectively, a_n, b_n, c_n, and d_n.

Definition 6.2. We denote by $\mathcal{P}^{\le k}$ the set of parallelogram polyominoes in which every diagonal has size at most k.

Consider $\alpha \in \mathcal{P}$ and a diagonal τ of α. Let u (resp. d) be the northwest (resp. southeast) corner of the topmost (resp. bottommost) square of τ. There are four cases for d. We say that d is an *NN-diagonal* (resp. *NE-diagonal*, *EN-diagonal*, and *EE-diagonal*) if the step in $U(\alpha)$ starting at u is a north (resp. north, east, and east) step and the step in $D(\alpha)$ starting at d is a north (resp. east, north, and east) step. See Figure 3.

For sequences $\{a_n\}_{n\ge 0}, \{b_n\}_{n\ge 0}, \{c_n\}_{n\ge 0}$, and $\{d_n\}_{n\ge 0}$, define the *weight* $\mathrm{wt}(\alpha; a, b, c, d)$ of $\alpha \in \mathcal{P}$ to be the product of a_n (resp. b_n, c_n, and d_n) for each NN-diagonal (resp. EE-diagonal, NE-diagonal, and EN-diagonal) of size $n+1$.

Now we review Flajolet's theory [13] on continued fraction expressions for Motzkin path generating functions.

Definition 6.3. A *Motzkin path* is a lattice path from $(0,0)$ to $(n,0)$ consisting of up steps $(1,1)$, down steps $(1,-1)$, and horizontal steps $(1,0)$ that never goes below the x-axis. A *2-Motzkin path* is a Motzkin path in which every horizontal step is colored red or blue. The *height* of a 2-Motzkin path is the largest integer y for which (x,y) is a point in the path.

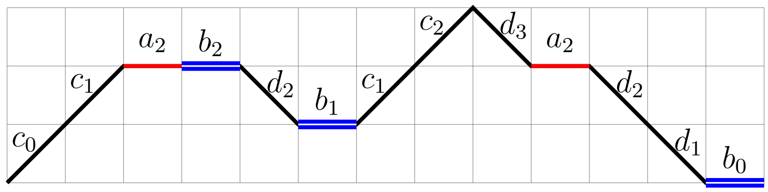

FIGURE 4. A 2-Motzkin path p in $\mathrm{Motz}_2^{\le 3}$ with $\mathrm{wt}(p;a,b,c,d) = a_2^2 b_0 b_1 b_2 c_0 c_1^2 c_2 d_1 d_2^2 d_3$. The blue horizontal edges are represented by double edges.

Denote by Motz_2 the set of all 2-Motzkin paths and by $\mathrm{Motz}_2^{\le m}$ the set of all 2-Motzkin paths with height at most m.

For sequences $\{a_n\}_{n\ge 0}, \{b_n\}_{n\ge 0}, \{c_n\}_{n\ge 0}$, and $\{d_n\}_{n\ge 0}$, define the *weight* $\mathrm{wt}(p;a,b,c,d)$ of a 2-Motzkin path p to be the product of a_n (resp. b_n, c_n, and d_n) for each red horizontal step (resp. blue horizontal step, up step, and down step) starting at height n; see Figure 4.

Flajolet's theory [13] proves the following lemma for a finite continued fraction.

Lemma 6.4. *Given sequences $\{a_n\}_{n\ge 0}, \{b_n\}_{n\ge 0}, \{c_n\}_{n\ge 0}$, and $\{d_n\}_{n\ge 0}$, we have*

$$\sum_{p\in \mathrm{Motz}_2^{\le m}} \mathrm{wt}(p;a,b,c,d) = \cfrac{1}{1-a_0-b_0-\cfrac{c_0 d_1}{1-a_1-b_1-\ddots-\cfrac{c_{m-1}d_m}{1-a_m-b_m}}}.$$

There is a well-known bijection between 2-Motzkin paths and parallelogram polyominoes.

Definition 6.5 (The map $\phi : \mathrm{Motz}_2^{\le m} \to \mathcal{P}^{\le m+1}$)**.** Let $p \in \mathrm{Motz}_2^{\le m}$. Then $\phi(p) = \alpha$ is the parallelogram polyomino whose upper and lower boundary paths U, D are constructed by the following algorithm.

(1) The first step of U (resp. D) is a north (resp. east) step.
(2) For $i = 1, 2, \dots, n$, where n is the number of steps in p, the $(i+1)^{\text{st}}$ steps of U and D are defined as follows.
  (a) If the i^{th} step of p is an up step, then the $(i+1)^{\text{st}}$ step of U (resp. D) is a north (resp. east) step.
  (b) If the i^{th} step of p is a down step, then the $(i+1)^{\text{st}}$ step of U (resp. D) is a east (resp. north) step.
  (c) If the i^{th} step of p is a red horizontal step, then the $(i+1)^{\text{st}}$ steps of U and D are both north steps.
  (d) If the i^{th} step of p is a blue horizontal step, then the $(i+1)^{\text{st}}$ steps of U and D are both east steps.
(3) Finally, the last step of U (resp. D) is an east (resp. north) step.

For example, if p is the 2-Motzkin path in Figure 4, then $\phi(p)$ is the parallelogram polyomino α in Figure 1.

It is easy to see from the construction that $\phi : \mathrm{Motz}_2^{\le m} \to \mathcal{P}^{\le m+1}$ is a bijection such that if $\phi(p) = \alpha$, then $\mathrm{wt}(\alpha;a,b,c,d) = d_0\,\mathrm{wt}(p;a,b,c,d)$.

Therefore we obtain the following proposition from Lemma 6.4, which changes the weighted 2-Motzkin paths into weighted parallelogram polyominoes.

Proposition 6.6. *Given sequences $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}, \{c_n\}_{n\ge0}$, and $\{d_n\}_{n\ge0}$, we have*

$$\sum_{\alpha\in\mathcal{P}^{\le m+1}} \mathrm{wt}(\alpha; a,b,c,d) = \cfrac{d_0}{1-a_0-b_0-\cfrac{c_0d_1}{1-a_1-b_1-\ddots-\cfrac{c_{m-1}d_m}{1-a_m-b_m}}}.$$

As a special case in Proposition 6.6, if $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}, \{c_n\}_{n\ge0}$, and $\{d_n\}_{n\ge0}$ are the sequences given by $a_n = q^{n+1}Y$, $b_n = q^{n+1}X$, $c_n = q^{n+1}XY$, and $d_n = q^{n+1}$, then one can easily check that

$$XY\cdot \mathrm{wt}(\alpha; a,b,c,d) = X^{\mathrm{col}(\alpha)}Y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)}.$$

Thus we obtain the following corollary.

Corollary 6.7. *We have*

$$\sum_{\alpha\in\mathcal{P}^{\le m+1}} X^{\mathrm{col}(\alpha)}Y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = \cfrac{qXY}{1-q(X+Y)-\cfrac{q^3XY}{1-q^2(X+Y)-\ddots-\cfrac{q^{2m+1}XY}{1-q^{m+1}(X+Y)}}}.$$

For the rest of this section we will find a finite version of the following result due to Bousquet-Mélou and Viennot [4].

Theorem 6.8 ([9] for $\nu=0$ and [4] for general ν). *The trivariate generating function for parallelogram polyominoes is*

$$\sum_{\alpha\in\mathcal{P}} (q^\nu x)^{\mathrm{col}(\alpha)}(q^\nu)^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = -q^\nu x^{1/2}\frac{J^{(3)}_{\nu+1}(x^{1/2};q^{-1})}{J^{(3)}_{\nu}(x^{1/2};q^{-1})}.$$

In fact Delest and Fédou [9] (for $\nu=0$), and Bousquet-Mélou and Viennot [4] state their results in the following equivalent form:

$$\sum_{\alpha\in\mathcal{P}} x^{\mathrm{col}(\alpha)}y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = \frac{qxy}{1-qy}\cdot\frac{{}_1\phi_1\left(0;q^2y;q,q^2x\right)}{{}_1\phi_1\left(0;qy;q,qx\right)}.$$

Bousquet-Mélou and Viennot [4] also showed that

$$\sum_{\alpha\in\mathcal{P}} x^{\mathrm{col}(\alpha)}y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = \cfrac{qxy}{1-q(x+y)-\cfrac{q^3xy}{1-q^2(x+y)-\cfrac{q^5xy}{\ddots}}}. \tag{6.1}$$

We note that in [4, Corollary 4.6] the sequence of the coefficients of $(x+y)$ in the continued fraction (6.1) was inadvertently written $q, q^3, q^5, \dots$, where the correct sequence is $q, q^2, q^3, \dots$ We also note that there are similar results in [1].

For a sequence $s = \{s_n\}_{n\ge0}$, define $\delta s = \{s_{n+1}\}_{n\ge0}$. Kim and Stanton [18, Eq. (5.4)] showed that for given sequences $b = \{b_n\}_{n\ge0}$, $a = \{a_n\}_{n\ge0}$, and $\lambda = \{\lambda_n\}_{n\ge0}$, and for a nonnegative integer k,

$$\frac{x^mP_m(x^{-1};\delta b,\delta a,\delta\lambda)}{x^{m+1}P_{m+1}(x^{-1};b,a,\lambda)} = \frac{1}{-a_0x-\lambda_0x^2}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-a_ix-\lambda_ix^2}{1-b_ix}\right). \tag{6.2}$$

Now we are ready to prove a finite version of Theorem 6.8.

Theorem 6.9. *The trivariate generating function for bounded diagonal parallelogram polyominoes is*

$$\sum_{\alpha\in\mathcal{P}^{\le m+1}} (q^\nu x)^{\mathrm{col}(\alpha)}(q^\nu)^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = \frac{q^{2\nu+1}}{1-q^{\nu+1}}\cdot\frac{r_m^{(3)}(x^{-1};q^{\nu+2},q)}{r_{m+1}^{(3)}(x^{-1};q^{\nu+1},q)}. \tag{6.3}$$

Proof. Let $b=\{b_i\}_{i\ge0}$, $a=\{a_i\}_{i\ge0}$, and $\lambda=\{\lambda_i\}_{i\ge0}$, where

$$b_i=\frac{q^{\nu+i+1}}{1-q^{\nu+i+1}},\qquad a_i=\frac{q^{2\nu+2i+1}}{(1-q^{\nu+i})(1-q^{\nu+i+1})},\qquad \lambda_i=0.$$

Then $P_m(x;b,a,\lambda)=r_m^{(3)}(x;q^{\nu+1},q)$ and $P_m(x;\delta b,\delta a,\delta\lambda)=r_m^{(3)}(x;q^{\nu+2},q)$. By (6.2),

$$\frac{r_m^{(3)}(x^{-1};q^{\nu+2},q)}{xr_{m+1}^{(3)}(x^{-1};q^{\nu+1},q)}=\frac{x^mP_m(x^{-1};\delta b,\delta a,\delta\lambda)}{x^{m+1}P_{m+1}(x^{-1};b,a,\lambda)}=\frac{1}{-a_0x}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-a_ix}{1-b_ix}\right).$$

By Lemma 5.9 with $c_i=1-q^{\nu+i+1}$,

$$\begin{aligned}\frac{1}{-a_0x}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-a_ix}{1-b_ix}\right)&=\frac{1}{-a_0x}\frac{1}{c_{-1}}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-a_ic_{i-1}c_ix}{c_i-b_ic_ix}\right)\\&=\frac{1-q^{\nu+1}}{-q^{2\nu+1}x}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-q^{2\nu+2i+1}x}{1-q^{\nu+i+1}-q^{\nu+i+1}x}\right).\end{aligned}$$

Letting $X=q^\nu x$ and $Y=q^\nu$, and combining the above equations, we obtain

$$\frac{q^{2\nu+1}}{1-q^{\nu+1}}\cdot\frac{r_m^{(3)}(x^{-1};q^{\nu+2},q)}{r_{m+1}^{(3)}(x^{-1};q^{\nu+1},q)}=-\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-q^{2i+1}XY}{1-q^{i+1}(X+Y)}\right).$$

Corollary 6.7 then completes the proof. ∎

Remark 6.10. It is well known that the generating function for bounded Motzkin paths is given by a ratio of orthogonal polynomials; see e.g. Flajolet [13], Viennot [28, Ch. V, Eq. (27)], or Krattenthaler [24, Theorem 10.11.1]. The argument in this section implies that the left-hand side of (6.3) is the generating function for certain weighted bounded Motzkin paths. Theorem 6.9 shows that this generating function is also equal to a ratio of type R_I polynomials.

By (3.4), taking the limit $m\to\infty$ in Theorem 6.9 we obtain Theorem 6.8. We may also use Theorem 4.11 to write the finite continued fraction as an explicit rational function.

Corollary 6.11. *The trivariate generating function for bounded diagonal parallelogram polyominoes is*

$$\sum_{\alpha\in\mathcal{P}^{\le n+1}} x^{\mathrm{col}(\alpha)}y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)}=-\frac{x\sum_{k=0}^{n}\sum_{a=0}^{n-k}(-1)^kx^ay^{-k-a}q^{-\binom{k}{2}-2k}\begin{bmatrix}k+a\\a\end{bmatrix}_{q^{-1}}\begin{bmatrix}n-a\\k\end{bmatrix}_{q^{-1}}}{\sum_{k=0}^{n+1}\sum_{a=0}^{n+1-k}(-1)^kx^ay^{-k-a}q^{-\binom{k}{2}-k}\begin{bmatrix}k+a\\a\end{bmatrix}_{q^{-1}}\begin{bmatrix}n+1-a\\k\end{bmatrix}_{q^{-1}}}.$$

Cigler and Krattenthaler [7] found a different finite version of Theorem 6.8.

Theorem 6.12 ([7, Corollary 55]). *For any integer $k \geq 1$, we have*

$$\sum_{\alpha \in \mathcal{P}_1^{\leq k}} x^{\mathrm{col}(\alpha)} y^{\mathrm{row}(\alpha)} q^{\mathrm{area}(\alpha)} = -\frac{y \sum_{j=1}^{k} (-1)^j x^j q^{\binom{j+1}{2}} \sum_{i=0}^{k-j} (yq)^i \begin{bmatrix} k-i-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ j-1 \end{bmatrix}_q}{\sum_{j=0}^{k} (-1)^j x^j q^{\binom{j+1}{2}} \sum_{i=0}^{k-j} (yq)^i \begin{bmatrix} k-i \\ j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ j-1 \end{bmatrix}_q},$$

where $\mathcal{P}_1^{\leq k}$ is the set of parallelogram polyominoes such that each column has length at most k and $\begin{bmatrix} i \\ -1 \end{bmatrix}_q = \delta_{i,-1}$.

Remark 6.13. The second odd-even trick (7.1) with $\lambda_{2k-1} = q^k y$ and $\lambda_{2k} = q^k$ gives

$$1 + \cfrac{qy}{1 - q(x+y) - \cfrac{q^3 xy}{1 - q^2(x+y) - \cfrac{q^5 xy}{\ddots}}} = \cfrac{1}{1 - \cfrac{qy}{1 - \cfrac{qx}{1 - \cfrac{q^2 y}{1 - \cfrac{q^2 x}{\ddots}}}}}.$$

Remark 6.14. There are also finite versions of Theorem 6.9 for the classical q-Lommel polynomials and the even-odd q-Lommel polynomials. The rational function is again a quotient of orthogonal polynomials while the weights on $\mathcal{P}^{\leq m+1}$ depend upon the diagonals.

Here are the infinite continued fractions for these two cases. For the classical q-Lommel polynomials, Theorem 5.3 becomes

$$\frac{{}_2\phi_1(0,0;q^2y;q;-qx)}{{}_2\phi_1(0,0;qy;q;-qx)} = \cfrac{1-qy}{1 - qy - \cfrac{q^2 xy}{1 - q^2 y - \cfrac{q^3 xy}{1 - q^3 y - \cfrac{q^4 xy}{\ddots}}}}.$$

For the even-odd q-Lommel polynomials, Theorem 5.4 becomes

$$\frac{{}_1\phi_1(0;q^2y;q;q^2x)}{{}_1\phi_1(0;qy;q;qx)} = \cfrac{1-qy}{1 - qy - \cfrac{A_1}{1 - q^2 y - \cfrac{A_2}{1 - q^3 y - \cfrac{A_3}{\ddots}}}}$$

where $A_{2k-1} = xq^k$ and $A_{2k} = xyq^{3k/2+1}$.

7. Concurrence of moments

Recall the notation (5.2) for the moments $\mu_n\Big(\{b_k\}_{k\geq 0}, \{a_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 0}\Big)$, which we also write $\mu_n(\{b_k\}, \{a_k\}, \{\lambda_k\})$, or $\mu_n(\{b_k\}, \{\lambda_k\})$ when $a_k = 0$. There is a concurrence of moments (see Propositions 4.5 and 4.6), which we call the first and second *odd-even tricks*

$$\begin{aligned} \mu_{2n}(\{0\}, \{\lambda_k\}) &= \mu_n(\{\lambda_{2k} + \lambda_{2k+1}\}, \{\lambda_{2k}\lambda_{2k-1}\}), \\ \mu_{2n+2}(\{0\}, \{\lambda_k\}) &= \lambda_1 \mu_n(\{\lambda_{2k+2} + \lambda_{2k+1}\}, \{\lambda_{2k}\lambda_{2k+1}\}). \end{aligned} \tag{7.1}$$

The classical orthogonal polynomial moments are a special case of type R_I moments

$$\mu_n(\{b_k\},\{0\},\{\lambda_k\}) = \mu_n\left(\{b_k\},\{\lambda_k\}\right).$$

There is another concurrence of moments, which follows from [18, Corollary 3.7]

$$\mu_{2n}(\{0\},\{a_k\}) = \mu_n(\{0\},\{a_k\},\{0\}). \tag{7.2}$$

It is known [18] that a type R_I moment $\mu_n(\{b_k\},\{a_k\},\{\lambda_k\})$ is a nonnegative polynomial in the recurrence coefficients. Besides (7.2) Theorem 4.14 is another example of classical orthogonal polynomial moments being equal to type R_I moments

$$\mu_{2n}\left(\{0\},\{\Lambda_k\}\right) = \mu_n\left(\{b_k\},\{a_k\},\{0\}\right). \tag{7.3}$$

The main result in this section is Theorem 7.2, which expresses the Λ_k as a function of the sequences a_k and b_k, thereby providing the concurrence (7.3).

To prove Theorem 7.2 we need to recall a classical result and notation. The Hankel determinant [6, Theorem 4.2] will be used:

$$\det(\mu_{i+j}\left(\{b_k\}_{k\geq 0},\{\lambda_k\}_{k\geq 0}\right))_{i,j=0}^{n} = \lambda_1^n\lambda_2^{n-1}\cdots\lambda_n^1.$$

Recall that for a sequence $a=\{a_k\}_{k\geq 0}$ we write $\delta a = \{a_{k+1}\}_{k\geq 0}$. We also define $\delta^{-1}a = \{a_{k-1}\}_{k\geq 0}$, where $a_{-1}=1$ (the value of a_{-1} is irrelevant for our purpose).

Definition 7.1. A *Schröder path* is a lattice path from $(r,0)$ to $(s,0)$, for some integers r,s, consisting of northeast steps $(1,1)$, east steps $(1,0)$, and south steps $(0,-1)$ that never goes below the x-axis. Given sequences $b=\{b_k\}_{k\geq 0}$ and $a=\{a_k\}_{k\geq 0}$, the *weight* $\mathrm{wt}(P)$ of a Schröder path P is the product of b_i for each east step starting at height i and a_i for each south step starting at height i.

Our main theorem of this section is the next theorem.

Theorem 7.2. *Suppose that sequences $b=\{b_k\}_{k\geq 0}$, $a=\{a_k\}_{k\geq 0}$, and $\Lambda=\{\Lambda_k\}_{k\geq 0}$ satisfy*

$$\mu_{2n}\left(\{0\},\{\Lambda_k\}\right) = \mu_n\left(\{b_k\},\{a_k\},\{0\}\right).$$

Then

$$\Lambda_1\Lambda_2\cdots\Lambda_{2n} = \frac{f_n(a,b)}{f_{n-1}(a,b)},$$

where

$$f_n(a,b) = \sum_p \mathrm{wt}(p),$$

and the sum is over all n-tuples $p=(P_0,P_1,\ldots,P_n)$ of non-intersecting Schröder paths, $P_k:(-k,0)\to(k,0), 0\leq k\leq n$. Moreover,

$$\Lambda_1\Lambda_2\cdots\Lambda_{2n-1} = a_0^{-1}\frac{f_n(\delta^{-1}a,\delta^{-1}b)}{f_{n-1}(\delta^{-1}a,\delta^{-1}b)},$$

and if $a_k=b_k=1$ then

$$f_n(\{1\},\{1\}) = 2^{\binom{n+1}{2}}.$$

Proof. Let

$$\begin{aligned}\rho_n &:= \mu_{2n}\left(\{0\},\{\Lambda_k\}\right) = \mu_n\left(\{b_k\},\{a_k\},\{0\}\right),\\ \Delta_n &:= \det(\rho_{i+j})_{0\le i,j\le n}.\end{aligned}$$

Using the odd-even trick $B_n = \Lambda_{2n+1} + \Lambda_{2n}$ and $\Theta_n = \Lambda_{2n-1}\Lambda_{2n}$, we have

$$\rho_n = \mu_{2n}\left(\{0\},\{\Lambda_k\}\right) = \mu_n(\{B_k\},\{\Theta_k\}).$$

Therefore

$$\Delta_n = \det(\mu_{i+j}(\{B_k\},\{\Theta_k\}))_{0\le i,j\le n} = \Theta_1^n\Theta_2^{n-1}\cdots\Theta_n^1 = \Lambda_1^n\Lambda_2^n\Lambda_3^{n-1}\Lambda_4^{n-1}\cdots\Lambda_{2n-1}^1\Lambda_{2n}^1,$$

which shows $\Lambda_1\Lambda_2\cdots\Lambda_{2n} = \Delta_n/\Delta_{n-1}$.

Kim and Stanton [18, Corollary 3.7] showed that $\mu_n\left(\{b_k\},\{a_k\},\{0\}\right)$ is the sum of weights of all Schröder paths from $(0,0)$ to $(n,0)$. Since $\Delta_n = \det(\mu_{i+j}\left(\{b_k\},\{a_k\},\{0\}\right))_{0\le i,j\le n}$, the $(n+1)\times(n+1)$ determinant Δ_n is the signed generating function for $(n+1)$-tuples of Schröder paths $(P_0,\dots,P_n)$, $P_k : (-k,0)\to(\sigma(k),0)$, for some permutation σ of $\{0,1,\dots,n\}$. Because there are no SE edges ($\lambda_k = 0$), any two paths which intersect do so at integer coordinates. Thus we may apply the Lindström–Gessel–Viennot lemma of tail swapping to reduce this sum to non-intersecting paths, σ = identity, $P_k : (-k,0)\to(k,0)$. Thus $\Delta_n = f_n(a,b)$ and we obtain the identity for $\Lambda_1\Lambda_2\cdots\Lambda_{2n}$.

Now using the second odd-even trick $B_n' = \Lambda_{2n+2} + \Lambda_{2n+1}$ and $\Lambda_n' = \Lambda_{2n+1}\Lambda_{2n}$, we have

$$\rho_{n+1} = \mu_{2n+2}\left(\{0\},\{\Lambda_k\}\right) = \Lambda_1\mu_n(\{B_k'\},\{\Lambda_k'\}).$$

Then

$$\begin{aligned}\Delta_n' := \det(\rho_{i+j+1})_{0\le i,j\le n-1} &= \Lambda_1^n\det(\mu_{i+j}(\{B_k'\},\{\Lambda_k'\}))_{0\le i,j\le n-1}\\ &= \Lambda_1^n\Lambda_2^{n-1}\Lambda_3^{n-1}\cdots\Lambda_{2n-2}^1\Lambda_{2n-1}^1,\end{aligned}$$

so

$$\Lambda_1\Lambda_2\cdots\Lambda_{2n-1} = \Delta_n'/\Delta_{n-1}'.$$

As in the even case, $\Delta_n' = \det(\mu_{i+j+1}\left(\{b_k\},\{a_k\},\{0\}\right))_{0\le i,j\le n-1}$ is the generating function for n-tuples non-intersecting Schröder paths $p' = (P_1',\dots,P_n')$, $P_k' : (-k+1,0)\to(k,0)$. For $1\le k\le n$, let P_k be the path from $(-k,-1)$ to $(k,-1)$ obtained from P_k' by adding a northeast step at the beginning and a south step at the end, and let P_0 be the empty path from $(0,-1)$ to $(0,-1)$. This gives a bijection from n-tuples non-intersecting Schröder paths $p' = (P_1',\dots,P_n')$, $P_k' : (-k+1,0)\to(k,0)$ to $(n+1)$-tuples non-intersecting Schröder paths $p = (P_0,P_1,\dots,P_n)$, $P_k : (-k,-1)\to(k,-1)$. Note that the starting point of P_k has height -1, which shifts the indices of a_k and b_k down by one. This shows that

$$\Delta_n' = a_0^{-n}\det(\mu_{i+j}\left(\{b_{k-1}\},\{0\},\{a_{k-1}\}\right))_{0\le i,j\le n} = a_0^{-n}f_n(\delta^{-1}a,\delta^{-1}b),$$

and we obtain the identity for $\Lambda_1\Lambda_2\cdots\Lambda_{2n-1}$.

Finally, the fact that $\Delta_n = 2^{\binom{n+1}{2}}$ and $\Delta_n' = 2^{\binom{n+1}{2}}$ if $a_k = b_k = 1$ for all k follows from [18, Theorem 6.15, $A = B = 1, C = 0$]. □

The first few values of $\Lambda_1\cdots\Lambda_k$ in Theorem 7.2 are

$$\begin{aligned}
\Lambda_1 &= a_0^{-1}\frac{f_1(\delta^{-1}a,\delta^{-1}b)}{f_0(\delta^{-1}a,\delta^{-1}b)} = \frac{a_1+b_0}{1},\\
\Lambda_1\Lambda_2 &= \frac{f_1(a,b)}{f_0(a,b)} = a_1\frac{a_2+b_1}{1},\\
\Lambda_1\Lambda_2\Lambda_3 &= a_0^{-1}\frac{f_2(\delta^{-1}a,\delta^{-1}b)}{f_1(\delta^{-1}a,\delta^{-1}b)}\\
&= a_1\frac{a_1a_2a_3+a_2^2b_0+a_2a_3b_0+2a_2b_0b_1+b_0b_1^2+a_1a_2b_2+a_2b_0b_2}{a_1+b_0},\\
\Lambda_1\Lambda_2\Lambda_3\Lambda_4 &= \frac{f_2(a,b)}{f_1(a,b)}\\
&= a_1a_2\frac{a_2a_3a_4+a_3^2b_1+a_3a_4b_1+2a_3b_1b_2+b_1b_2^2+a_2a_3b_3+a_3b_1b_3}{a_2+b_1}.
\end{aligned}$$

Remark 7.3. Eu and Fu [11] used the idea relating Δ_n and Δ'_{n-1} in the proof of Theorem 7.2 to give a simple proof of the Aztec diamond theorem, which is equivalent to the result $\Delta_n = 2^{\binom{n+1}{2}}$ when $a_k = b_k = 1$.

8. Open problems

Recall that Kishore's theorem is a statement about the power series coefficients of the ratio $J_{\nu+1}(x)/J_\nu(x)$ of two Bessel functions.

Theorem 8.1 (Kishore [19])**.** *We have*

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \sum_{n=1}^{\infty}\frac{N_{n,\nu}}{D_{n,\nu}}\left(\frac{z}{2}\right)^{2n-1},$$

where

$$D_{n,\nu} = \prod_{k=1}^{n}(k+\nu)^{\lfloor n/k\rfloor},$$

and $N_{n,\nu}$ is a polynomial in ν with nonnegative integer coefficients.

We conjecture the following finite version of Kishore's theorem on a ratio of Lommel polynomials $L_{m,\nu}(x)$ defined in Section 3.

Conjecture 8.2. *Let*

$$\frac{L_{m,\nu+2}(x)}{L_{m+1,\nu+1}(x)} = \sum_{n=0}^{\infty}\frac{N_{n,\nu}^{(m)}}{D_{n,\nu}^{(m)}}\left(\frac{x}{2}\right)^{2n+1},$$

where

$$D_{n,\nu}^{(m)} = \prod_{k=0}^{m}(\nu+k+1)^{f(m,n,k)},$$

$$f(m,n,k) = \begin{cases}\max\left(\left\lfloor\dfrac{n+1}{k+1}\right\rfloor, \left\lfloor\dfrac{n+m-2k+1}{m-k+1}\right\rfloor\right), & \textit{if } k\neq m/2,\\ 1, & \textit{if } k = m/2.\end{cases}$$

Then $N_{n,\nu}^{(m)}$ is a polynomial in ν with nonnegative integer coefficients.

In Section 5 we saw that the ratio

$$\frac{J^{(3)}_{\nu+1}(z;q^{-1})}{J^{(3)}_{\nu}(z;q^{-1})} = \frac{-q^{\nu+1}z}{1-q^{\nu+1}} \cdot \frac{{}_1\phi_1\left(0;q^{\nu+2};q,q^{\nu+2}z^2\right)}{{}_1\phi_1\left(0;q^{\nu+1};q,q^{\nu+1}z^2\right)}$$

has two generalizations, the q-Nörlund continued fraction and Heine's continued fraction. These two generalizations seem to have a similar property as follows.

Conjecture 8.3. *Let*

$$\sum_{n\ge 0}\gamma_n(a,b,c)z^n = \frac{{}_2\phi_1\left(aq,bq;cq;q,z\right)}{{}_2\phi_1\left(a,b;c;q,z\right)}.$$

Then

$$\frac{\gamma_n(a,b,c)}{1-c} = \frac{P_n(a,b,c)}{\prod_{k=0}^{n}(1-cq^k)^{\left\lfloor\frac{n+1}{k+1}\right\rfloor}},$$

for some polynomial $P_n(a,b,c)$ in a,b,c,q with integer coefficients.

Conjecture 8.4. *Let*

$$\sum_{n\ge 0}\gamma'_n(a,b,c)z^n = \frac{{}_2\phi_1\left(aq,b;cq;q,z\right)}{{}_2\phi_1\left(a,b;c;q,z\right)}.$$

Then

$$\frac{\gamma'_n(a,b,c)}{1-c} = \frac{P'_n(a,b,c)}{\prod_{k=0}^{n}(1-cq^k)^{\left\lfloor\frac{n+1}{k+1}\right\rfloor}},$$

for some polynomial $P'_n(a,b,c)$ in a,b,c,q with integer coefficients.

Note that, between these two conjectures, only the second argument of the ${}_2\phi_1$ in the numerator differs (namely, bq vs b).

Problem 8.5. *Find a combinatorial proof of Theorem 5.12.*

Problem 8.6. *Find a combinatorial proof of Theorem 6.9, which contains the Bousquet-Mélou–Viennot result Theorem 6.8.*

Problem 8.7. *Find an Askey scheme whose top element is the associated Askey–Wilson polynomial which contains the q-Lommel polynomials. As an alternative, a referee has suggested that an Askey scheme with non-polynomial entries may exist which contains the q-Lommel polynomials.*

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ON q-ORDER STATISTICS

MALVINA VAMVAKARI[1]

[1]Harokopio University of Athens, Department of Informatics and Telematics, Athens, Greece; https://dblp.org/pid/46/6916.html

Abstract. Building on the notion of q-integral introduced by Thomae in 1869, we introduce q-order statistics (that, is q-analogues of the classical order statistics, for $0 < q < 1$) which arise from dependent and not identically distributed q-continuous random variables and to study their distributional properties. We study the q-distribution functions and the q-density functions of the relative q-ordered random variables. We focus on q-ordered variables arising from dependent and not identically q-uniformly distributed random variables and we derive their q-distributions, including q-power law, q-beta and q-Dirichlet distributions.

Keywords: q-order statistics, q-multinomial formulae, univariate and multivariate q-continuous random variables, q-uniform distribution, q-power law distribution, q-beta distribution, q-Dirichlet distribution, waiting times of the Heine process.

1. Introduction

Order statistics and their properties have been studied thoroughly the last decades. The literature devoted to order statistics from independent and identically distributed random variables is very extensive. The study of order statistics arising from independent or dependent and not identically distributed, random variables, is of great research interest. Excellent references devoted to order statistics are, among others, the work of Arnold, Balakrishnan and Nagaraja [2], Balakrishnan [3], David and Nagaraja [8], or Papadatos [13].

In the field of discrete q-distributions, Charalambides [6, p.167] has presented the order statistics arising from independent and identically distributed random variables, with common distribution a discrete q-uniform distribution. Charalambides [4, 5] also has studied the distributions of the record statistics in q-factorially increasing populations.

The main objective of this work is to introduce q-order statistics, for $0 < q < 1$, arising from dependent and not identically distributed q-continuous random variables and to study their distributional properties. We introduce q-order statistics as q-analogues of the classical order statistics. We study the q-distribution functions and q-density functions of the relative q-ordered random variables. We focus on q-ordered variables arising from dependent and not identically q-uniformly distributed random variables and we derive their q-distributions, including q-power law, q-beta and q-Dirichlet distributions. Moreover, we consider the Heine process, which had been introduced by Kyriakoussis and Vamvakari [12]; see also the work of Kemp [11]. Note that our notion of q-distribution is not related to the q-Gaussian distribution, or to other Tsallis distributions [14].

We prove that a conditional q-joint distribution of the waiting times of the Heine process coincides with the joint q-density function of q-ordered random variables arising from dependent q-continuous random variables.

This work contains three sections along with the introductory Section 1. In the preliminary Section 2, we present all our q-definitions. In the main Section 3, we state and prove our results concerning the q-order statistics and their distributional properties.

2. Preliminaries, definitions and notation

In this section, we define the q-series, the univariate and multivariate q-continuous random variables, the Heine process, and the q-uniform distribution. It will allow us to study q-order statistics in the next section.

2.1. **q-Series preliminaries.** The q-shifted factorials are

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}), \quad n = 1, 2, \ldots, \text{ or } \infty.$$

The multiple q-shifted factorials are defined by

$$(a_1, \ldots, a_k; q)_n := \prod_{j=1}^{k}(a_j;q)_n$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} \nu \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{[n]_q!}{[k]_q![n-k]_q}, \;\; k = 0, 1, \ldots, n,$$

where

$$[n]_q! = [1]_q[2]_q \cdots [n]_q = \frac{(q;q)_n}{(1-q)^n} = \frac{\prod_{k=1}^{n}(1-q^k)}{(1-q)^n}$$

is the q-factorial number of order n with $[t]_q = \frac{1-q^t}{1-q}$.

The kth-order factorial of the number $[n]_q$ is called *q-factorial* of n of order k and is given by

$$[n]_k = [n]_q[n-1]_q \cdots [n-k+1]_q, \quad k = 1, 2, \ldots, n\,.$$

Note that

$$[n]_{q^{-1}} = q^{-n+1}[n]_q, \;\; [n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q! \;\; \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} = q^{-k(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}_q.$$

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$, equals the k-combinations $\{m_1, \ldots, m_k\}$ of the set $\{1, \ldots, n\}$, weighted by $q^{m_1+\cdots+m_k-\binom{k+1}{2}}$,

$$\sum_{1 \le m_1 < \cdots < m_k \le n} q^{m_1+\cdots+m_k-\binom{k+1}{2}} = \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{2.1}$$

Let n be a positive integer and let x, y and q be real numbers, with $q \neq 1$. Then, a version of q-Vandermonde's formula is

$$\begin{bmatrix} x+y \\ n \end{bmatrix}_q = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(y-n+k)}[x]_k[y]_{n-k}.$$

An interesting q-identity deduced by the above version of q-Vandermonde's formula is

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}-n(y+k)} \frac{[y]_q}{[y+k]_q} = \frac{1}{\begin{bmatrix} y+n \\ n \end{bmatrix}_q}. \tag{2.2}$$

Note that from the above equation we have the corresponding q^{-1}-identity

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}+ny} \frac{[y]_q}{[y+k]_q} = \frac{1}{\begin{bmatrix} y+n \\ n \end{bmatrix}_{\frac{1}{q}}}. \tag{2.3}$$

The q-binomial formula is

$$\prod_{i=1}^{n}(1+tq^{i-1}) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k.$$

The above q-binomial formula, by replacing q by q^{-1} and t by $-t$, becomes

$$\prod_{i=1}^{n}(1-tq^{-(i-1)}) = \sum_{k=0}^{n}(-1)^k q^{-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} t^k = \sum_{k=0}^{n}(-1)^k q^{-\binom{k}{2}-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k. \tag{2.4}$$

The q-multinomial coefficient is defined for nonnegative integers n and k_i's by

$$\begin{bmatrix} n \\ k_1,\dots,k_r \end{bmatrix}_q = \frac{[n]_q!}{[k_1]_q \cdots [k_r]_q![n-k_1-\cdots-k_r]_q!}.$$

We then have the two following equivalent expressions for the q^{-1}-multinomial coefficient

$$\begin{aligned} \begin{bmatrix} n \\ k_1,\dots,k_r \end{bmatrix}_{\frac{1}{q}} &= q^{-\binom{n}{2}+\sum_{j=1}^{r+1}\binom{k_j}{2}} \begin{bmatrix} n \\ k_1,\dots,k_r \end{bmatrix}_q \\ &= q^{-\sum_{j=1}^{r} k_j(n-k_1-\cdots-k_j)} \begin{bmatrix} n \\ k_1,\dots,k_r \end{bmatrix}_q. \end{aligned} \tag{2.5}$$

An ordered set partition of A is a sequence $(A_1,\dots,A_m)$ of non-empty disjoint subsets of A, such that $A_1 \cup \cdots \cup A_m = A$. Using the notation from Flajolet and Sedgewick's book *Analytic Combinatorics* [9], ordered set partitions are accordingly defined by the symbolic formula $\mathrm{Seq}(\mathrm{Set}_{\geq 1})$, and thus have the (exponential) generating function $1/(2-\exp(t))$ of Fubini numbers $\{F_n\}_{n\geq 0} = \{1,1,3,13,75,541,\dots\}$. E.g., there are 13 ordered set partitions of {1,2,3}.

Charalambides [7] showed that the q-multinomial coefficient $\begin{bmatrix} n \\ k_1,\dots,k_r \end{bmatrix}_q$ equals the number of ordered partitions of the set $\{1,\dots,n\}$ into $r+1$ subsets, $(A_1,\dots,A_{r+1})$ of size $(k_1,\dots,k_{r+1})$, if one associates a specific q-weight to each subset. Writing $A_j = \{m_{j,1},\dots,m_{j,k_j}\}$, this weight is $q^{m_{j,1}+\cdots+m_{j,k_j}-\binom{k_j+1}{2}}$, and one has

$$\begin{bmatrix} n \\ k_1,\dots,k_r \end{bmatrix}_q = \sum_{A_1,\dots,A_r} \prod_{j=1}^{r} q^{m_{j,1}+\cdots+m_{j,k_j}-\binom{k_j+1}{2}}, \tag{2.6}$$

where the summation is over all the above mentioned ordered partitions of $\{1,\dots,n\}$.

Vamvakari [15] earlier proved the following alternative summation expansion of the q-multinomial coefficient

$$\sum \prod_{j=1}^{r} q^{k_{j,1}+2k_{j,2}+\cdots+nk_{j,n}-\binom{k_j+1}{2}} = \begin{bmatrix} n \\ k_1, \ldots, k_r \end{bmatrix}_q,$$

where the summation is over all $k_{j,i} = 0, 1$ such that $\sum_{i=1}^{n} k_{j,i} = k_j$ (for $j = 1, \ldots, r$).

We shall also use the following q-difference operator (which we also call "q-derivative")

$$d_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}. \tag{2.7}$$

We refer to [1, Chapter 10.2] or [6] for a more thorough discussion of its properties. It is clear that it is a discrete analogue of the derivative; it satisfies e.g.

$$d_q x^n = \frac{1-q^n}{1-q} x^{n-1} = [n]_q x^{n-1}$$

and $d_q(f(x) \cdot g(x)) = g(x) d_q f(x) + f(qx) d_q g(x)$. What is more, for differentiable functions, one has

$$\lim_{q \to 1} d_q f(x) = f'(x).$$

Now, following [1, Chapter 10.1], we define the q-integral by

$$\int_0^a f(x) d_q x := \sum_{n=0}^{\infty} [aq^n - aq^{n+1}] f(aq^n), \tag{2.8}$$

$$\int_a^b f(x) d_q x := \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

In this context, d_q is sometimes called the Fermat measure, and should not be confused with the above q-derivative, even if they are, in some sense, related. The q-integral over $[0, \infty)$ uses the division points $\{q^n : -\infty < n < \infty\}$ and is

$$\int_0^{\infty} f(x) d_q x := (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

2.2. **The Heine process.** Kyriakoussis and Vamvakari [12] introduced the Heine process as a q-analogue of the Poisson process. The Heine process is defined as follows.

Definition 2.1 (Heine Process)**.** A continuous time process $\{X(t), t > 0\}$, where $X(t)$ expresses the number of arrivals in a time interval $(0, t]$, is called *Heine process* with parameters $0 < q < 1$ and $\lambda > 0$, if the following three assumptions hold

(a) The process starts at time 0 with $X(0) = 0$.

(b) In each time interval of length $\delta = (1-q)t$, one has 1 arrival with probability $p(t)$, and 0 arrival with probability $1 - p(t)$, where

$$p(t) := \frac{\lambda(1-q)t}{1 + \lambda(1-q)t}.$$

That is,

$$P\left(X(t) - X(qt) = 1\right) = p(t) \qquad \text{and} \qquad P\left(X(t) - X(qt) = 0\right) = 1 - p(t).$$

This implies, for any $k \geq 1$:

$$P\left(X(q^{k-1}t) - X(q^k t) = 1\right) = \frac{\lambda(1-q)q^{k-1}t}{1+\lambda(1-q)q^{k-1}t},$$
$$P\left(X(q^{k-1}t) - X(q^k t) = 0\right) = \frac{1}{1+\lambda(1-q)q^{k-1}t}.$$

Also, the Heine process has the Heine distribution:

$$P(X(t) = k) = e_q(-\lambda t)\frac{q^{\binom{k}{2}}(\lambda t)^k}{[k]_q!},$$

for $k \in \mathbb{N}$, with $e_q(z) = \prod_{i=1}^{\infty}\left(1-(1-q)zq^{i-1}\right)^{-1}$, $|z| < 1/(1-q)$.

2.3. **Univariate and multivariate q-continuous random variables.** Kyriakoussis and Vamvakari [12] presented the following definition of q-continuous random variables. For clarity, let us begin by presenting this concept for one random variable.

Definition 2.2 (q-continuous)**.** A random variable X is called *q-continuous* (or "Fermat integrable", as we integrate over the Fermat measure defined in (2.8)) if there exists a non-negative function $f_q(x)$ (for $x \geq 0$) such that

$$P(\alpha < X \leq \beta) = \int_{\alpha}^{\beta} f_q(x) d_q x.$$

The function $f_q(x)$ is called *q-density function* of the random variable X.

Note that, in particular, one has

$$\int_0^{\infty} f_q(x) d_q x = 1.$$

For the corresponding distribution function

$$F(x) = P(X \leq x),$$

we have by definition

$$P(\alpha < X \leq \beta) = F(\beta) - F(\alpha),$$

and, for $x \geq 0$,

$$F(x) = \int_0^x f_q(t) d_q t.$$

Taking the q-derivative of the above relation we have

$$d_q F(x) = f_q(x)$$

and by the definition of the q-derivative we obtain

$$f_q(x) = \frac{F(x) - F(qx)}{(1-q)x} = \frac{P(qx < X \leq x)}{(1-q)x}.$$

Let us now present the case of tuples.

Definition 2.3 (multivariate q-continuous)**.** A k-variate random variable $\mathcal{X} = (X_1, \ldots, X_k)$ is called q-*continuous* (or "Fermat integrable", as we integrate over the Fermat measure defined in (2.8)) if there exists a non-negative function $f_q(x_1, \ldots, x_k)$ such that

$$P\left(\alpha_1 < X_1 \leq \beta_1, \ldots, \alpha_k < X_k \leq \beta_k\right) = \int_{\alpha_k}^{\beta_k} \cdots \int_{\alpha_1}^{\beta_1} f_q(x_1, \ldots, x_k) d_q x_1 \cdots d_q x_k.$$

The function $f_q(x_1, \ldots, x_k)$ is called q-*density function* of the k-variate random variable $\mathcal{X} = (X_1, \ldots, X_k)$ or joint q-density function of the random variables $X_1, \ldots, X_k$.

In particular, we have

$$\int_0^\infty \cdots \int_0^\infty f_q(x_1, \ldots, x_k) d_q x_1 \cdots d_q x_k = 1.$$

For the corresponding joint distribution function

$$F(x_1, \ldots, x_k) = P\left(X_1 \leq x_1, \ldots, X_k \leq x_k\right)$$

we have

$$F(x_1, \ldots, x_k) = \int_0^{x_k} \cdots \int_0^{x_1} f_q(t_1, \ldots, t_k) d_q t_1 \cdots d_q t_k. \tag{2.9}$$

Building on the notation (2.7), let us define the partial q-derivatives by

$$\frac{\partial F(x_1, \ldots, x_k)}{\partial_q x_k \cdots \partial_q x_1} = (d_q x_k) \cdots (d_q x_1) F(x_1, \ldots, x_k).$$

Then, taking the partial q-derivatives of the relation (2.9), we have

$$\frac{\partial F(x_1, \ldots, x_k)}{\partial_q x_k \cdots \partial_q x_1} = f_q(x_1, \ldots, x_k), \ x_i > 0, \ i = 1, \ldots, k$$

and by the definition of the partial q-derivative we obtain

$$f_q(x_1, \ldots, x_k) = \frac{P\left(q x_1 < X_1 \leq x_1, \ldots, q x_k < X_k \leq x_k\right)}{(1-q)x_1 \cdots (1-q)x_k}. \tag{2.10}$$

The marginal q-density functions of the random variables $X, \ i = 1, \ldots, k$, are given by

$$f_{X_i}(x_i) = \int_0^\infty \cdots \int_0^\infty f_{\mathcal{X}}(x_1, \ldots, x_k) d_q x_1 \cdots d_q x_{i-1} d_q x_{i+1} \cdots d_q x_k, \ i = 1, \ldots, k.$$

For the needs of this work, we also define the conditional q-density function. Let (X, Y) be a bivariate q-continuous random variable, with q-density function $f_q(x, y) \geq 0$, $x, y > 0$ and $f_q(y) > 0$, $y > 0$ the marginal q-density function of Y. Then the function

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, x > 0$$

is a q-density function because

$$f_{X|Y}(x|y) \geq 0, x > 0$$

and

$$\int_0^\infty f_{X|Y}(x|y) d_q x = \frac{1}{f_Y(y)} \int_0^\infty f_{X,Y}(x, y) d_q x = \frac{f_Y(y)}{f_Y(y)} = 1.$$

Since

$$P\left(qx < X \leq x | qy < Y \leq y\right) = \frac{P\left(qx < X \leq x, qy < Y \leq y\right)}{P\left(qy < Y \leq y\right)}$$

we confirm that

$$f_{X|Y}(x|y) = \frac{P\left(qx < X \leq x | qy < Y \leq y\right)}{(1-q)x} = \frac{\frac{P(qx<X\leq x, qy<Y<y)}{(1-q)x(1-q)y}}{\frac{P(qy<Y\leq y)}{(1-q)y}} = \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{2.11}$$

and we give the following definition of conditional q-density function.

Definition 2.4 (conditional q-density)**.** Let (X, Y) be a bivariate q-continuous random variable. Let $f_{X,Y}(x, y)$ be its q-density function and $f_Y(y)$ the marginal q-density function of Y. If $f_Y(y) > 0$ for $y > 0$, the function

$$f_{X|Y}(x|y) := \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

is called *conditional q-density function* of the random variable X given that $qy < Y < y$.

Let $(X_1, \ldots, X_k)$ be a q-continuous k-variate random variable, with joint q-density function $f(x_1, \ldots, x_k) \geq 0$, $x_i > 0$, $i = 1, \ldots, k$. The *conditional q-density function* of a q-continuous r-variate random variable $(X_1, \ldots, X_r)$ given a q-continuous $(k-r)$-variate random variable $(X_{r+1}, X_{r+2}, \ldots, X_k)$ is expressed as

$$h_{(X_1,\ldots,X_r)|(X_{r+1},X_{r+2},\ldots,X_k)}(x_1, \ldots, x_r | x_{r+1}, \ldots, x_k) = \frac{f(x_1, \ldots, x_r)}{g(x_{r+1}, x_{r+2}, \ldots, x_k)}, \tag{2.12}$$

where $g(x_{r+1}, x_{r+2}, \ldots, x_k) > 0$ is the marginal q-density function of the $(k-r)$-variate random variable $(X_{r+1}, X_{r+2}, \ldots, X_k)$.

2.4. **On the q-continuous q-uniform distribution.** For the needs of this work we give the definition of the q-uniform distribution and derive easily its main characteristics and properties. The q-uniform distribution is defined as follows.

Definition 2.5 (q-uniform)**.** Let X be a q-continuous random variable with q-density function

$$f_q(x) = \begin{cases} \frac{1}{\beta}, & 0 \leq x \leq \beta, \\ 0, & x < 0 \text{ or } x > \beta, \end{cases} \tag{2.13}$$

where $\beta > 0$. The distribution of the random variable X is called q-uniform distribution with parameter β.

Note that by the function (2.13) and the definition of the q-integral,

$$\int_0^\beta f_q(x) d_q x = \sum_{n=0}^{\infty} \beta \left(q^n - q^{n+1}\right) f_q(\beta q^n) = 1,$$

as required by the definition of a q-density function.

Proposition 2.6. *The r-th q-moments of the q-uniform distribution is given by*

$$\mu_r = E\left(X^r\right) = \frac{\beta^r}{[r+1]_q}. \tag{2.14}$$

In particular its q-mean and q-variance are given respectively by

$$\mu_q = E\left(X\right) = \frac{\beta}{[2]_q} \quad and \quad \sigma_q^2 = \frac{\beta^2 q}{(1+q+q^2)(1+2q+q^2)}.$$

Proof. Using the q-density function (2.13) and the definition of the q-integral, the rth q-moment of the q-uniform distribution,

$$\mu_r = E\left(X^r\right) = \int_0^\beta x^r f_q(x) d_q x,$$

is easily obtained in the form (2.14). The q-mean and q-variance of X follows. □

Remark 2.7. Let X be a q-continuous random variable obeying a q-uniform distribution with parameter β, then the linearly transformed q-continuous random variable $Y = \frac{X}{\beta}$ obeys the q-uniform distribution with parameter $\beta = 1$. Indeed

$$F_Y(y) = P\left(Y \leq y\right) = P\left(X \leq \beta y\right) = F_X\left(\beta y\right) = \int_\alpha^{\beta y} f_q(x) d_q x = y, \;\; 0 \leq y \leq 1.$$

So

$$f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1, \\ 0, & y < 0 \text{ or } y > 1. \end{cases}$$

In the following proposition, we show that the linear transformation $Y = \frac{X}{\beta}$ can be generalized by considering the transformation $Y = F_X(X)$, where $F_X(x)$ is a distribution function of a q-continuous random variable X.

Proposition 2.8. *Let X be a q-continuous random variable with probability function $F_X(x)$, $x \in R$. Then the distribution of the q-continuous random variable $Y = F_X\left(X\right)$ is the q-uniform distribution with parameter $\beta = 1$.*

Proof. The distribution function of the q-continuous random variable Y is given, for $0 \leq y \leq 1$, by

$$F_Y(y) = P\left(Y \leq y\right) = P\left(F_X(X) \leq y\right) = P\left(X \leq F_X^{-1}(y)\right) = F_X\left(F_X^{-1}(y)\right) = y.$$

So

$$f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1, \\ 0, & y < 0 \text{ or } y > 1 \end{cases}$$

and the proposition follows. □

3. Main results

3.1. **On the distributions of q-ordered random variables.** Let a ν-variate q-continuous random variable $\mathcal{X} = (X_1, \ldots, X_\nu)$ be defined in a sample space Ω. Then for the values $x_1 = X_1(\omega), \ldots, x_\nu = X_\nu(\omega)$, $\omega \in \Omega$ there is a permutation $(i_1, \ldots, i_\nu)$ of the ν indices $\{1, \ldots, \nu\}$, such that $x_{i_1} \leq \cdots < x_{i_{\nu-1}} \leq x_{i_\nu}$. The k-th ordered random variable is denoted by $X_{(k)}$ and defined by

$$X_{(k)}(\omega) = x_{(k)}, \ \omega \in \Omega,$$

where $x_{(k)} = x_{i_k}$, $k = 1, \ldots, \nu$. In particular, for $k = 1$ this gives $X_{(1)} = \min\{X_1, \cdots, X_\nu\}$ and, for $k = \nu$ this gives $X_{(\nu)} = \max\{X_1, \ldots, X_\nu\}$. Generally, the following inequalities hold:

$$0 \leq X_{(1)} \leq \cdots \leq X_{(\nu)} \leq \beta,$$

for a positive real number β.

We now introduce the following definition of q-ordered random variables.

Definition 3.1 (q-ordered)**.** Let $\mathcal{Y} = (Y_1, \ldots, Y_\nu)$ be a ν-variate q-continuous random variable and $Y_{(k)}$, $1 \leq k \leq \nu$, be the corresponding k-th ordered random variables. Assume that $Y_{(k)}$, $1 \leq k \leq \nu$, satisfy the inequalities

$$0 \leq Y_{(1)} < qY_{(2)} < Y_{(2)} < \cdots < Y_{(\nu-1)} < qY_{(\nu)} < Y_{(\nu)} \leq \beta, \tag{3.1}$$

for a positive real number β. Then, $Y_{(k)}$ (for any k such that $1 \leq k \leq \nu$) is called the k-th *q-ordered* random variables.

Let $Y_{(k)}$, $1 \leq k \leq \nu$, be the k-th q-ordered random variables, where the non-ordered q-continuous random variables $Y_1, \ldots, Y_\nu$, are *dependent and not identically distributed*. The non-ordered, dependent and not identically distributed, random variables $Y_i, i = 1, \ldots, \nu$, are randomly selected from the same sample space and the corresponding k-th q-ordered random variables, $Y_{(k)}$, $1 \leq k \leq \nu$, satisfy inequalities (3.1). Each non-ordered random variable Y_i is thus defined on the set

$$\begin{aligned} &R_{Y_i} := [0, q^{(i-1)}\beta] = \cup_{j=i}^{\nu} R_j, \\ &\text{where } R_j := (q^j\beta, q^{j-1}\beta] \text{ for } j = 1, \ldots, \nu-1 \text{ and } R_\nu := [0, q^{\nu-1}\beta]. \end{aligned} \tag{3.2}$$

In particular, one has

$$\cup_{j=1}^{\nu} R_j = [0, \beta] \text{ and } R_i \cap R_j = \emptyset \text{ for } i \neq j.$$

Moreover, we assume that the non-ordered random variables Y_i's are not identically distributed according to their definitions sets but they are distributed with the same functional form. Furthermore, the stochastic dependencies satisfied by the non-ordered random variables Y_i's are explicitly defined hereafter.

For any integer r between 1 and ν, let $\{i_1, \ldots, i_r\}$ be an r-combination of $\{1, \ldots, \nu\}$ satisfying $i_1 < \cdots < i_r$, and let $\{i_{r+1}, i_{r+2}, \ldots, i_\nu\}$ be its complementary combination (i.e., one has $\{i_1, \ldots, i_r\} \cup \{i_{r+1}, i_{r+2}, \ldots, i_\nu\} = \{1, \ldots, \nu\}$) satisfying $i_{r+1} < i_{r+2} < \cdots < i_\nu$. Then, we assume that the non-ordered random variables Y_i's satisfy the following dependence relations for $y \in [0, \beta]$:

$$P\left(Y_{i_r} \leq y | Y_{i_1} \leq y, \ldots, Y_{i_{r-1}} \leq y\right) = P\left(Y_{i_r} \leq q^{r-1} y\right), \tag{3.3}$$

$$P\left(Y_{i_r} \leq y | Y_{i_1} > y, \ldots, Y_{i_{r-1}} > y\right) = P\left(Y_{i_r} \leq y\right), \tag{3.4}$$

and

$$\begin{aligned} &P\left(Y_{i_m} \leq y | Y_{i_1} \leq y, \ldots, Y_{i_r} \leq y, Y_{i_{r+1}} > y, \ldots, Y_{i_{m-1}} > y\right) \\ &\quad = P\left(Y_{i_m} \leq y | Y_{i_1} \leq y, Y_{i_r} \leq y\right) \\ &\quad = P\left(Y_{i_m,q} \leq q^{i_m-(m-r)} y\right), m = r+1, r+2, \ldots, \nu. \end{aligned} \tag{3.5}$$

The q-distribution functions of the maximum, minimum, and k-th q-ordered random variables (respectively $Y_{(1)}$, $Y_{(\nu)}$, and $Y_{(k)}$) are derived in the following lemma.

Lemma 3.2. *Let $Y_1, \ldots, Y_\nu$ be dependent q-continuous random variables, where*

(a) *Each Y_i is defined on the set R_{Y_i} from Formula (3.2).*
(b) *Each Y_i has a q-distribution function $F_{Y_i}(y) = P(Y_i \leq y)$, for $y \in R_{Y_i}$, of the same functional form and satisfies the dependence relations (3.3), (3.4), (3.5).*

Then, the q-distribution function of the maximum q-ordered random variable $Y_{(\nu)} = \max \{Y_1, \ldots, Y_\nu\}$, where $Y_{(i)}$, $i = 1, \ldots, \nu$, satisfy inequalities (3.1), is given for $y \in [0, \beta]$ by

$$F_{Y_{(\nu)}}(y) = \prod_{i=1}^{\nu} F_{Y_i}(q^{i-1} y). \tag{3.6}$$

Moreover, the q-distribution function of the minimum q-ordered random variable $Y_{(1)} = \min\{Y_1, \ldots, Y_\nu\}$, where $Y_{(i)}$, $i = 1, \ldots, \nu$, satisfy inequalities (3.1), is given by

$$F_{Y_{(1)}}(y) = 1 - \prod_{i=1}^{\nu} \left(1 - F_{Y_i}(y)\right). \tag{3.7}$$

Furthermore, the q-distribution function of k-th q-ordered random variable $Y_{(k)}$, $1 \leq k \leq \nu$, where $Y_{(i)}$, $i = 1, \ldots, \nu$, satisfy inequalities (3.1), is given for $y \in [0, \beta]$ by

$$F_{Y_{(k)}}(y) = \sum_{r=k}^{\nu} \sum_{1 \leq i_1 < \ldots < i_r \leq \nu} \prod_{j=1}^{r} F_{Y_{i_j}}\left(q^{j-1} y\right) \prod_{m=r+1}^{\nu} \left(1 - F_{Y_{i_m}}\left(q^{i_m-(m-r)} y\right)\right), \tag{3.8}$$

where the inner summation is over all r-combinations $\{i_1, \ldots, i_r\}$ of the set $\{1, \ldots, \nu\}$.

Proof. Let $F_{Y_{(\nu)}}(y)$ be the q-distribution function of $Y_{(\nu)} = \max\{Y_1, \ldots, Y_\nu\}$, then

$$\begin{aligned} F_{Y_{(\nu)}}(y) &= P\left(Y_{(\nu)} \le y\right) = P\left(\max\{Y_1, \ldots, Y_\nu\} \le y\right) \\ &= P\left(Y_1 \le y, Y_2 \le y, \ldots, Y_\nu \le y\right) \\ &= P\left(Y_1 \le y\right) P(Y_2 \le y | Y_1 \le y) \cdots P(Y_\nu \le y | Y_1 \le y, \ldots, Y_{\nu-1} \le y). \end{aligned} \tag{3.9}$$

By assumptions (a) and (b), still for $y \in [0, \beta]$, the above equation (3.9) becomes

$$F_{Y_{(\nu)}}(y) = \prod_{i=1}^{\nu} F_{Y_i}(q^{i-1} y).$$

Let also $F_{Y_{(1)}}(y), y \in [0, \beta]$, be the q-distribution function of $Y_{(1)} = \min\{Y_1, \ldots, Y_\nu\}$, then

$$\begin{aligned} &F_{Y_{(1)}}(y) = P\left(Y_{(1)} \le y\right) = 1 - P\left(Y_{(1)} > y\right) = 1 - P\left(\min\{Y_1, \ldots, Y_\nu\} > y\right) \\ &= 1 - P\left(Y_1 > y, Y_2 > y, \ldots, Y_\nu > y\right) \\ &= 1 - P\left(Y_1 > y\right) P(Y_2 > y | Y_1 > y) \cdots P(Y_\nu > y | Y_1 > y, Y_2 > y, \ldots, Y_{\nu-1} > y) \\ &= 1 - \left(1 - P\left(Y_1 < y\right)\right)\left(1 - P(Y_2 < y | Y_1 > y\right) \cdots (1 - P(Y_\nu < y | Y_1 > y, \ldots, Y_{\nu-1} > y)). \end{aligned} \tag{3.10}$$

By assumptions (a) and (b), the above equation (3.10) becomes

$$F_{Y_{(1)}}(y) = 1 - \prod_{i=1}^{\nu} \left(1 - F_{Y_i}(y)\right), y \in [0, \beta].$$

Now, let $F_{Y_{(k)}}(y)$ be the q-distribution function of $Y_{(k)}$. Then, the event $Y_{(k)} \le y$ occurs if and only if at least k random variables from $\{Y_1, Y_\nu\}$ take values in the set $[0, y]$ while the remaining ones $\nu - k$ take values in the set $(y, \beta]$. More precisely, consider an r-combination $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, \nu, \}$, with $i_1 < \ldots < i_r$, and its complementary combination $\{i_{r+1}, i_{r+2}, \ldots, i_\nu\}$, with $i_{r+1} < i_{r+2} < \ldots < i_\nu$. Then the q-distribution function of $Y_{(k)}$ is expressed as

$$\begin{aligned} &F_{Y_{(k)}}(y) = P\left(Y_{(k)} \le y\right) \\ &= \sum_{r=k}^{\nu} \sum_{1 \le i_1 < \ldots < i_r \le \nu} P\left(Y_{i_1} \le y, \ldots, Y_{i_r} \le y, Y_{i_{r+1}} > y, \ldots, Y_{i_\nu} > y\right) \\ &= \sum_{r=k}^{\nu} \sum_{1 \le i_1 < \ldots < i_r \le \nu} P\left(Y_{i_1} \le y, \ldots, Y_{i_r} \le y\right) P\left(Y_{i_{r+1}} > y, \ldots, Y_{i_\nu} > y | Y_{i_1} \le y, \ldots, Y_{i_r} \le y\right) \\ &= \sum_{r=k}^{\nu} \sum_{1 \le i_1 < \ldots < i_r \le \nu} P\left(Y_{i_1} \le y) \cdots P(Y_{i_r} \le y | Y_{i_1} \le y, \ldots, Y_{i_{r-1}} \le y\right) \\ &\quad \cdot P(Y_{i_{r+1}} > y | Y_{i_1} \le y, \ldots, Y_{i_r} \le y) \cdots P(Y_{i_\nu} > y | Y_{i_1} \le y, \ldots, Y_{i_r} \le y, Y_{i_{r+1}} > y, \ldots, Y_{i_{\nu-1}} > y), \end{aligned} \tag{3.11}$$

still with an inner summation over all r-combinations $\{i_1, \ldots, i_r\}$ of the set $\{1, \ldots, \nu\}$. By assumptions (a) and (b), Equation (3.11) becomes

$$F_{Y_{(k)}}(y) = \sum_{r=k}^{\nu} \sum_{1 \le i_1 < \ldots < i_r \le \nu} \prod_{j=1}^{r} F_{Y_{i_j}}\left(q^{j-1} y\right) \prod_{m=r+1}^{\nu} \left(1 - F_{Y_{i_m}}\left(q^{i_m - (m-r)} y\right)\right),$$

where the inner summation is over all r-combinations $\{i_1, \ldots, i_r\}$ of the set $\{1, \ldots, \nu\}$. $\square$

In the next theorem, we assume that the non ordered random variables $Y_i, i = 1, 2, \dots$ are dependent, q-uniformly distributed on the sets $[0, q^{i-1}t], t > 0$, $i = 1, \dots, \nu$, respectively, and we use the above lemma 3.2, to derive the q-distribution function and the q-density function of the corresponding maximum, minimum and k-th q-ordered random variables $Y_{(k)}, k = 1, \dots, \nu$.

Theorem 3.3. *Let $Y_1, \dots, Y_\nu$ be dependent q-continuous random variables, q-uniformly distributed on the sets $[0, q^{i-1}t]$, $t > 0$, $i = 1, \dots, \nu$, respectively. Assume that the random variables Y_i, $i = 1, \dots, \nu$, satisfy the dependence relations (3.3), (3.4), (3.5). Then, for $y \in [0, t]$, we have the following q-distribution functions and q-density functions:*

- *For the maximum q-ordered random variable $Y_{(\nu)} = \max\{Y_1, \dots, Y_\nu\}$ we have*
$$F_{Y_{(\nu)}}(y) = \frac{y^\nu}{t^\nu}$$
and
$$f_{Y_{(\nu)}}(y) = [\nu]_q \frac{y^{\nu-1}}{t^\nu}. \tag{3.12}$$

- *For the q-distribution function and q-density function of the minimum q-ordered random variable $Y_{(1)} = \min\{Y_1, \dots, Y_\nu\}$ we have*
$$F_{Y_{(1)}}(y) = 1 - \prod_{i=1}^{\nu} \left(1 - \frac{y}{q^{i-1}t}\right)$$
and
$$f_{Y_{(1)}}(y) = \frac{[\nu]_q}{q^{\nu-1}t} \prod_{i=1}^{\nu-1} \left(1 - \frac{y}{q^{i-1}t}\right). \tag{3.13}$$

- *For the q-distribution function and the q-density function of the k-th q-ordered random variable $Y_{(k)}$ we have*
$$F_{Y_{(k)}}(y) = \sum_{r=k}^{\nu} \begin{bmatrix} \nu \\ r \end{bmatrix}_{\frac{1}{q}} \frac{y^r}{t^r} \prod_{i=1}^{\nu-r} \left(1 - \frac{y}{q^{i-1}t}\right)$$
and
$$f_{Y_{(k)}}(y) = \frac{[\nu]_q! q^{\binom{\nu-k}{2}}}{[k-1]_q! [\nu-k]_q! q^{\binom{\nu}{2} - \binom{k}{2}}} \frac{y^{k-1}}{t^k} \prod_{j=1}^{\nu-k} \left(1 - \frac{y}{q^{i-1}t}\right). \tag{3.14}$$

Proof. The theorem assumptions allow us to use Equation (3.6); the q-distribution function of $Y_{(\nu)}$ is thus
$$F_{Y_{(\nu)}}(y) = \prod_{i=1}^{\nu} F_{Y_i}(q^{i-1}y) = \frac{y}{t} \frac{qy}{qt} \cdots \frac{q^{\nu-1}}{q^{\nu-1}t} = \frac{y^\nu}{t^\nu}.$$

Taking the q-derivative of the above relation we have that the q-density function of $Y_{(\nu)}$ is straightforwardly given for $y \in [0,t]$ by

$$f_{Y_{(\nu)}}(y) = d_q F_{Y_{(\nu)}}(y) = [\nu]_q \frac{y^{\nu-1}}{t^{\nu}}.$$

Note that

$$\int_0^t f_{Y_{(\nu)}}(y) d_q y = \int_0^t [\nu]_q \frac{y^{\nu-1}}{t^{\nu}} d_q y = 1,$$

which is coherent with the fact we have here a q-density function.
Also, the q-distribution function of $Y_{(1)}$, by Equation (3.7) of the previous lemma 3.2, is straightforwardly given for $y \in [0,t]$ by

$$F_{Y_{(1)}}(y) = 1 - \prod_{i=1}^{\nu} (1 - F_{Y_i}(y)) = 1 - \prod_{i=1}^{\nu} \left(1 - \frac{y}{t}\right).$$

Taking the q-derivative of the above relation and using the q-binomial formula (2.4), we have that the q-density function of $Y_{(\nu)}$ is expressed as

$$\begin{aligned} f_{Y_{(1)}}(y) = d_q F_{Y_{(1)}}(y) &= -\sum_{k=0}^{\nu} (-1)^k q^{-\binom{k}{2}} \begin{bmatrix} \nu \\ k \end{bmatrix}_{\frac{1}{q}} \frac{[k]_q y^{k-1}}{t^k} \\ &= \frac{[n]_q}{t} \sum_{k=0}^{\nu-1} (-1)^{k-1} q^{-\binom{k}{2}} q^{-k(\nu-k)} \begin{bmatrix} \nu-1 \\ k-1 \end{bmatrix}_q \frac{y^{k-1}}{t^{k-1}} \\ &= \frac{[n]_q}{q^{\nu-1} t} \sum_{k=0}^{\nu-1} (-1)^{k-1} q^{-\binom{k-1}{2}} q^{-(k-1)(\nu-k)} \begin{bmatrix} \nu-1 \\ k-1 \end{bmatrix}_q \frac{y^{k-1}}{t^{k-1}} \\ &= \frac{[n]_q}{q^{\nu-1} t} \sum_{k=0}^{\nu-1} (-1)^{k-1} q^{-\binom{k-1}{2}} \begin{bmatrix} \nu-1 \\ k-1 \end{bmatrix}_{\frac{1}{q}} \frac{y^{k-1}}{t^{k-1}} = \frac{[\nu]_q}{q^{\nu-1} t} \prod_{i=1}^{\nu-1} \left(1 - \frac{y}{q^{i-1} t}\right). \end{aligned}$$

Note that using the q-binomial formula (2.4) and the q-identity (2.2), we obtain

$$\begin{aligned} \int_0^t f_{Y_{(1)}}(y) d_q y &= \frac{[\nu]_q}{q^{\nu-1}} \sum_{j=0}^{\nu-1} (-1)^j q^{-\binom{j}{2}} \begin{bmatrix} \nu-1 \\ j \end{bmatrix}_{\frac{1}{q}} \int_0^t \frac{y^j}{t^{j+1}} d_q y \\ &= \frac{[\nu]_q}{q^{\nu-1}} \sum_{j=0}^{\nu-1} (-1)^j q^{-\binom{j}{2}} q^{-j(\nu-1-j)} \begin{bmatrix} \nu-1 \\ j \end{bmatrix}_q \frac{1}{[j+1]_q} \\ &= [\nu]_q \sum_{j=0}^{\nu-1} (-1)^j q^{\binom{j+1}{2}} q^{-(j+1)(\nu-1)} \begin{bmatrix} \nu-1 \\ j \end{bmatrix}_q \frac{1}{[j+1]_q} = [\nu]_q \frac{1}{\begin{bmatrix} \nu \\ \nu-1 \end{bmatrix}_q} = 1, \end{aligned}$$

which is coherent with the fact we have here a q-density function.

Moreover, thanks to Equation (3.8), the q-distribution function of $Y_{(k)}$ is given by

$$\begin{aligned}
F_{Y_{(k)}}(y) &= \sum_{r=k}^{\nu} \sum_{1\le i_1<\ldots<i_r\le\nu} \prod_{j=1}^{r} F_{Y_{i_j}}\left(q^{j-1}y\right) \prod_{m=r+1}^{\nu}\left(1 - F_{Y_{i_m}}\left(q^{i_m-(m-r-1)}y\right)\right) \\
&= \sum_{r=k}^{\nu} \sum_{1\le i_1<\ldots<i_r\le\nu} \frac{y}{q^{i_1-1}t}\frac{qy}{q^{i_2-1}t}\cdots\frac{q^{r-1}y}{q^{i_r-1}t}\left(1-\frac{y}{t}\right)\left(1-\frac{y}{qt}\right)\cdots\left(1-\frac{y}{q^{\nu-r-1}t}\right) \\
&= \sum_{r=k}^{\nu} \frac{y^r}{t^r}\prod_{i=1}^{\nu-r}\left(1-\frac{y}{q^{i-1}t}\right)\sum_{1\le i_1<\ldots<i_r\le\nu} q^{-i_1-\cdots-i_r+\binom{r+1}{2}},
\end{aligned}$$

where the inner summation is over the r-combinations $\{i_1,\ldots,i_r\}$ of the set $\{1,\ldots,\nu\}$.

Applying the formula of the q-binomial coefficient (2.1) in the above equation, we obtain

$$F_{Y_{(k)}}(y) = \sum_{r=k}^{\nu} \begin{bmatrix}\nu\\ r\end{bmatrix}_{\frac{1}{q}} \frac{y^r}{t^r}\prod_{i=1}^{\nu-r}\left(1-\frac{y}{q^{i-1}t}\right), y\in[0,t].$$

Taking the q-derivative of the above q-distribution function, using suitably the q-binomial formula (2.4) and conducting all the needed algebraic manipulations, we have that the q-density function of $Y_{(k)}$ for $1\le k\le\nu$ is expressed (for $y\in[0,t]$) as

$$\begin{aligned}
f_{Y_{(k)}}(y) = d_q F_{Y_{(k)}}(y) &= \sum_{r=k}^{\nu} q^{-r(\nu-r)}\begin{bmatrix}\nu\\ r\end{bmatrix}_q [r]_q\frac{y^{r-1}}{t^r}\prod_{i=1}^{\nu-r}\left(1-\frac{y}{q^{i-1}t}\right) \\
&+\sum_{r=k}^{\nu} q^{-r(\nu-r)}\begin{bmatrix}\nu\\ r\end{bmatrix}_q \frac{q^r y^r}{t^r}\sum_{j=0}^{\nu-r}(-1)^j q^{-\binom{j}{2}-j(\nu-r-j)}\begin{bmatrix}\nu-r\\ j\end{bmatrix}_q [j]_q\frac{y^{j-1}}{t^j} \\
&= \frac{[\nu]_q}{t}\sum_{r=k}^{\nu} q^{-(\nu-r)}q^{-(r-1)(\nu-r)}\begin{bmatrix}\nu-1\\ r-1\end{bmatrix}_q\frac{y^{r-1}}{t^{r-1}}\prod_{i=1}^{\nu-r}\left(1-\frac{y}{q^{i-1}t}\right) \\
&-\frac{1}{t}\sum_{r=k}^{\nu} q^{-r(\nu-r)}\begin{bmatrix}\nu\\ r\end{bmatrix}_q\frac{q^r y^r}{t^r}[\nu-r]_q q^{-(\nu-r-1)} \\
&\qquad\times\sum_{j=0}^{\nu-r}(-1)^{j-1}q^{-\binom{j-1}{2}-(j-1)(\nu-r-j)}\begin{bmatrix}\nu-r-1\\ j-1\end{bmatrix}_q\frac{y^{j-1}}{t^{j-1}} \\
&= \frac{[\nu]_q}{t}\sum_{r=k}^{\nu} q^{-(\nu-r)}q^{-(r-1)(\nu-r)}\begin{bmatrix}\nu-1\\ r-1\end{bmatrix}_q\frac{y^{r-1}}{t^{r-1}}\prod_{i=1}^{\nu-r}\left(1-\frac{y}{q^{i-1}t}\right) \\
&-\frac{[\nu]_q}{t}\sum_{r=k}^{\nu-1} q^{-(\nu-r-1)}q^{-r(\nu-1-r)}\begin{bmatrix}\nu-1\\ r\end{bmatrix}_q\frac{y^r}{t^r}\prod_{i=1}^{\nu-r-1}\left(1-\frac{y}{q^{i-1}t}\right) \\
&= \frac{[\nu]_q}{t}q^{-(\nu-k)}q^{-(k-1)(\nu-k)}\begin{bmatrix}\nu-1\\ k-1\end{bmatrix}_q\frac{y^{k-1}}{t^{k-1}}\prod_{i=1}^{\nu-k}\left(1-\frac{y}{q^{i-1}t}\right) \\
&= q^{-k(\nu-k)}\frac{[\nu]_q!}{[k-1]_q![\nu-k]_q!}\frac{y^{k-1}}{t^k}\prod_{j=1}^{\nu-k}\left(1-\frac{y}{q^{i-1}t}\right) \\
&= \frac{[\nu]_q!q^{\binom{\nu-k}{2}}}{[k-1]_q![\nu-k]_q!q^{\binom{\nu}{2}-\binom{k}{2}}}\frac{y^{k-1}}{t^k}\prod_{j=1}^{\nu-k}\left(1-\frac{y}{q^{i-1}t}\right).
\end{aligned}$$

Note that using suitably the q-binomial formula (2.4), the q-identity (2.2) and carrying out all the needed algebraic manipulations, we obtain

$$\begin{aligned}
\int_0^t f_{Y_{(k)}}(y)d_q y &= \frac{[\nu]_q! q^{\binom{\nu-k}{2}}}{[k-1]_q![\nu-k]_q! q^{\binom{\nu}{2}-\binom{k}{2}} t} \int_0^t \frac{y^{k-1}}{t^{k-1}} \prod_{j=1}^{\nu-k}\left(1-\frac{y}{q^{i-1}t}\right) d_q y \\
&= q^{-k(\nu-k)} \frac{[\nu]_q!}{[k-1]_q![\nu-k]_q!} \sum_{j=0}^{\nu-k}(-1)^j q^{-\binom{j}{2}} \begin{bmatrix} \nu-k \\ j \end{bmatrix}_{\frac{1}{q}} \int_0^t \frac{y^{k+j-1}}{t^{j+k}} d_q y \\
&= q^{-k(\nu-k)} \frac{[\nu]_q!}{[k]_q![\nu-k]_q!} \sum_{j=0}^{\nu-k}(-1)^j q^{-\binom{j}{2}} q^{-j(\nu-k-j)} \begin{bmatrix} \nu-k \\ j \end{bmatrix}_q \frac{[k]_q}{[k+j]_q} \\
&= \frac{[\nu]_q!}{[k]_q![\nu-k]_q!} \sum_{j=0}^{\nu-k}(-1)^j q^{\binom{j+1}{2}-(\nu-k)(k+j)} \begin{bmatrix} \nu-k \\ j \end{bmatrix}_q \frac{[k]_q}{[k+j]_q} \\
&= \frac{[\nu]_q!}{[k]_q![\nu-k]_q!} \frac{1}{\begin{bmatrix} \nu \\ \nu-k \end{bmatrix}_q} = 1,
\end{aligned}$$

which is coherent with the fact we have here a q-density function. □

Remark 3.4. The random variables $Y_{(1)}$ and $Y_{(\nu)}$ follow q-power law distributions (see Formulas (3.12) and (3.13)) while the random variables $Y_{(2)}, \ldots, Y_{(\nu-1)}$ follow q-beta distributions (see Formula (3.14)).

In the following lemma, we consider the non-ordered q-continuous random variables, $Y_1, \ldots, Y_\nu$, being dependent and not identically distributed, and we derive the joint q-distribution function of the q-ordered random variables, $Y_{(1)}$ and $Y_{(\nu)}$ that satisfy inequalities (3.1).

Lemma 3.5. *Let $Y_1, \ldots, Y_\nu$ be dependent q-continuous random variables, where*

(a) *Each Y_i is defined on the set R_{Y_i} from Formula (3.2).*
(b) *Each Y_i has a q-distribution function $F_{Y_i}(y) = P(Y_i \leq y)$, for $y \in R_{Y_i}$, of the same functional form and satisfy the dependence relations (3.3), (3.4), (3.5).*

Then, the joint q-distribution function of the q-ordered random variables

$$Y_{(1)} = \min\{Y_1, \ldots, Y_\nu\} \quad \textit{and} \quad Y_{(\nu)} = \max\{Y_1, \ldots, Y_\nu\},$$

is given by

$$F_{Y_{(1)},Y_{(\nu)}}(y,z) = \prod_{i=1}^{\nu} F_{Y_i}(q^{i-1}z) - \prod_{i=1}^{\nu}\Big(F_{Y_i}(q^{i-1}z) - F_{Y_i}(y)\Big) \tag{3.15}$$

with $y < q^{\nu-1}z, \nu \geq 1, y, z \in [0, \beta]$.

Proof. Let $F_{Y_{(1)},Y_{(\nu)}}(y,z)$, $y<q^{\nu-1}z, \nu\geq 1, y,z\in[0,\beta]$, be the joint q-distribution function of the random variables $Y_{(1)}$ and $Y_{(\nu)}$. Using the expression

$$P\left(Y_{(1)}\leq y, Y_{(\nu)}\leq z\right)=P\left(Y_{(\nu)}\leq z\right)-P\left(Y_{(1)}>y, Y_{(\nu)}\leq z\right)$$

we then have

$$\begin{aligned}
&F_{Y_{(1)},Y_{(\nu)}}(y,z)=P\left(Y_{(1)}\leq y, Y_{(\nu)}\leq z\right)=P\left(Y_{(\nu)}\leq z\right)-P\left(Y_{(1)}>y, Y_{(\nu)}\leq z\right)\\
&=P\left(Y_1\leq z, Y_2\leq z,\dots,Y_\nu\leq z\right)-P\left(y<Y_1\leq z, y<Y_2\leq z,\dots,y<Y_\nu\leq z\right)\\
&=P\left(Y_1\leq z\right)P(Y_2\leq z|Y_1\leq z)\cdots P(Y_\nu\leq z|Y_1\leq z,Y_2\leq z,\dots,Y_{\nu-1}\leq z)\\
&\quad -P(y<Y_1\leq z)P(y<Y_2\leq z|y<Y_1\leq z)\cdots\\
&\qquad \cdot P(y<Y_\nu\leq z|y<Y_1\leq z, y<Y_2\leq z,\dots,y<Y_{\nu-1}\leq z).
\end{aligned}\tag{3.16}$$

By assumptions (a) and (b), Equation (3.16) becomes (for $y,z\in[0,\beta]$ such that $y<q^{\nu-1}z$):

$$F_{Y_{(1)},Y_{(\nu)}}(y,z)=\prod_{i=1}^{\nu}F_{Y_i}(q^{i-1}z)-\prod_{i=1}^{\nu}\left(F_{Y_i}(q^{i-1}z)-F_{Y_i}(y)\right). \qquad \square$$

In the next theorem, we assume that the non ordered random variables Y_i are dependent and q-uniformly distributed on the sets $[0,q^{i-1}t]$ (for $t>0$), and we use the above lemma 3.5, to derive the joint q-distribution function and the joint q-density function of the q-ordered random variables $Y_{(1)}$ and $Y_{(\nu)}$.

Theorem 3.6. *Let $Y_1,\dots,Y_\nu$ be dependent q-continuous random variables, q-uniformly distributed on the sets $[0,q^{i-1}t], t>0$, $i=1,\dots,\nu$, respectively. Assume that the random variables Y_i satisfy the dependence relations* (3.3)*,* (3.4)*,* (3.5)*. Then, the joint q-distribution function and the joint q-density function of the q-ordered random variables, $Y_{(1)}=\min\{Y_1,\dots,Y_\nu\}$ and $Y_{(\nu)}=\max\{Y_1,\dots,Y_\nu\}$ are given respectively by*

$$F_{Y_{(1)},Y_{(\nu)}}(y,z)=\frac{z^\nu}{t^\nu}-\frac{z^\nu}{t^\nu}\prod_{i=1}^{\nu}\left(1-\frac{y}{q^{i-1}z}\right)$$

and

$$f_{Y_{(1)},Y_{(\nu)}}(y,z)=q^{-\nu+1}[\nu]_q[\nu-1]_q\frac{z^{\nu-2}}{t^\nu}\prod_{i=1}^{\nu-2}\left(1-\frac{y}{q^iz}\right)$$

with $y<q^{\nu-1}z, \nu\geq 1, y,z\in[0,t]$.

Proof. With the conditions of the theorem, by Equation (3.15), the q-distribution function of the random variables $Y_{(1)}$ and $Y_{(\nu)}$ becomes

$$\begin{aligned}
F_{Y_{(1)},Y_{(\nu)}}(y,z)&=\prod_{i=1}^{\nu}F_{Y_i}(q^{i-1}z)-\prod_{i=1}^{\nu}\left(F_{Y_i}(q^{i-1}z)-F_{Y_i}(y)\right)\\
&=\frac{z}{t}\frac{qz}{qt}\cdots\frac{q^{\nu-1}z}{q^{\nu-1}t}-\left(\frac{z}{t}-\frac{y}{t}\right)\left(\frac{qz}{qt}-\frac{y}{qt}\right)\left(\frac{q^2z}{q^2t}-\frac{y}{q^2t}\right)\cdots\left(\frac{q^{\nu-1}z}{q^{\nu-1}t}-\frac{y}{q^{\nu-1}t}\right)\\
&=\frac{z^\nu}{t^\nu}-\frac{z^\nu}{t^\nu}\prod_{i=1}^{\nu}\left(1-\frac{y}{q^{i-1}z}\right).
\end{aligned}$$

Taking the partial q-derivatives of the above joint q-distribution function and using the q-binomial formula (2.4), we have that the joint q-density function of the random variables $Y_{(1)}$ and $Y_{(\nu)}$ is expressed as

$$\begin{aligned}
f_{Y_{(1)},Y_{(\nu)}}(y,z) &= \frac{\partial F_{Y_{(1)},Y_{(\nu)}}(y,z)}{\partial_q z \partial_q y} = -\frac{1}{\partial_q z \partial_q y}\frac{z^\nu}{t^\nu}\prod_{i=1}^{\nu}\left(1-\frac{y}{q^{i-1}z}\right) \\
&= -\frac{1}{\partial_q z \partial_q y}\frac{z^\nu}{t^\nu}\sum_{r=0}^{\nu}(-1)^r q^{-\binom{r}{2}}\begin{bmatrix}\nu\\ r\end{bmatrix}_{\frac{1}{q}}\frac{y^r}{z^r} \\
&= -\frac{1}{t^\nu}\sum_{r=0}^{\nu}(-1)^r q^{-\binom{r}{2}}q^{-r(\nu-r)}\begin{bmatrix}\nu\\ r\end{bmatrix}_q [r]_q[\nu-r]_q y^{r-1}z^{\nu-r-1} \\
&= \frac{z^{\nu-2}}{t^\nu}[\nu]_q[\nu-1]_q\sum_{r=0}^{\nu}(-1)^{r-1}q^{-\binom{r}{2}}q^{-r(\nu-r)}\begin{bmatrix}\nu-2\\ r-1\end{bmatrix}_q\frac{y^{r-1}}{z^{r-1}} \\
&= \frac{z^{\nu-2}}{q^{\nu-1}t^\nu}[\nu]_q[\nu-1]_q\sum_{r=0}^{\nu}(-1)^{r-1}q^{-\binom{r-1}{2}}q^{-(r-1)(\nu-r-1)}\begin{bmatrix}\nu-2\\ r-1\end{bmatrix}_q\frac{y^{r-1}}{(qz)^{r-1}} \\
&= \frac{z^{\nu-2}}{q^{\nu-1}t^\nu}[\nu]_q[\nu-1]_q\sum_{j=0}^{\nu}(-1)^{j}q^{-\binom{j}{2}}q^{-j(\nu-2-j)}\begin{bmatrix}\nu-2\\ j\end{bmatrix}_q\frac{y^{j}}{(qz)^{j}} \\
&= q^{-\nu+1}[\nu]_q[\nu-1]_q\frac{z^{\nu-2}}{t^\nu}\prod_{i=1}^{\nu-2}\left(1-\frac{y}{q^i z}\right).
\end{aligned}$$

Note that using suitably the q-binomial formula (2.4), the q^{-1}-identity (2.3) and carrying out all the needed algebraic manipulations, we obtain

$$\begin{aligned}
&\int_0^t\int_0^{q^{\nu-1}z} f_{Y_{(1)},Y_{(\nu)}}(y,z)d_qyd_qz \\
&= \frac{q^{-\nu+1}}{t^\nu}[\nu]_q[\nu-1]_q\sum_{j=0}^{\nu}(-1)^j q^{-\binom{j}{2}}q^{-j(\nu-2-j)-j}\begin{bmatrix}\nu-2\\ j\end{bmatrix}_q\int_0^t\int_0^{q^{\nu-1}z} y^j z^{\nu-2-j}d_qyd_qz \\
&= \frac{q^{-\nu+1}}{t^\nu}[\nu]_q[\nu-1]_q\sum_{j=0}^{\nu}(-1)^j q^{-\binom{j}{2}}q^{-j(\nu-2-j)-j}\begin{bmatrix}\nu-2\\ j\end{bmatrix}_q\frac{q^{(\nu-1)(j+1)}t^\nu}{[j+1]_q[\nu]_q} \\
&= q^{-\nu+2}[\nu-1]_q\sum_{j=0}^{\nu}(-1)^j q^{\binom{j+1}{2}+\nu-2}\begin{bmatrix}\nu-2\\ j\end{bmatrix}_q\frac{[1]_q}{[1+j]_q} \\
&= q^{-\nu+2}[\nu-1]_q\frac{1}{\begin{bmatrix}\nu-1\\ \nu-2\end{bmatrix}_{\frac{1}{q}}} = 1
\end{aligned}$$

which is coherent with the fact we have here a q-density function. □

In the following lemma, we consider the non-ordered q-continuous random variables, $Y_1,\dots,Y_\nu$, being dependent and not identically distributed, and we derive the joint q-distribution function of the q-ordered random variables, $Y_{(k)}$ and $Y_{(r)}$, $1\le k<r\le\nu$.

Lemma 3.7. *Let $Y_1, \ldots, Y_\nu$ be dependent q-continuous random variables, where*

(a) *Each Y_i is defined on the set R_{Y_i} from Formula (3.2).*
(b) *Each Y_i has a q-distribution function $F_{Y_i}(y) = P(Y_i \le y), y \in R_{Y_i}$ of the same functional form and satisfies the dependence relations (3.3), (3.4), (3.5).*

Then, the joint q-distribution function of the q-ordered random variables $Y_{(k)}$ and $Y_{(r)}$ for $1 \le k < r \le \nu$, where $Y_{(i)}$, $i = 1, \ldots, \nu$, satisfy inequalities (3.1), is given by

$$\begin{aligned} &F_{Y_{(k)},Y_{(r)}}(y,z) \\ &= \sum_{j=r}^{\nu}\sum_{s=k}^{j}\sum\prod_{n_1=1}^{s} F_{Y_{i_{n_1}}}\left(q^{n_1-1}y\right)\prod_{ni_1,\ldots,i_r=s+1}^{j}\left(F_{Y_{i_{n_2}}}\left(q^{n_2-s-1}z\right)-F_{Y_{i_{n_2}}}(y)\right) \\ &\cdot\prod_{n_3=j+1}^{\nu}\left(1-F_{Y_{i_{n_3}}}\left(q^{i_{n_3}-(n_3-j)}z\right)\right), y<q^{r-k}z, 1\le k<r\le\nu, y,z\in[0,\beta], \end{aligned} \tag{3.17}$$

where the inner summation is over all pairwise disjoint subsets $\{i_1, \ldots, i_s\}$ and $\{i_{s+1}, \ldots, i_j\}$ of the set $\{1, \ldots, \nu\}$ with $1 \le i_1 < \cdots < i_s \le \nu$ and $1 \le i_{s+1} < i_{s+2} < \cdots < i_j \le \nu$.

Proof. Let $F_{Y_{(k)},Y_{(r)}}(y,z) = P\left(Y_{(k)} \le y, Y_{(r)} \le z\right)$, $y < q^{r-k}z, 1 \le k < r \le \nu, y, z \in [0,\beta]$, be the joint q-distribution function of the random variables $Y_{(k)}$ and $Y_{(r)}$ with $1 \le k < r \le \nu$. Then, the events $Y_{(k)} \le y$ and $Y_{(r)} \le z$ occur if and only if at least k random variables in $\{Y_1, \ldots, Y_\nu\}$ take values in the set $[0, y]$, while $r-k$ random other variables take values in the set $(y, z]$, and the remaining ones take values in the set $(z, \beta]$, $1 \le k < r \le \nu$. So, for $y < q^{r-k}z, 1 \le k < r \le \nu, y, z \in [0,\beta]$, we have

$$\begin{aligned} &F_{Y_{(k)},Y_{(r)}}(y,z) = P\left(Y_{(k)} \le y, Y_{(r)} \le z\right) \\ &= \sum_{j=r}^{\nu}\sum_{s=k}^{j}\sum_{\substack{1\le i_1<\ldots<i_s\le\nu \\ 1\le i_{s+1}<i_{s+2}<\ldots<i_j\le\nu}} P\Big(\{Y_{i_\ell}\le y\}_{\ell=1,\ldots,s}, \{y<Y_{i_\ell}\le z\}_{\ell=s+1,\ldots,j}, \{Y_{i_\ell}>z\}_{\ell=j+1,\ldots,\nu}\Big) \\ &= \sum_{j=r}^{\nu}\sum_{s=k}^{j}\sum_{\substack{1\le i_1<\ldots<i_s\le\nu \\ 1\le i_{s+1}<i_{s+2}<\ldots<i_j\le\nu}} \\ &P(Y_{i_1}\le y)P(Y_{i_2}\le y|Y_{i_1}\le y)\cdots P(Y_{i_s}\le y|Y_{i_1}\le y, Y_{i_2}\le y,\ldots,Y_{i_{s-1}}\le y) \\ &\cdot P\Big(y<Y_{i_{s+1}}\le z|Y_{i_1}\le y,\ldots,Y_{i_s}\le y\Big) \\ &\cdot P\Big(y<Y_{i_{s+2}}\le z|Y_{i_1}\le y,\ldots,Y_{i_s}\le y, y<Y_{i_{s+1}}\le z\Big)\cdots \\ &\cdot P\Big(y<Y_{i_j}\le z|Y_{i_1}\le y,\ldots,Y_{i_s}\le y, y<Y_{i_{s+1}}\le z,\ldots,y<Y_{i_{j-1}}\le z\Big) \\ &\cdot P\Big(Y_{i_{j+1}}>z|Y_{i_1}\le y,\ldots,Y_{i_s}\le y, y<Y_{i_{s+1}}\le z,\ldots,y<Y_{i_j}\le z\Big)\cdots \\ &\cdot P\Big(Y_{i_\nu}>z|Y_{i_1}\le y,\ldots,Y_{i_s}\le y, y<Y_{i_{s+1}}\le z,\ldots,y<Y_{i_j}\le z, Y_{i_{j+1}}>z,\ldots,Y_{i_{\nu-1}}>z\Big), \end{aligned} \tag{3.18}$$

where the inner summation is over all pairwise disjoint subsets $\{i_1, \ldots, i_s\}$ and $\{i_{s+1}, \ldots, i_j\}$ of the set $\{1, \ldots, \nu\}$ with $1 \le i_1 < \ldots < i_s \le \nu$ and $1 \le i_{s+1} < i_{s+2} < \ldots < i_j \le \nu$.

By assumptions (a) and (b), Equation (3.18) becomes (for $y < q^{r-k}z$, $1 \leq k < r \leq \nu$, and $y, z \in [0, \beta]$)

$$F_{Y_{(k)},Y_{(r)}}(y,z) = \sum_{j=r}^{\nu}\sum_{s=k}^{j} \sum_{\substack{1\leq i_1<\ldots<i_s\leq\nu \\ 1\leq i_{s+1}<i_{s+2}<\ldots<i_j\leq\nu}} \prod_{n_1=1}^{s} F_{Y_{i_{n_1}}}\left(q^{n_1-1}y\right)$$
$$\cdot \prod_{n_2=s+1}^{j}\left(F_{Y_{i_{n_2}}}\left(q^{n_2-s-1}z\right) - F_{Y_{i_{n_2}}}(y)\right) \prod_{n_3=j+1}^{\nu}\left(1 - F_{Y_{i_{n_3}}}\left(q^{i_{n_3}-(n_3-j)}z\right)\right),$$

where the inner summation is over all pairwise disjoint subsets $\{i_1, \ldots, i_s\}$ and $\{i_{s+1}, \ldots, i_j\}$ of the set $\{1, \ldots, \nu\}$ with $1 \leq i_1 < \cdots < i_s \leq \nu$ and $1 \leq i_{s+1} < i_{s+2} < \cdots < i_j \leq \nu$. □

In the next theorem, we use the above lemma 3.7 to derive the joint q-distribution function and the joint q-density function of the ordered random variables.

Theorem 3.8. *Let $Y_1, \ldots, Y_\nu$ be dependent q-continuous random variables, q-uniformly distributed on the sets $[0, q^{i-1}t], t > 0$, $i = 1, \ldots, \nu$, respectively. Assume that the random variables Y_i, $i = 1, \ldots, \nu$, satisfy the dependence relations (3.3), (3.4), (3.5). Then, the joint q-distribution function and the joint q-density function of the q-ordered random variables, $Y_{(k)}$ and $Y_{(r)}$, for $1 \leq k < r \leq \nu$, are given respectively by*

$$F_{Y_{(k)},Y_{(r)}}(y,z) = \sum_{j=r}^{\nu}\sum_{s=k}^{j} \begin{bmatrix} \nu \\ s, j-s \end{bmatrix}_{\frac{1}{q}} \frac{y^s}{t^s}\frac{z^{j-s}}{t^{j-s}} \prod_{i=1}^{j-s}\left(1 - \frac{y}{q^{i-1}z}\right)\prod_{m=1}^{\nu-j}\left(1 - \frac{z}{q^{m-1}t}\right) \tag{3.19}$$

and

$$f_{Y_{(k)},Y_{(r)}}(y,z) = \frac{q^{-r(\nu-r)}q^{-k(r-k)}[\nu]_q!}{[k-1]_q![r-k-1]_q![\nu-r]_q!}\frac{y^{k-1}}{t^r} z^{r-k-1}\prod_{i=1}^{r-k-1}\left(1 - \frac{y}{q^i z}\right)\prod_{m=1}^{\nu-r}\left(1 - \frac{z}{q^{m-1}t}\right) \tag{3.20}$$

with $y < q^{r-k}z, 1 \leq k < r \leq \nu, y, z \in [0, t]$.

Proof. By Equation (3.17) and the q-multinomial formulas (2.6) and (2.5), the joint q-distribution function of $Y_{(k)}$ and $Y_{(r)}$ satisfies

$$F_{Y_{(k)},Y_{(r)}}(y,z) = \sum_{j=r}^{\nu}\sum_{s=k}^{j} \sum_{\substack{1\leq i_1<\cdots<i_s\leq\nu \\ 1\leq m_1<m_2<\cdots<m_{j-s}\leq\nu}} q^{\binom{s+1}{2}}q^{\binom{j-s+1}{2}}q^{-i_1-\cdots-i_s}q^{-m_1-\cdots-m_{j-s}}$$
$$\cdot \frac{y^s}{t^s}\frac{z^{j-s}}{t^{j-s}}\left(1-\frac{y}{z}\right)\left(1-\frac{y}{qz}\right)\cdots\left(1-\frac{y}{q^{j-s-1}z}\right)\left(1-\frac{z}{t}\right)\left(1-\frac{z}{qt}\right)\cdots\left(1-\frac{z}{q^{\nu-j-1}t}\right)$$
$$= \sum_{j=r}^{\nu}\sum_{s=k}^{j} \begin{bmatrix} \nu \\ s, j-s \end{bmatrix}_{\frac{1}{q}} \frac{y^s}{t^s}\frac{z^{j-s}}{t^{j-s}} \prod_{i=1}^{j-s}\left(1 - \frac{y}{q^{i-1}z}\right)\prod_{m=1}^{\nu-j}\left(1 - \frac{z}{q^{m-1}t}\right), \tag{3.21}$$

where the inner summation of the first equality, is over all pairwise disjoint subsets $\{i_1, i_2, \ldots, i_s\}$ and $\{m_1, m_2, \ldots, m_{j-s}\}$ of the set $\{1, \ldots, \nu\}$ with $1 \leq i_1 < \ldots < i_s \leq \nu$ and $1 \leq m_1 < m_2 < \ldots < m_{j-s} \leq \nu$.

The above joint q-distribution (3.21), of the random variables $Y_{(k)}$ and $Y_{(r)}$, for $1 \le k < r \le \nu$ and $y < q^{r-k}z$, $y, z \in [0,t]$ can be written as

$$F_{Y_{(k)},Y_{(r)}}(y,z) = \sum_{j=r}^{\nu} \frac{q^{-j(\nu-j)}}{t^j} \begin{bmatrix} \nu \\ j \end{bmatrix}_q \prod_{m=1}^{\nu-j} \left(1 - \frac{z}{q^{m-1}t}\right) \sum_{s=k}^{j} \begin{bmatrix} j \\ s \end{bmatrix}_q \left(\frac{z}{q^s}\right)^{j-s} \prod_{i=1}^{j-s} \left(1 - \frac{y}{q^{i-1}z}\right).$$

Taking the partial q-derivative of the inner sum over y, using suitably the q-binomial formula (2.4) and carrying out all needed algebraic manipulations, we obtain

$$\begin{aligned}
&\frac{\partial_q}{\partial_q y} \sum_{s=k}^{j} q^{-s(j-s)} \begin{bmatrix} j \\ s \end{bmatrix}_q z^{j-s} \prod_{i=1}^{j-s} \left(1 - \frac{y}{q^{i-1}z}\right) \\
&= [j]_q \left(\sum_{s=k}^{j} q^{-s(j-s)} \begin{bmatrix} j-1 \\ s-1 \end{bmatrix}_q y^{s-1} z^{j-s} \prod_{i=1}^{j-s} \left(1 - \frac{y}{q^{i-1}z}\right) \right) - [j]_q \left(\sum_{s=k}^{j} q^{-(s+1)(j-s-1)} \begin{bmatrix} j-1 \\ s \end{bmatrix}_q \right. \\
&\qquad \left. y^s z^{j-s-1} \prod_{i=1}^{j-s-1} \left(1 - \frac{y}{q^{i-1}z}\right) q^{-k(j-k)} [j]_q \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_q y^{k-1} z^{j-k} \prod_{i=1}^{j-k} \left(1 - \frac{y}{q^{i-1}z}\right) \right).
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{\partial_q F_{Y_{(k)},Y_{(r)}}(y,z)}{\partial_q y} \\
&= \sum_{j=r}^{\nu} q^{-j(\nu-j)} \begin{bmatrix} \nu \\ j \end{bmatrix}_q \frac{1}{t^j} \prod_{m=1}^{\nu-j} \left(1 - \frac{z}{q^{m-1}t}\right) q^{-k(j-k)} [j]_q \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_q y^{k-1} z^{j-k} \prod_{i=1}^{j-k} \left(1 - \frac{y}{q^{i-1}z}\right) \\
&= \sum_{j=r}^{\nu} q^{-j(\nu-j)} q^{-k(j-k)} \begin{bmatrix} \nu \\ j \end{bmatrix}_q \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_q [j]_q \frac{1}{t^j} y^{k-1} z^{j-k} \prod_{i=1}^{j-k} \left(1 - \frac{y}{q^{i-1}z}\right) \prod_{m=1}^{\nu-j} \left(1 - \frac{z}{q^{m-1}t}\right) \\
&= \frac{[\nu]_q! y^{k-1}}{[k-1]_q! [\nu-k]_q!} \sum_{j=r}^{\nu} q^{-j(\nu-j)} q^{-k(j-k)} \begin{bmatrix} \nu-k \\ j-k \end{bmatrix}_q \frac{z^{j-k}}{t^j} \prod_{i=1}^{j-k} \left(1 - \frac{y}{q^{i-1}z}\right) \prod_{m=1}^{\nu-j} \left(1 - \frac{z}{q^{m-1}t}\right).
\end{aligned}$$

In the last sum of this equation, taking the partial q-derivative over z, and using suitably q-binomial formula (2.4), we get

$$\begin{aligned}
&\frac{\partial_q}{\partial_q z} \sum_{j=r}^{\nu} q^{-j(\nu-j)} q^{-k(j-k)} \begin{bmatrix} \nu-k \\ j-k \end{bmatrix}_q \frac{1}{t^j} z^{j-k} \prod_{i=1}^{j-k} \left(1 - \frac{y}{q^{i-1}z}\right) \prod_{m=1}^{\nu-j} \left(1 - \frac{z}{q^{m-1}t}\right) \\
&= \sum_{j=r}^{\nu} q^{-j(\nu-j)} q^{-k(j-k)} \begin{bmatrix} \nu-k \\ j-k \end{bmatrix}_q [j-k]_q \frac{1}{t^j} z^{j-k-1} \prod_{i=1}^{j-k-1} \left(1 - \frac{y}{q^i z}\right) \prod_{m=1}^{\nu-j} \left(1 - \frac{z}{q^{m-1}t}\right) \\
&\quad - \sum_{j=r}^{\nu} q^{-j(\nu-j)-k(j-k)-(\nu-j-1)+j-k} \begin{bmatrix} \nu-k \\ j-k \end{bmatrix}_q [\nu-j]_q \frac{z^{j-k}}{t^{j+1}} \prod_{i=1}^{j-k} \left(1 - \frac{y}{q^i z}\right) \prod_{m=1}^{\nu-j-1} \left(1 - \frac{z}{q^{m-1}t}\right).
\end{aligned}$$

It follows that

$$\frac{\partial_q}{\partial_q z}\sum_{j=r}^{\nu} q^{-j(\nu-j)}q^{-k(j-k)}\begin{bmatrix}\nu-k\\ j-k\end{bmatrix}_q \frac{1}{t^j}z^{j-k}\prod_{i=1}^{j-k}\left(1-\frac{y}{q^{i-1}z}\right)\prod_{m=1}^{\nu-j}\left(1-\frac{z}{q^{m-1}t}\right)$$

$$=[\nu-k]_q\sum_{j=r}^{\nu} q^{-j(\nu-j)}q^{-k(j-k)}\begin{bmatrix}\nu-k-1\\ j-k-1\end{bmatrix}_q \frac{1}{t^j}z^{j-k-1}\prod_{i=1}^{j-k-1}\left(1-\frac{y}{q^{i}z}\right)\prod_{m=1}^{\nu-j}\left(1-\frac{z}{q^{m-1}t}\right)$$

$$-[\nu\text{-}k]_q\sum_{j=r}^{\nu} q^{-j(\nu-j)-k(j-k)-(\nu-j-1)+j-k}\begin{bmatrix}\nu\text{-}k\text{-}1\\ j\text{-}k\end{bmatrix}_q \frac{z^{j-k}}{t^{j+1}}\prod_{i=1}^{j-k}\left(1\text{-}\frac{y}{q^{i}z}\right)\prod_{m=1}^{\nu-j-1}\left(1\text{-}\frac{z}{q^{m-1}t}\right)$$

$$=q^{-r(\nu-r)}q^{-k(r-k)}[\nu-k]_q\begin{bmatrix}\nu-k-1\\ r-k-1\end{bmatrix}_q \frac{1}{t^r}z^{r-k-1}\prod_{i=1}^{r-k-1}\left(1-\frac{y}{q^{i}z}\right)\prod_{m=r}^{\nu-j}\left(1-\frac{z}{q^{m-1}t}\right).$$

From this identity, we get that the joint q-density function given by

$$f_{Y_{(k)},Y_{(r)}}(y,z)=\frac{\partial_q^2 F_{Y_{(k)},Y_{(r)}}(y,z)}{\partial_q z\partial_q y}$$

$$=\frac{q^{-r(\nu-r)}q^{-k(r-k)}[\nu]_q!}{[k-1]_q![r-k-1]_q![\nu-r]_q!}\frac{y^{k-1}}{t^r}z^{r-k-1}\prod_{i=1}^{r-k-1}\left(1\text{-}\frac{y}{q^{i}z}\right)\prod_{m=r}^{\nu-j}\left(1\text{-}\frac{z}{q^{m-1}t}\right)$$

with $y<q^{r-k}z$, $y,z\in[0,t]$.
Note that using suitably the q-binomial formula (2.4), the q^{-1} and q-identities (2.2), (2.3) and carrying out all the needed algebraic manipulations, we obtain

$$\int_0^t\int_0^{q^{r-k}z} f_{Y_{(k)},Y_{(r)}}(y,z)d_qy\,d_qz$$

$$=\frac{q^{-r(\nu-r)}q^{-k(r-k)}[\nu]_q!t^{-r}}{[k-1]_q![r-k-1]_q![\nu-r]_q!}$$

$$\times\int_0^t\left(\int_0^{q^{r-k}z} y^{k-1}\prod_{i=1}^{r-k-1}\left(1-\frac{y}{q^iz}\right)d_qy\right)z^{r-k-1}\prod_{m=r}^{\nu-j}\left(1-\frac{z}{q^{m-1}t}\right)d_qz$$

$$=\frac{q^{-r(\nu-r)}q^{-k(r-k)}[\nu]_q!t^{-r}}{[k-1]_q![r-k-1]_q![\nu-r]_q!}\frac{q^k}{[k]_q}$$

$$\times\sum_{m=0}^{r-k-1}(-1)^m q^{\binom{m+1}{2}+(r-k-1)k}\begin{bmatrix}r-k-1\\ m\end{bmatrix}_q\frac{[k]_q}{[k+m]_q}\int_0^t z^{r-1}\prod_{m=r}^{\nu-j}\left(1-\frac{z}{q^{m-1}t}\right)d_qz$$

$$=\frac{q^{-k(r-k)}[\nu]_q!}{[k\text{-}1]_q![r\text{-}k\text{-}1]_q![\nu\text{-}r]_q!}\frac{1}{\begin{bmatrix}r\text{-}1\\ r\text{-}k\text{-}1\end{bmatrix}_{\frac{1}{q}}}\frac{q^k}{[k]_q[r]_q}\sum_{i=0}^{\nu-r}(-1)^i q^{\binom{i+1}{2}-(i+r)(\nu-r)}\begin{bmatrix}\nu\text{-}r\\ i\end{bmatrix}_q\frac{[r]_q}{[r+i]_q}$$

$$=\frac{[\nu]_q!}{[k]_q![r-k-1]_q![\nu-r]_q!}\frac{[r-k-1]_q![k]_q!}{[r]_q!}\frac{1}{\begin{bmatrix}\nu\\ \nu-r\end{bmatrix}_q}=1,$$

which is coherent with the fact we have here a q-density function. Note also that the joint q-distribution function and q-density function of the random variables $Y_{(1)}$ and $Y_{(\nu)}$ are given respectively by (3.19) and (3.20), for $k=1$ and $r=\nu$. □

Remark 3.9. The bivariate random variables $\left(Y_{(k)}, Y_{(r)}\right)$ for $1 \leq k < r \leq \nu$ with joint q-density function (3.20), follow q-Dirichlet distributions.

In the following proposition, we consider the non-ordered q-continuous random variables, $Y_1, \ldots, Y_\nu$, being dependent and not identically distributed and we derive the joint distribution function of the q-ordered random variables, $Y_{(1)}, \ldots, Y_{(\nu)}$.

Proposition 3.10. *Let $(Y_1, \ldots, Y_\nu)$ be a q-continuous ν-variate random vector with joint q-density function $f(y_1, \ldots, y_\nu)$. Then the q-density function of the q-ordered random vector $\mathcal{Y} = (Y_{(1)}, \ldots, Y_{(\nu)})$ is given by*

$$f_{\mathcal{Y}}(y_{(1)}, \ldots, y_{(\nu)}) = \sum f_{Y_{i_\nu}}(y_{(\nu)})\ f_{Y_{i_{\nu-1}}|Y_{i_\nu}}(y_{(\nu-1)}|y_{(\nu)})\ \cdots\ f_{Y_{i_1}|(Y_{i_2},\ldots,Y_{i_\nu})}(y_{(1)}|y_{(2)}, \ldots, y_{(\nu)}),$$
$$0 < y_{(1)} < qy_{(2)} < y_{(2)} < qy_{(3)} < \cdots < y_{(\nu-1)} < qy_{(\nu)} < y_{(\nu)} < \beta, \tag{3.22}$$

where the summation is over all permutations $(i_1, \ldots, i_\nu)$ of $\{1, \ldots, \nu\}$.

Proof. The joint q-density function is

$$\begin{aligned}
f_{\mathcal{Y}}(y_{(1)}, \ldots, y_{(\nu)}) &= \frac{P\left(qy_{(1)} < Y_{(1)} \leq y_{(1)}, \ldots, qy_{(\nu)} < Y_{(\nu)} \leq y_{(\nu)}\right)}{(1-q)y_{(1)}(1-q)y_{(2)}\cdots(1-q)y_{(\nu)}} \\
&= (1-q)^{-\nu} \prod_{i=1}^{\nu} y_{(i)}^{-1} \sum P\left(qy_{(1)} < Y_{i_1} \leq y_{(1)}, \ldots, qy_{(\nu)} < Y_{i_\nu} \leq y_{(\nu)}\right) \\
&= (1-q)^{-\nu} \prod_{i=1}^{\nu} y_{(i)}^{-1} \sum P\left(qy_{(\nu)} < Y_{i_\nu} \leq y_{(\nu)}\right) P\left(qy_{(\nu-1)} < Y_{i_{\nu-1}} \leq y_{(\nu-1)}|qy_{(\nu)} < Y_{i_\nu} \leq y_{(\nu)}\right) \\
&\qquad \cdots P\left(qy_{(\nu)} < Y_{i_1} \leq y_{(1)}|qy_{(1)} < Y_{i_1} \leq y_{(1)}, \ldots, qy_{(\nu)} < Y_{i_\nu} \leq y_{(\nu)}\right),
\end{aligned} \tag{3.23}$$

where the summation is over all permutations $(i_1, \ldots, i_\nu)$ of $\{1, \ldots, \nu\}$.
Applying Definition 2.4 on the dependent q-density function and the relations (2.11), (2.12), to the above equation (3.23), we obtain 3.22. □

Next, we assume that the non ordered random variables $Y_i, i = 1, \ldots, \nu$ are dependent and q-uniformly distributed on the sets $[0, q^{i-1}t], t > 0$, $i = 1, \ldots, \nu$, respectively, and the joint q-density function of the q-ordered random variables $Y_{(1)}, \ldots, Y_{(\nu)}$, is obtained in the following corollary of Proposition 3.10.

Corollary 3.11. *Let $Y_1, \ldots, Y_\nu$ be dependent q-continuous random variables, q-uniformly distributed on the sets $[0, q^{i-1}t], t > 0$ $i = 1, \ldots, \nu$, respectively. Assume that the random variables Y_i, $i = 1, \ldots, \nu$, satisfy the dependence relations (3.3), (3.4), (3.5). Then the joint q-density function of the ν-variate q-continuous random vector $\mathcal{Y} = (Y_{(1)}, \ldots, Y_{(\nu)})$ with $Y_{(k)}$, $k = 1, \ldots, \nu$, the k-th q-ordered random variables, is given by*

$$f_{\mathcal{Y}}(y_1, \ldots, y_\nu) = \frac{[\nu]_q!}{q^{\binom{\nu}{2}} t^\nu},\ 0 < y_1 < qy_2 < y_2 < qy_3 < \cdots < y_{\nu-1} < qy_\nu < y_\nu < t. \tag{3.24}$$

Proof. Let $Y_1, \ldots, Y_\nu$ be dependent q-continuous random variables, with each Y_i (for $i = 1, \ldots, \nu$) q-uniformly distributed on the set $[0, q^{i-1}t]$ (for some $t > 0$). Applying (3.22) of the previous proposition 3.10, the joint q-density function of the ν-variate q-continuous random vector $\mathcal{Y} = (Y_{(1)}, \ldots, Y_{(\nu)})$ (where each $Y_{(k)}$, for $k = 1, \ldots, \nu$, is the k-th q-ordered random variable) is given (for $0 < y_1 < qy_2 < y_2 < qy_3 < \cdots < y_{\nu-1} < qy_\nu < y_\nu < t$) by

$$\begin{aligned} f_{\mathcal{Y}}(y_1, \ldots, y_\nu) &= \sum \frac{1}{q^{i_\nu - 1}t} \frac{1}{q^{i_{\nu-1}-1}t} \cdots \frac{1}{q^{i_1 - 1}t} \\ &= \frac{1}{t^\nu} \sum \frac{1}{\prod_i^\nu q^{i_j - 1}}, \end{aligned}$$

where the summation is over all permutations $(i_1, \ldots, i_\nu)$ of $\{1, \ldots, \nu\}$.

So,

$$f_{\mathcal{Y}}(y_1, \ldots, y_\nu) = \frac{[\nu]_{q^{-1}}!}{t^\nu} = \frac{[\nu]_q!}{q^{\binom{\nu}{2}} t^\nu}.$$

Note that

$$\begin{aligned} &\int_0^t \int_0^{qy_\nu} \int_0^{qy_{\nu-1}} \cdots \int_0^{qy_3} \int_0^{qy_2} f_{\mathcal{Y}}(y_1, \ldots, y_\nu) d_q y_1\, d_q y_2 \cdots d_q y_{\nu-2}\, d_q y_{\nu-1}\, d_q y_\nu \\ &= \int_0^t \int_0^{qy_\nu} \int_0^{qy_{\nu-1}} \cdots \int_0^{qy_3} \int_0^{qy_2} \frac{[\nu]_q!}{q^{\binom{\nu}{2}} t^\nu} d_q y_1\, d_q y_2 \cdots d_q y_{\nu-2}\, d_q y_{\nu-1}\, d_q y_\nu = 1, \end{aligned}$$

which confirms that Equation (3.24) is a joint q-density function. □

3.2. **On a conditional joint q-distribution of the waiting times of the Heine process and q-order statistics.** Let T_k be the waiting time of the kth arrival in the Heine process $\{X(t),\, t > 0\}$ with parameters λ and q. Let us stop the process at T_ν, for some integer $\nu \geq 1$. Now, we study the joint q-density function of the waiting times $T_1, \ldots, T_\nu$. In the next theorem we prove that this conditional joint q-density function coincides with the joint q-density function of a q-ordered random sample of size ν, from the q-continuous uniform distribution in the set $[0, q^{i-1}t]$, $i = 1, \ldots, \nu$.

Theorem 3.12. *Let T_k be the waiting time of the kth arrival of the Heine process $\{X(t),\, t > 0\}$ with parameters λ and q. Then the joint q-density function of the waiting times $T_1, \ldots, T_\nu$, in which the first ν events occur given that $X(t) = \nu$, $0 < t_1 < \cdots < t_\nu < t$ with $t_i \in (q^{\nu-i+1}t, q^{\nu-i}t]$, $i = 1, \ldots, \nu - 1$, is given by*

$$f_q\left(t_1, \ldots, t_\nu | X(t) = \nu\right) = \frac{[\nu]_q!}{q^{\binom{\nu}{2}} t^\nu},$$

that is the joint q-density function of a q-ordered random sample of size ν, from the q-continuous uniform distribution in the set $[0, q^{i-1}t]$, $i = 1, \ldots, \nu$.

Proof. By using the expression (2.10) and the three basic assumptions of Definition 2.1, the conditional joint q-density function of the Heine process satisfies the equation

$$
\begin{aligned}
& f_q\left(t_1, \ldots, t_\nu | X(t)=\nu\right) q^{\nu-1}(1-q) t q^{\nu-2}(1-q) t \cdots q(1-q) t \\
& = P\left(q^\nu t<T_1 \leq q^{\nu-1} t, \ldots, q^2 t<T_\nu \leq q t \;\mid\; X(t)=\nu\right) \\
& = P\left(X(q^\nu t)=0\right)\left(\prod_{i=1}^{\nu-1} P\left(X(q^i(1-q) t)=1\right)\right) \frac{P\left(X((1-q) t)=0\right)}{P\left(X(t)=\nu\right)} \\
& = e_q(-\lambda q^\nu t) \frac{\lambda q^{\nu-1}(1-q) t}{1+\lambda q^{\nu-1}(1-q) t} \frac{\lambda q^{\nu-2}(1-q) t}{1+\lambda q^{\nu-2}(1-q) t} \cdots \frac{\lambda q(1-q) t}{1+\lambda q(1-q) t} \frac{1}{1+\lambda(1-q) t} \\
& \quad \cdot\left(e_q(-\lambda t) \frac{q^{\binom{\nu}{2}}(\lambda t)^\nu}{[\nu]_q!}\right)^{-1} .
\end{aligned}
$$

So,

$$
f_q\left(t_1, \ldots, t_\nu | X(t)=\nu\right)=\frac{[\nu]_q!}{q^{\binom{\nu}{2}} t^\nu} .
$$

Therefore, by Corollary 3.11, this conditional joint q-density function coincides with q-ordered density from the claim of the theorem. □

4. Concluding remarks

In this work we have introduced q-order statistics, for $0<q<1$, arising from dependent and not identically q-continuous random variables, as q-analogues of the classical order statistics. We have studied their main properties concerning the q-distribution functions and q-density functions of the relative q-ordered random variables. We have concentrated on the q-ordered variables arising from dependent and not identically q-uniformly distributed random variables. The derived q-distributions include q-power law, q-beta and q-Dirichlet distributions. The motivation for introducing q-order statistics was given by studying the properties of the waiting times of the Heine process.

As further study we propose the introduction of q-order statistics arising from dependent and not identically discrete q-distributed random variables. Last but not least, in link with lattice paths combinatorics, we intend to study the relations between q-order statistics and q-random walks in $\mathbb{Z}^d$, building on [10].

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DIFFERENTIAL ALGEBRAIC GENERATING FUNCTIONS OF WEIGHTED WALKS IN THE QUARTER PLANE

THOMAS DREYFUS[1]

[1]Institut de Recherche Mathématique Avancée, U.M.R. 7501 Université de Strasbourg et C.N.R.S. 7, rue René Descartes 67084 Strasbourg, France;
https://sites.google.com/site/thomasdreyfusmaths/

Abstract. In the present paper we study the nature of the trivariate generating functions of weighted walks in the quarter plane. Combining the results of this paper to previous ones, we complete the proof of the following theorem. The series satisfies a nontrivial algebraic differential equation in one of its variables, if and only if it satisfies a nontrivial algebraic differential equation in each of its variables.

Keywords: Random walks in quarter plane, elliptic functions, transcendence.

1. Introduction

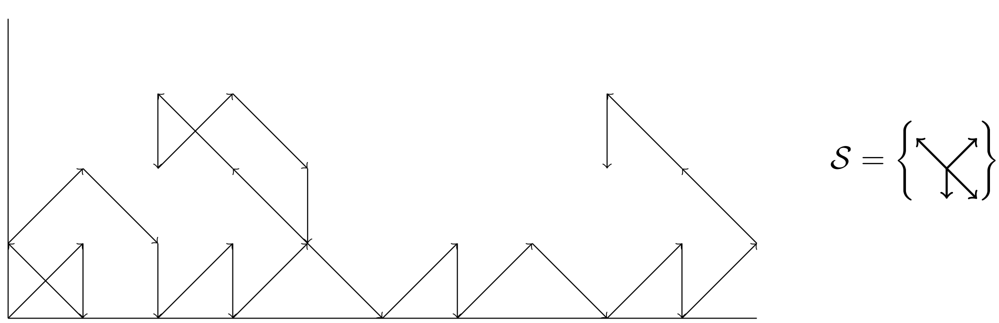
Framework. Consider a walk with small steps in the nonnegative quadrant $\mathbb{Z}^2_{\geq 0} = \{0, 1, 2, \ldots\}^2$ starting from $P_0 := (0, 0)$, that is a succession of points

$$P_0, P_1, \ldots, P_k,$$

where each P_n lies in the quarter plane, where the moves (or steps) $P_{n+1} - P_n$ belong to $\{0, \pm 1\}^2$, and the probability to move in the direction $P_{n+1} - P_n = (i, j)$ may be interpreted as some weight parameter $d_{i,j} \in [0, 1]$, with $\sum_{(i,j)\in\{0,\pm 1\}^2} d_{i,j} = 1$. The model of the walk (or model for short) is the data of the $d_{i,j}$ and the step set of the walk is the set of directions with nonzero weights, that is

$$\mathcal{S} = \{(i, j) \in \{0, \pm 1\}^2 \mid d_{i,j} \neq 0\}.$$

If $d_{0,0} = 0$ and if the nonzero $d_{i,j}$ all have the same value, we say that the model is unweighted. The following picture provides an example of a walk in the nonnegative quadrant:



Such objects are very natural both in combinatorics and probability theory: they are interesting for themselves and also because they are strongly related to other discrete structures; see [4,6] and references therein.

The weight of the walk is defined to be the product of the weights of its component steps. For any $(i,j) \in \mathbb{Z}^2_{\geq 0}$ and any $k \in \mathbb{Z}_{\geq 0}$, we let $q_{i,j,k}$ be the sum of the weights of all walks reaching the position (i,j) from the initial position $(0,0)$ after k steps. We introduce the corresponding trivariate generating function

$$Q(x,y;t) := \sum_{i,j,k \geq 0} q_{i,j,k} x^i y^j t^k.$$

Being the generating function of probabilities, $Q(x,y;t)$ converges for all $(x,y,t) \in \mathbb{C}^3$ such that $|x|, |y| \leq 1$ and $|t| < 1$. Note that in several papers, as in [4], it is not assumed that $\sum d_{i,j} = 1$. However, after a rescaling of the t variable, we may always reduce to this case.

Statement of the main result. As we will see in the sequel, this paper takes part in a long history of articles that study the algebraic and differential relations satisfied by $Q(x,y;t)$. For any choice of a variable $\star$ among x, y, t, we say that $Q(x,y;t)$ is $\partial_\star$-algebraic if there exists $n \in \mathbb{Z}_{\geq 0}$, such that there exists a nonzero multivariate polynomial $P_\star \in \mathbb{C}(x,y,t)[X_0, \ldots, X_n]$, such that

$$0 = P_\star(Q(x,y;t), \ldots, \partial_\star^n Q(x,y;t)).$$

We stress that in the above definition, it is equivalent to require $0 \neq P_\star \in \mathbb{Q}[X_0, \ldots, X_n]$; see Remark 3.1. Otherwise, we say that the series $Q(x,y;t)$ is $\partial_\star$-differentially transcendental.

Since the three variables x, y and t play a different role, we might expect the series to be of different nature with respect to the three derivatives. The main result of this paper, quite unexpected at first sight, shows that it is not the case. More precisely, using results of this paper and combining them to partial cases already known (see the discussion in the sequel), we complete the proof of the following main theorem.

Theorem 1.1. *The following facts are equivalent:*

- *The series $Q(x,y;t)$ is ∂_x-algebraic;*
- *The series $Q(x,y;t)$ is ∂_y-algebraic;*
- *The series $Q(x,y;t)$ is ∂_t-algebraic.*

Note that an algorithm is given in [16, Section 5] to decide whether the generating function is differentially algebraic in the x variable or not, but this does not provide the differential equation when it exists.

State of the art. More generally, the question of studying whether $Q(x,y;t)$ satisfies algebraic (resp. linear differential, resp. algebraic differential) equations attracted the attention of many authors in the last decade. In the unweighted case, the problem was first addressed in the seminal paper [4] and solved using several methods, such as combinatorics, computer algebra, complex analysis, and more recently, difference Galois theory; see [2,3,7–9,19–21]. We refer to the introduction of [12] for a history of the cited results, from which it follows that Theorem 1.1 is valid for the unweighted models.

The main difficulty in generalizing those results to weighted models is that, contrary to the unweighted framework, there are infinitely many weighted models. However, certain unweighted results are still valid in the weighted cases, while some others are proved by a case-by-case argument, and therefore cannot be generalized straightforwardly. So beyond the generalization, we believe that replacing case-by-case proofs by systematic arguments has its own interest since it shows that the unweighted version of Theorem 1.1 has not appeared by accident in a finite number of cases, and illustrates a general phenomenon.

In many situations, the equivalence between the ∂_x-algebraicity and the ∂_y-algebraicity can be straightforwardly deduced in this weighted context from the proof of [8, Proposition 3.10]. In [7, Theorem 2] it was proved that the ∂_t-algebraicity implies the ∂_x-algebraicity. So it remains to show the converse. In [2], the authors show that all ∂_x-differentially algebraic unweighted models have a decoupling function. They use this property to prove the ∂_t-algebraicity in that case. In [8], using difference Galois theory, the authors show that such unweighted models admit a telescoping relation. We refer to [16] for precise definitions of the two latter notions. In [11], it is proved that the ∂_x-algebraic weighted models also have a telescoping relation. Finally in [16] the equivalence between the existence of a telescoping relation and the existence of decoupling functions is shown. This implies that a ∂_x-algebraic series admits a certain decomposition into elliptic functions.

The main difficulty is that the existence of such decompositions is proved for fixed values of t, so nothing is known about the dependence in t of the coefficients. For instance, the function $x\Gamma(t)$, seen as a function of x is simple for all fixed value of t (it is rational!) but it is differentially transcendental with respect to t, due to Hölder's result. We then have to make a careful study of the t-dependence of such elliptic relations, and use some results of ∂_t-algebraicity of the Weierstrass function in [2]. Finally, we are able to show that the ∂_x-algebraicity implies the ∂_t-algebraicity. The following diagram summarizes the various contributions toward the proof of Theorem 1.1.

$$\begin{tikzcd}
\partial_y\text{-algebraicity} \arrow[r, Leftrightarrow, "{[8]}"'] & \partial_x\text{-algebraicity} \arrow[d, Rightarrow, "{[11]}"] & \partial_t\text{-algebraicity} \arrow[l, Rightarrow, "{[7]}"] \\
& \text{Telescoping relation} \arrow[r, Leftrightarrow, "{[16]}"] & \text{Decoupling function} \arrow[u, Rightarrow, "\text{This paper} + [2]"']
\end{tikzcd}$$

Structure of the paper. The paper is organized as follows. In Section 2 we provide some reminders of objects appearing in the study of models of walks in the quarter plane. More precisely, we will study well-known properties of the kernel curve and explain how the generating function may be continued. We will also explain why Theorem 1.1 is correct in some degenerate cases that we may withdraw. In Section 3 we prove technical results on differential algebraicity. Some intermediate results stay valid in the framework of algebraic functions and/or solution of linear differential equations, but to simplify the exposition, we chose to present this section in a unified framework, making some intermediate results suboptimal. Finally Section 4 is devoted to the proof of Theorem 1.1. We split our study in two cases depending on whether the so-called group of the walk is finite or not.

2. Kernel of the walk

2.1. **Functional equation.** The kernel of the walk is the polynomial defined by

$$K(x,y;t) := xy(1 - tS(x,y)),$$

where $S(x,y)$ denotes the jump polynomial

$$\begin{aligned} S(x,y) &= \sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j}x^i y^j \\ &= A_{-1}(x)\frac{1}{y} + A_0(x) + A_1(x)y \\ &= B_{-1}(y)\frac{1}{x} + B_0(y) + B_1(y)x, \end{aligned}$$

with $A_i(x) \in x^{-1}\mathbb{R}[x]$, $B_i(y) \in y^{-1}\mathbb{R}[y]$ (we recall that we consider weights $d_{i,j} \in [0,1]$). The kernel plays an important role in the so-called kernel method and the techniques we are going to apply will vary depending on its algebraic properties, that have been studied in [14] (when $t = 1$), and in [10, 11] (when $t \in (0,1)$). The starting point is the following fundamental functional equation.

Lemma 2.1. *The generating function $Q(x,y;t)$ satisfies the functional equation*

$$K(x,y;t)Q(x,y;t) = xy + K(x,0;t)Q(x,0;t) + K(0,y;t)Q(0,y;t) - K(0,0;t)Q(0,0;t).$$

Proof. As a walk is either empty, or a smaller walk to which one added a step (removing the cases leaving the quarter-plane), one has the following combinatorial functional equation

$$Q(x,y;t) = 1 + tS(x,y)Q(x,y;t) - t\tfrac{B_{-1}(y)}{x}Q(0,y;t) - t\tfrac{A_{-1}(x)}{y}Q(x,0;t) + t\tfrac{d_{-1,-1}}{xy}Q(0,0;t),$$

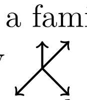
where the last summand removes the corresponding double counting. Multiplying by xy, we get Lemma 2.1. $\square$

2.2. **Degenerate cases.** Like in [10], we will discard the following degenerate cases.

Definition 2.2 (Degenerate model)**.** Let us fix $t \in (0,1)$. A model of walk is called degenerate if one of the following holds:

- $K(x,y;t)$ factors in non-constant polynomials in $\mathbb{C}[x,y]$,
- $K(x,y;t)$ has x-degree (or y-degree) less than or equal to 1.

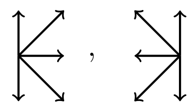
The notion of degeneracy thus seems to depend upon the parameter t. However, we will see in Proposition 2.3 below that the model is degenerate for a value of $t \in (0,1)$ if and only if it is degenerate for all values of $t \in (0,1)$. So, to lighten the terminology, we prefer not to stress this t-dependence and we say "degenerate" rather than "t-degenerate".

In what follows we will sometimes represent a family of models of walks with arrows. For instance, the family of models represented by  or, equivalently, $\{\nearrow, \searrow, \swarrow, \uparrow\}$ corresponds to models with $d_{1,0} = d_{0,-1} = d_{-1,1} = d_{-1,0} = 0$ and nothing is assumed on the other $d_{i,j}$. We stress the fact that the other $d_{i,j}$ (the weight of the arrows above) are allowed to be 0. In the following results, the behavior of the kernel curve *never* depends on $d_{0,0}$. This is the reason why, to reduce the amount of notations, we have decided not to associate an arrow to $d_{0,0}$.

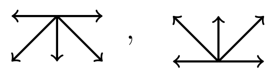
The following proposition has been proved in [14, Lemma 2.3.2] for $t = 1$, in [10, Proposition 1.2] for t is transcendental over $\mathbb{Q}(d_{i,j})$, and in [11, Proposition 1.3] for the other values of t in $(0, 1)$.

Proposition 2.3. *Let us fix $t \in (0, 1)$. A model of walk is degenerate if and only if at least one of the following holds:*

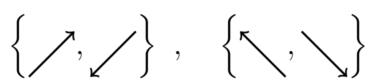
(a) *There exists $i \in \{-1, 1\}$ such that $d_{i,-1} = d_{i,0} = d_{i,1} = 0$. This corresponds to the following families of models*



(b) *There exists $j \in \{-1, 1\}$ such that $d_{-1,j} = d_{0,j} = d_{1,j} = 0$. This corresponds to the following families of models*



(c) *All the weights are 0 except maybe $\{d_{-1,-1}, d_{0,0}, d_{1,1}\}$ or $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$. This corresponds to the following families of models*



In virtue of the following lemma, Theorem 1.1 is valid for the degenerate models of walks. Therefore we will focus on models that are not degenerate.

Lemma 2.4. *Assume that the model of walk is degenerate. Then $Q(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$ (and thus is differentially algebraic in its three variables).*

Proof. We use Proposition 2.3. Consider the cases (a), (b), and first configuration of the case (c). In the unweighted case it is proved in [4, Section 1.2] that $Q(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$. The proof is the same in the weighted context but, to be self-contained, let us sketch the proof here. In the first configuration of case (a) the generating function is the same as the corresponding generating function of a model in the upper half-plane $\mathbb{Z} \times \mathbb{N}$. The latter is classically known to be algebraic over $\mathbb{C}(x, y, t)$, see for instance [5, Proposition 2]. In the second configuration of case (a), we have a unidimensional walk on the y-axis and such series is known to be rational, and therefore algebraic over $\mathbb{C}(x, y, t)$. The case (b) is similar. In the first configuration of case (c), we are considering a unidimensional walk on the half-line $\{(x, x), x \in \mathbb{N}\}$, and the generating function is algebraic. Since in all these cases, $Q(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$, it is differentially algebraic in its three variables. In the last configuration of case (c), all the weights are 0 except maybe $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$, so the walk cannot leave $(0, 0)$ and we have

$$Q(x, y; t) = \sum_{k=0}^{\infty} d_{0,0}^k t^k = \frac{1}{1 - d_{0,0} t}.$$

Therefore the result holds in that case too. $\square$

2.3. **Genus of the walk.** The kernel curve E_t is the complex affine algebraic curve defined by

$$E_t = \{(x,y) \in \mathbb{C} \times \mathbb{C} \mid K(x,y;t) = 0\}.$$

We shall now consider a compactification of this curve. We let $\mathbb{P}^1(\mathbb{C})$ be the complex projective line, that is the quotient of $(\mathbb{C} \times \mathbb{C}) \setminus \{(0,0)\}$ by the equivalence relation $\sim$ defined by

$$(x_0, x_1) \sim (x_0', x_1') \Leftrightarrow \exists \lambda \in \mathbb{C}^*, (x_0', x_1') = \lambda (x_0, x_1).$$

The equivalence class of $(x_0, x_1) \in (\mathbb{C} \times \mathbb{C}) \setminus \{(0,0)\}$ is denoted by $[x_0 : x_1] \in \mathbb{P}^1(\mathbb{C})$. The map $x \mapsto [x : 1]$ embeds $\mathbb{C}$ inside $\mathbb{P}^1(\mathbb{C})$. The latter map is not surjective: its image is $\mathbb{P}^1(\mathbb{C}) \setminus \{[1:0]\}$; the missing point $[1:0]$ is usually denoted by ∞. Now, we embed E_t inside $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ via $(x,y) \mapsto ([x:1],[y:1])$. The kernel curve $\overline{E}_t$ is the closure of this embedding of E_t. In other words, the kernel curve $\overline{E}_t$ is the algebraic curve defined by

$$\overline{E}_t = \{([x_0 : x_1],[y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid \overline{K}(x_0, x_1, y_0, y_1; t) = 0\}$$

where $\overline{K}(x_0, x_1, y_0, y_1; t)$ is the following degree two homogeneous polynomial in the four variables x_0, x_1, y_0, y_1

$$\overline{K}(x_0, x_1, y_0, y_1; t) = x_1^2 y_1^2 K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}; t\right) = x_0 x_1 y_0 y_1 - t \sum_{i,j=0}^{2} d_{i-1,j-1} x_0^i x_1^{2-i} y_0^j y_1^{2-j}.$$

Although it may seem more natural to take the closure of $\overline{E}_t$ in $\mathbb{P}^2(\mathbb{C})$, the above definition allows us to avoid unnecessary singularities.

The following proposition has been proved in [10, Proposition 2.1 and Corollary 2.6], when t is transcendental over $\mathbb{Q}(d_{i,j})$ and has been extended for a general $0 < t < 1$ in [11, Proposition 1.9].

Proposition 2.5. *Let us fix $t \in (0,1)$ and assume that the model of the walk is not degenerate. The following facts are equivalent:*

(1) $\overline{E}_t$ is an elliptic curve;
(2) The set of authorized directions $\mathcal{S}$ is not included in any half-space with boundary passing through the origin.

Let us now discuss the case where for $t \in (0,1)$ fixed, the model is not degenerate and $\overline{E}_t$ is not an elliptic curve. By Proposition 2.3 and Proposition 2.5, this corresponds to nondegenerate models that belong to one of the four families in Figure 1.

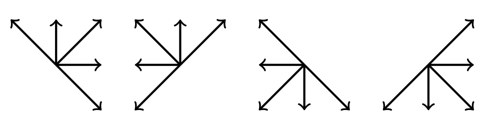


FIGURE 1. Our four nondegenerate models

Note that although the third configuration in Figure 1 is called nondegenerate, it leads to walks that never escape from $(0,0)$ and thus their generating function is trivial.

The following lemma yields that Theorem 1.1 is valid for the families of models in Figure 1.

Lemma 2.6. *The following holds:*

(a) *Assume that the model of the walk is not degenerate and belongs to the first family in Figure 1. Then $Q(x, y; t)$ is differentially transcendental in its three variables.*
(b) *Assume that the model of the walk belongs to the second, third or the fourth family in Figure 1. Then $Q(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$, and thus is differentially algebraic in its three variables.*

Proof. (a) This is [7, Corollary 2.2]; see also [9, Theorem 3.1].
(b) Consider the second family. We have $Q(x, 0; t) = Q(0, 0; t)$ and $K(x, 0; t) = K(0, 0; t)$. Then by Lemma 2.1,

$$K(x, y; t)Q(x, y; t) = K(0, y; t)Q(0, y; t) + xy. \tag{2.1}$$

Let us see that with the same arguments as for the walks in the half-plane, we deduce that $Q(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$. The idea is to locally write $K(\phi(y; t), y; t) = 0$. Evaluating at $(\phi(y; t), y; t)$ we then have for convenient y and t, $0 = K(0, y; t)Q(0, y; t) + \phi(y; t)y$, proving that $Q(0, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$. The functional equation (2.1) allows then to conclude that $Q(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$. As in the proof of Lemma 2.4, we may deduce that $Q(x, y; t)$ is differentially algebraic in its three variables. The reasoning for the fourth family is similar. For the third family, the walk has to stay at $(0, 0)$ and we have

$$Q(x, y; t) = \sum_{k=0}^{\infty} d_{0,0}^k t^k = \frac{1}{1 - d_{0,0}t}.$$

Therefore the result holds in that case too. □

2.4. **Group of the walk.** From now on, we may focus on the case where $\overline{E}_t$ is an elliptic curve. Recall that we have seen in Proposition 2.3, that $K(x, y; t)$ has degree two in x and y, and nonzero coefficient of degree 0 in x and y. Hence, $A_1(x), A_{-1}(x), B_1(y), B_{-1}(y)$ are not identically zero.

Following [4, Section 3], [17, Section 3] or [14], and using the notations introduced in Section 2.3, we consider the rational involutions given by

$$i_1([x_0 : x_1], [y_0 : y_1]) = \left(\frac{x_0}{x_1}, \frac{A_1(\frac{x_0}{x_1})}{A_{-1}(\frac{x_0}{x_1})\frac{y_0}{y_1}}\right) \quad \text{and} \quad i_2([x_0 : x_1], [y_0 : y_1]) = \left(\frac{B_{-1}(\frac{y_0}{y_1})}{B_1(\frac{y_0}{y_1})\frac{x_0}{x_1}}, \frac{y_0}{y_1}\right).$$

Note that we have $i_1([x_0/x_1 : 1], [y_0/y_1 : 1]) = i_1([x_0 : x_1], [y_0 : y_1])$ and the same holds for i_2. Note also that i_1, i_2 are a priori not defined when the denominators vanish but we will see in the sequel that we may overcome this problem when we will restrict to $\overline{E}_t$.

For a fixed value of x, there are at most two possible values of y such that $(x, y) \in \overline{E}_t$. The involution i_1 corresponds to interchanging these values. A similar interpretation can be given for i_2. Therefore the kernel curve $\overline{E}_t$ is left invariant by the natural action of i_1, i_2.

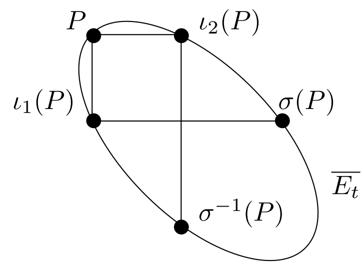


FIGURE 2. The maps i_1 and i_2 restricted to the kernel curve $\overline{E}_t$ are denoted by ι_1 and ι_2, respectively.

We denote by ι_1, ι_2 the restriction of i_1, i_2 to $\overline{E}_t$; see Figure 2. Again, these functions are a priori not defined where the denominators vanish. However, by [10, Proposition 4.1], this is only an "apparent problem". To be precise, the authors proved this for t transcendental over $\mathbb{Q}(d_{i,j})$ but the proof is still valid when $\overline{E}_t$ is an elliptic curve. We then obtain that ι_1 and ι_2 can be extended to morphisms of $\overline{E}_t$. We recall that a rational map f from $\overline{E}_t$ to $\overline{E}_t$ is a morphism if it is regular at any $P \in \overline{E}_t$, i.e. if f can be represented in suitable affine charts containing P and $f(P)$ by a rational function with nonvanishing denominator at P.

Let us finally define $\sigma = \iota_2 \circ \iota_1$. Note that such a map is called a QRT-map and has been widely studied; see [13].

Definition 2.7 (Group of the walk)**.** We call G the group generated by ι_1 and ι_2 and we call G_t the specialization of this group for a fixed value of $0 < t < 1$.

In the unweighted case, the algebraic nature of the generating series depends on whether σ has finite or infinite order. More precisely, G is finite if and only if the generating function is holonomic, i.e. satisfies a nontrivial linear differential equation with coefficients in $\mathbb{C}(x, y, t)$ in each of its three variables. On the other hand, when G is infinite, G_t can be either finite or infinite; see [15] for concrete examples. However, in that situation, the set of values of t such that G_t is finite is countable, see [8, Proposition 2.6].

2.5. **Uniformization of the curve.** In this section, we consider the uniformization problem in the genus one context, that has been solved in [14] for the case $t = 1$, and [11] for the case $0 < t < 1$. Let us consider a nondegenerate model of walk and assume that for all $t \in (0, 1)$, $\overline{E}_t$ is an elliptic curve. By Proposition 2.5, this corresponds to the situation where the step set is not included in any half-plane whose boundary passes through $(0, 0)$. By [11, Proposition 2.1], the elliptic curve $\overline{E}_t$ is biholomorphic to $\mathbb{C}/(\omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z})$ for some lattice $\omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z}$ of $\mathbb{C}$ via some $(\omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z})$-periodic holomorphic map Λ

$$\begin{array}{rccl} \Lambda : & \mathbb{C} & \rightarrow & \overline{E}_t \\ & \Lambda(\omega) & := & (x(\omega; t), y(\omega; t)), \end{array}$$

where x, y are rational functions of $\wp$ and its derivative $\partial_\omega \wp$, and $\wp$ is the Weierstrass function associated with the lattice $\omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z}$:

$$\wp(\omega; t) = \frac{1}{\omega^2} + \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(\omega + \ell_1\omega_1(t) + \ell_2\omega_2(t))^2} - \frac{1}{(\ell_1\omega_1(t) + \ell_2\omega_2(t))^2} \right).$$

Then, the field of meromorphic functions on $\overline{E}_t$ is isomorphic to the field of meromorphic functions on $\mathbb{C}/(\omega_1(t)\mathbb{Z}+\omega_2(t)\mathbb{Z})$, that is itself isomorphic to the field of meromorphic functions on $\mathbb{C}$ that are $(\omega_1(t),\omega_2(t))$-periodic. For $t\in(0,1)$ fixed, the latter field is equal to $\mathbb{C}(\wp,\partial_\omega\wp)$; see [23, Chapter 9, Theorem 1.8].

The maps ι_1, ι_2, and σ may be analytically lifted to the ω-plane $\mathbb{C}$ via the map Λ^{-1}. We denote these lifts by $\widetilde{\iota}_1$, $\widetilde{\iota}_2$, and $\widetilde{\sigma}$ respectively. So we have the commutative diagrams

$$\begin{array}{ccc} \overline{E}_t & \xrightarrow{\iota_k} & \overline{E}_t \\ \Lambda\uparrow & & \uparrow\Lambda \\ \mathbb{C} & \xrightarrow[\widetilde{\iota}_k]{} & \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} \overline{E}_t & \xrightarrow{\sigma} & \overline{E}_t \\ \Lambda\uparrow & & \uparrow\Lambda \\ \mathbb{C} & \xrightarrow[\widetilde{\sigma}]{} & \mathbb{C} \end{array}$$

For any $[x_0:x_1]$ in $\mathbb{P}^1(\mathbb{C})$, we denote by $\Delta_1([x_0:x_1];t)$ the discriminant of the degree two homogeneous polynomial given by $y\mapsto \overline{K}(x_0,x_1,y;t)$. Let us write

$$\Delta_1([x_0:x_1];t)=\sum_{i=0}^{4}\alpha_i(t)x_0^ix_1^{4-i}.$$

By [11, Theorem 1.11], the discriminant $\Delta_1([x_0:x_1];t)$ admits four distinct continuous real roots $a_1(t),\dots,a_4(t)$. They are numbered such that the cycle of $\mathbb{P}^1(\mathbb{R})$ starting from -1 to ∞ and from $-\infty$ to -1 crosses the a_i in the order $a_1(t),\dots,a_4(t)$. Furthermore, $[1:0]$ is a root if and only if $\alpha_4(t)=0$. In [11, Section 1.4], we see that $\alpha_4(t)=t^2(d_{1,0}^2-4d_{1,-1}d_{1,1})$. It follows that $[1:0]$ is a root of $\Delta_1([x_0:x_1];t)$ for one value of $t\in(0,1)$, if and only if $[1:0]$ is a root of $\Delta_1([x_0:x_1];t)$ for all $t\in(0,1)$.

Similarly, we denote by $b_1(t),\dots,b_4(t)$ the continuous real roots of the discriminant $x\mapsto \overline{K}(x,y_0,y_1;t)$, numbered in the same way, and we write $\Delta_2([y_0:y_1];t)=\sum_{i=0}^{4}\beta_i(t)y_0^iy_1^{4-i}$.

The following formulas have been proved

- in [14, Section 3.3] when $t=1$,
- in [22] in the unweighted case,
- in [11, Proposition 2.1 and (2.16)], in the weighted case.

Proposition 2.8 ([11], Proposition 2.1, Lemma 2.6, and (2.16))**.** *For $i=1,2$, let us set $D_i(\star;t):=\Delta_i([\star:1];t)$. An explicit expression of the periods is given by the elliptic integrals*

$$\omega_1(t)=\mathbf{i}\int_{a_3(t)}^{a_4(t)}\frac{dx}{\sqrt{|D_1(x;t)|}}\in\mathbf{i}\mathbb{R}_{>0}\quad\textit{and}\quad\omega_2(t)=\int_{a_4(t)}^{a_1(t)}\frac{dx}{\sqrt{D_1(x;t)}}\in\mathbb{R}_{>0}.$$

An explicit expression of the holomorphic map $\Lambda(\omega;t)=(x(\omega;t),y(\omega;t))$ is given by

- *If $a_4(t)\neq[1:0]$, then* $x(\omega;t)=\left[a_4(t)+\frac{D_1'(a_4(t);t)}{\wp(\omega;t)-\frac{1}{6}D_1''(a_4(t);t)}:1\right]$;
- *If $a_4(t)=[1:0]$, then* $x(\omega;t)=[\wp(\omega;t)-\alpha_2(t)/3:\alpha_3(t)]$;
- *If $b_4(t)\neq[1:0]$, then* $y(\omega;t)=\left[b_4(t)+\frac{D_2'(b_4(t);t)}{\wp(\omega-\omega_3(t)/2;t)-\frac{1}{6}D_2''(b_4(t);t)}:1\right]$;
- *If $b_4(t)=[1:0]$, then* $y(\omega;t)=[\wp(\omega-\omega_3(t)/2;t)-\beta_2(t)/3:\beta_3(t)]$.

An explicit expression of the involutions is given by

$$\widetilde{\iota}_1(\omega) = -\omega, \quad \widetilde{\iota}_2(\omega) = -\omega + \omega_3 \quad \textit{and} \quad \widetilde{\sigma}(\omega) = \omega + \omega_3,$$

where

$$\omega_3(t) = \int_{a_4(t)}^{X_\pm(b_4(t);t)} \frac{dx}{\sqrt{D_1(x;t)}} \in (0, \omega_2(t)), \tag{2.2}$$

and $X_\pm(y;t)$ *are the two roots of* $\overline{K}(X_\pm(y;t), y; t) = 0$.

2.6. **Meromorphic continuation of the generating function.** Let us summarize here the results of [11, Section 2.3]. Let us fix $t \in (0,1)$. The generating function $Q(x,y;t)$ converges for $|x|, |y| < 1$. The projection of this set inside $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ has a nonempty intersection with the kernel curve $\overline{E}_t$. In virtue of Lemma 2.1, we then find for $|x|, |y| < 1$ and $(x,y) \in \overline{E}_t$,

$$0 = K(x,0;t)Q(x,0;t) + K(0,y;t)Q(0,y;t) - K(0,0;t)Q(0,0;t) + xy.$$

To shorten several expressions hereafter, it is convenient to rewrite this equation introducing new auxiliary series F_1 and F_2:

$$0 = F_1(x;t) + F_2(y;t) - K(0,0;t)Q(0,0;t) + xy. \tag{2.3}$$

Since the series $F_1(x;t)$ and $F_2(y;t)$ converge for $|x|$ and $|y| < 1$ respectively, we then continue $F_1(x;t)$ for $(x,y) \in \overline{E}_t$ and $|y| < 1$ with the formula:

$$F_1(x;t) = -F_2(y;t) + K(0,0;t)Q(0,0;t) - xy.$$

We continue $F_2(y;t)$ for $(x,y) \in \overline{E}_t$ and $|x| < 1$ similarly. There exists a connected set $\mathcal{O} \subset \mathbb{C}$ such that

- $\Lambda(\mathcal{O}) = \{(x,y) \in \overline{E}_t \text{ such that } |x| < 1 \text{ or } |y| < 1\}$;
- $\widetilde{\sigma}^{-1}(\mathcal{O}) \cap \mathcal{O} \neq \varnothing$;
- $\bigcup_{\ell \in \mathbb{Z}} \widetilde{\sigma}^{\ell}(\mathcal{O}) = \mathbb{C}$.

There also exist meromorphic functions on $\mathcal{O}$, $r_x(\omega;t)$ and $r_y(\omega;t)$, such that $r_x(\omega;t) = F_1(x(\omega;t);t)$ and $r_y(\omega;t) = F_2(y(\omega;t);t)$.

Lemma 2.9 (Inclusion of poles)**.** *The set of poles of* $r_x(\omega;t)$ *inside* $\mathcal{O}$ *are contained in the set of poles of* $x(\omega;t)$ *with* $|y(\omega;t)| < 1$. *The set of poles of* $r_y(\omega;t)$ *inside* $\mathcal{O}$ *are contained in the set of poles of* $y(\omega;t)$ *with* $|x(\omega;t)| < 1$.

Proof. Let us use (2.3). On $\mathcal{O}$, we have

$$0 = r_x(\omega;t) + r_x(\omega;t) - K(0,0;t)Q(0,0;t) + x(\omega;t)y(\omega;t).$$

Let us focus on $r_x(\omega;t)$, the proof for $r_y(\omega;t)$ is similar. Recall that $F_1(x;t)$ has no poles for $|x| < 1$. Since $r_x(\omega;t) = F_1(x(\omega;t);t)$, we find that $r_x(\omega;t)$ has no poles when $|x(\omega;t)| < 1$. With $\Lambda(\mathcal{O}) = \{(x,y) \in \overline{E}_t | |x| < 1 \text{ or } |y| < 1\}$, we deduce that a pole of $r_x(\omega;t)$ inside $\mathcal{O}$ satisfies $|y(\omega;t)| < 1$. We use $r_y(\omega;t) = F_2(y(\omega;t);t)$, and the fact that $F_2(y;t)$ has no poles for $|y| < 1$ to deduce that $r_y(\omega;t)$ has no poles when $|y(\omega;t)| < 1$. Therefore, the poles of $r_x(\omega;t)$ inside $\mathcal{O}$ corresponds to the poles of $x(\omega;t)y(\omega;t)$ with $|y(\omega;t)| < 1$. The result follows. □

With $\bigcup_{\ell\in\mathbb{Z}} \widetilde{\sigma}^{\ell}(\mathcal{O}) = \mathbb{C}$ and $\widetilde{\sigma}^{-1}(\mathcal{O}) \cap \mathcal{O} \neq \varnothing$, we then extend $r_x(\omega;t)$ and $r_y(\omega;t)$ as meromorphic functions on $\mathbb{C}$ where they satisfy the functional equations

$$r_x(\omega + \omega_3(t);t) = r_x(\omega;t) + b_x(\omega;t), \tag{2.4}$$
$$r_x(\omega + \omega_1(t);t) = r_x(\omega;t), \tag{2.5}$$
$$r_y(\omega + \omega_3(t);t) = r_y(\omega;t) + b_y(\omega;t),$$
$$r_y(\omega + \omega_1(t);t) = r_y(\omega;t), \tag{2.6}$$

where $b_x(\omega;t) = y(-\omega;t)(x(\omega;t) - x(\omega+\omega_3(t);t))$ and $b_y(\omega;t) = x(\omega;t)(y(\omega;t) - y(-\omega;t))$.

From the functional equations (2.5) and (2.6), the set of poles of $\omega \mapsto r_x(\omega;t)$ and $\omega \mapsto r_y(\omega;t)$ are $\omega_1(t)$ periodic. With the other functional equations and $\bigcup_{\ell\in\mathbb{Z}} \widetilde{\sigma}^{\ell}(\mathcal{O}) = \mathbb{C}$, we may deduce the expression of a discrete set containing the poles of r_x and r_y.

Lemma 2.10. *Let $\mathcal{P}_x$ be the poles of r_x in $\mathcal{O}$ and $\mathcal{P}_{b,x}$ be the poles of b_x in $\mathbb{C}$. The set of poles of $\omega \mapsto r_x(\omega;t)$ is included in $(\mathcal{P}_x + \omega_3(t)\mathbb{Z}) \bigcup (\mathcal{P}_{b,x} + \omega_3(t)\mathbb{Z})$. A similar statement holds for $r_y(\omega;t)$.*

3. Preliminary results on differential algebraicity

In this section, we prove some results on differential algebraicity, and more specifically on ∂_t-algebraicity of the functions that appear in Section 2.

Let us begin by definitions. Let $f(x_1,\dots,x_n)$ be a multivalued Puiseux series. For $i = 1,\dots,n$, we say that f is ∂_{x_i}-algebraic if and only if it satisfies a nontrivial algebraic differential equation in the variable x_i, with coefficients in $\mathbb{Q}$. We say that f is differentially algebraic in all its variables (or differentially algebraic for short) if and only if for all $1 \leq i \leq n$, f is ∂_{x_i}-algebraic.

The following remark, proved e.g. in [18, Proposition 8, page 101], will be used several times in the sequel.

Remark 3.1. Let $f_1,\dots,f_n$ be differentially algebraic functions meromorphic on a common domain. A function satisfies a nontrivial algebraic differential equation with coefficients in $\mathbb{C}(f_1,\dots,f_n)$ if and only if it satisfies a nontrivial algebraic differential equation with coefficients in $\mathbb{Q}$.

The following lemma shows that the set of differentially algebraic functions is stable under many operations.

Lemma 3.2 (Closure properties)**.** *The set of differentially algebraic functions meromorphic on a domain is a field stable under derivations. If f and g are differentially algebraic and $f \circ g$ is well-defined then $f \circ g$ is differentially algebraic as well. If f is differentially algebraic and admits an inverse f^{-1}, then f^{-1} is also differentially algebraic.*

Proof. See [8, Lemma 6.4] for the inverse property in the univariate case. The proof extends straightforwardly to the multivariate case. The rest of the statements follows from [2, Corollary 6.4 and Proposition 6.5]. □

In what follows, we might also consider functions of t defined only on some intervals of $(0,1)$. Let $\mathfrak{D}$ be the field of multivalued functions that admit an expansion as convergent Puiseux series for all $t \in (0,1)$, and that are differentially algebraic. In the sequel, when we will say that a function of t defined (a priori) only of some intervals of $(0,1)$ is differentially algebraic, it will be implicit that it may be continued as an element of $\mathfrak{D}$.

The goal of the following results is to prove that various functions that appear in the uniformization of the elliptic curve are ∂_t-algebraic.

Lemma 3.3 ([2, Lemma 6.10])**.** *The functions $\omega_1(t), \omega_2(t), \omega_3(t)$ belong to $\mathfrak{D}$.*[1] *Moreover, they are analytic on a complex neighborhood of $(0,1)$.*

Proposition 3.4. *Functions of $\mathfrak{D}(\wp(\omega;t), \partial_\omega \wp(\omega;t))$ are differentially algebraic in t and ω.*

Proof. Since the differentially algebraic functions form a field stable under the derivations (see Lemma 3.2), it suffices to show that $\wp(\omega;t)$ is differentially algebraic. It is well known that for $t \in (0,1)$ fixed, $\wp(\omega;t)$ is ∂_ω-algebraic. More precisely, it satisfies an equation of the form $(\partial_\omega \wp)^2 = 4\wp^3 - g_2(t)\wp - g_3(t)$, where $g_2(t), g_3(t)$ are the invariants of the elliptic curve. Differentiating with respect of ω allows us to eliminate the invariants, and obtain $\partial_\omega^3 \wp(\omega;t) = 12\wp(\omega;t)\partial_\omega \wp(\omega;t)$; see [1, (18.6.5)]. Hence $\wp(\omega;t)$ is ∂_ω-algebraic.

Let us prove the ∂_t-algebraicity. In virtue of [2, Proposition 6.7], $\wp$ satisfies a nontrivial ∂_t-algebraic equation with coefficients in $\mathbb{C}(\omega_1(t), \omega_2(t))$. By Lemma 3.3, the periods $\omega_1(t)$ and $\omega_2(t)$ of $\wp$ are differentially algebraic, so in virtue of Remark 3.1, $\wp(\omega;t)$ is ∂_t-algebraic. $\square$

Remark 3.5. The same result holds with $\wp$ replaced by the Weierstrass function associated to the lattice $\omega_1(t)\mathbb{Z} + k\omega_2\mathbb{Z}$, or the lattice $\omega_1(t)\mathbb{Z} + \omega_3(t)\mathbb{Z}$.

Definition 3.6 (Principal part)**.** Let $f(\omega;t)$ be a meromorphic function at $\omega = a(t) \in \mathfrak{D}$, with Laurent series $f(\omega;t) = \sum_{\ell=\nu}^{\infty} a_\ell(t)(\omega - a(t))^\ell$. The principal part of f at $\omega = a(t)$ is the sum $\sum_{\ell=\nu}^{-1} a_\ell(t)(\omega - a(t))^\ell$ with the convention that it is 0 when $\nu \geq 0$. The coefficients of this principal part are $a_\nu(t), \dots, a_{-1}(t)$.

The following lemma will be used several times in the sequel.

Lemma 3.7. *The following statements hold:*

- *Let $d(t) \in \mathfrak{D}$ be an arbitrary function. We have $\wp(\omega;t) \in \mathfrak{D}((\omega + d(t)))$;*
- *$\wp(\omega;t) \in \omega^{-2}\mathfrak{D}[[\omega]]$;*
- *The coefficients of the principal parts of $\omega \mapsto \wp(\omega;t)$ belong to $\mathfrak{D}$.*

Proof. The last two assertions are straightforward consequences of the first one. Let us prove the first point. The function $d(t)$ and the poles of $\omega \mapsto \wp(\omega;t)$ are analytic on a convenient domain. So either $-d(t)$ is a pole of $\omega \mapsto \wp(\omega;t)$ with constant order with respect to t, or the set of t such that $-d(t)$ is a pole of $\omega \mapsto \wp(\omega;t)$ is discrete. It follows that the order of the pole of $\omega \mapsto \wp(\omega;t)$ at $-d(t)$ is constant except on a discrete set. Since for t fixed, $\omega \mapsto \wp(\omega;t)$ has pole of order at most two, we may write $\wp(\omega;t) = \sum_{\ell=k}^{\infty} c_\ell(t)(\omega + d(t))^\ell$.

[1] They even are solutions of linear differential equations.

Note that the coefficients $c_\ell(t)$ may have a pole when the order of the pole of $\omega \mapsto \wp(\omega;t)$ at $d(t)$ increases. In virtue of the field property of Lemma 3.2, combined with Proposition 3.4, we find that $(\omega+d(t))^{-k}\wp(\omega;t)$ is differentially algebraic. Note that $c_k(t)$ is the value at $-d(t)$ of the ∂_t-algebraic function $(\omega+d(t))^{-k}\wp(\omega;t)$. By the field property of Lemma 3.2 and Proposition 3.4, $(\omega+d(t))^{-k}\wp(\omega;t)$ is differentially algebraic in its two variables. By the composition property of Lemma 3.2 it follows that $c_k(t)\in\mathfrak{D}$. Let us fix $k\leq n$ and assume that for $\ell=k,\dots,n$, $c_\ell(t)\in\mathfrak{D}$. Let us show that $c_{n+1}(t)\in\mathfrak{D}$. This will prove the result by induction. Let us define $h_n(\omega;t)=\wp(\omega,t)-\sum_{\ell=k}^{n}c_\ell(t)(\omega+d(t))^\ell$. By Proposition 3.4, the field property of Lemma 3.2, and the induction hypothesis, the function $t\mapsto h_n(\omega;t)$ is differentially algebraic in its two variables. Note that $c_{n+1}(t)$ is the value at $-d(t)$ of $(\omega+d(t))^{-(n+1)}h_n(\omega;t)$. By the composition property of Lemma 3.2 it follows that $c_k(t)\in\mathfrak{D}$. □

As a consequence of what precedes, we deduce:

Corollary 3.8. *The following holds:*

- *The functions $x(\omega;t)$ and $y(\omega;t)$ are differentially algebraic in their two variables;*
- *For $d(t)\in\mathfrak{D}$, we have $x(\omega;t),y(\omega;t)\in\mathfrak{D}((\omega+d(t)))$;*
- *The poles and the coefficients of the principal parts of $\omega\mapsto x(\omega;t)$ and $\omega\mapsto y(\omega;t)$ belong to $\mathfrak{D}$.*

Proof. We use the expressions of $x(\omega;t)$ and $y(\omega;t)$ given in Proposition 2.8. The elements involved in the expression are meromorphic on some complex neighborhood of $(0,1)$ in the t-plane and are differentially algebraic by Proposition 3.4. Since the differentially algebraic elements form a field, see Lemma 3.2, the first point follows. Using Lemma 3.7, we deduce that $x(\omega;t),y(\omega;t)\in\mathfrak{D}((\omega+d(t)))$ for all $d(t)\in\mathfrak{D}$. Then the coefficients of the principal parts of $\omega\mapsto x(\omega;t)$ and $\omega\mapsto y(\omega;t)$ belong to $\mathfrak{D}$.

It remains to prove the differential algebraicity of the poles. Let $a(t)$ be a pole of $\omega\mapsto x(\omega;t)$ or $\omega\mapsto y(\omega;t)$. Then $a(t)$ is a continuous function solution of $\wp(a(t);t)=b(t)$, where $b(t)$ is ∂_t-algebraic.

Assume first that $\partial_\omega\wp(a(t);t)$ is identically zero or $a(t)$ is a pole of $\wp(\omega;t)$. By [23, page 270], this corresponds to the case where $a(t)\in\omega_1(t)\frac{\mathbb{Z}}{2}+\omega_2(t)\frac{\mathbb{Z}}{2}$. By Lemma 3.3, $a(t)$ is meromorphic on a complex neighborhood of $(0,1)$ and is ∂_t-algebraic. Then, it belongs to $\mathfrak{D}$.

Assume now that $\partial_\omega\wp(a(t);t)$ is not identically zero and $\wp(a(t);t)=b(t)$. Then, by the implicit function theorem, $a(t)$ admits an expansion as a meromorphic function on a complex neighborhood of any $t\in(0,1)$ with $\partial_\omega\wp(a(t);t)\neq 0$. On that domain $\wp$ is locally invertible and its inverse is differentially algebraic in its two variables by Lemma 3.2. So we may write $\wp^{-1}(b(t);t)=a(t)$, where $\wp^{-1}$ is the local inverse of $\wp$. With the composition and inverse properties of Lemma 3.2, we deduce that $a(t)$ is ∂_t-algebraic. Furthermore, by the implicit function theorem, it admits an expansion as a convergent series on a complex neighborhood of any $t\in(0,1)$. The set of t such that $\partial_\omega\wp(a(t);t)\neq 0$ being dense, we find that the differential equation holds everywhere. This concludes the proof. □

Recall, see Section 2.6, that

$b_x(\omega;t) = y(-\omega;t)(x(\omega;t) - x(\omega+\omega_3(t);t))$ and $b_y(\omega;t) = x(\omega;t)(y(\omega;t) - y(-\omega;t))$.

Corollary 3.9. *The following holds:*

- *The functions $b_x(\omega;t)$ and $b_y(\omega;t)$ are differentially algebraic in their two variables;*
- *For $d(t) \in \mathfrak{D}$, we have $b_x(\omega;t), b_y(\omega;t) \in \mathfrak{D}((\omega + d(t)))$;*
- *The poles and the coefficients of the principal parts of $\omega \mapsto b_x(\omega;t)$ and $\omega \mapsto b_y(\omega;t)$ belong to $\mathfrak{D}$.*

Proof. By Lemma 3.3, $\omega_3(t)$ belongs to $\mathfrak{D}$. This is now a straightforward application of Corollary 3.8, combined with the field property of Lemma 3.2. □

Toward the proof of Theorem 1.1, we are going to face to many situations where the series is known to be ∂_x-algebraic (or ∂_y-algebraic) for all fixed values t. More precisely the differential algebraicity of the series will be proved to be equivalent to the existence of functions that are for all t fixed, elliptic functions. Unfortunately, few things are known about the t-dependence of the coefficients. The following result will be the main ingredient in the proof of Theorem 1.1 since it gives a framework where we can state that the elliptic functions are differentially algebraic in all their variables.

Theorem 3.10. *Let $\omega \mapsto f(\omega;t)$ be a function such that:*

- *For all $t \in (0,1)$, $\omega \mapsto f(\omega;t) \in \mathbb{C}(\wp(\omega;t), \partial_\omega \wp(\omega;t))$.*
- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega \mapsto f(\omega;t)$.*
- *The coefficients of the principal parts of $\omega \mapsto f(\omega;t)$ are in $\mathfrak{D}$.*
- *There exists $a(t) \in \mathfrak{D}$ such that $f(a(t);t) \in \mathfrak{D}$.*

Then, $f(\omega;t)$ is differentially algebraic in its two variables.

Remark 3.11. At first sight, nothing is explicitly assumed on the t-dependence of $t \mapsto f(\omega;t)$. However, the assumptions on the poles, on the principal parts, and on the special value $f(a(t);t)$, will imply that $t \mapsto f(\omega;t)$ is analytic on a convenient domain.

Proof. If f is constant in the ω variable, then the result is clear. Assume that $f(\omega;t)$ is not constant. Let $a \in \mathbb{C}$. By the field property in Lemma 3.2, $f(\omega+a;t)$ satisfies the assumptions of Theorem 3.10. By the composition property, $f(\omega;t)$ is differentially algebraic in its two variables if and only if $f(\omega+a;t)$ is differentially algebraic in its two variables. Then, without loss of generality, we may reduce to the case where for any pole $b(t)$ of $\omega \mapsto f(\omega;t)$, $\partial_\omega \wp(b(t);t)$ is not identically zero. We may also assume that $a(t)$ is not a pole of $\omega \mapsto \wp(\omega;t)$.

Let us begin with the case where $\omega \mapsto f(\omega;t)$ is an even function. As we can see in the proof of [23, Lemma 1.9], we may write

$$f(\omega;t) = c(t) \prod_{i=1}^{\kappa_z} f_i(\omega;t) \prod_{j=1}^{\kappa_p} g_j(\omega;t),$$

where

- $c(t)$ is a function that does not depend upon ω;
- $f_i(\omega;t)$ are of the form $\wp(\omega;t) - \wp(a(t);t)$, where $a(t)$ are zeros of $\omega \mapsto f(\omega;t)$;
- $g_j(\omega;t)$ are of the form $(\wp(\omega;t) - \wp(b(t);t))^{-1}$, where $b(t)$ are poles of $\omega \mapsto f(\omega;t)$.

Then, a partial fraction decomposition yields a sum of the form

$$f(\omega;t)=\widetilde{c}(t)+\sum_{i=1}^{n_\infty}a_{i,\infty}(t)\wp(\omega;t)^i+\sum_j\sum_{i=1}^{n_j}\frac{a_{i,j}(t)}{(\wp(\omega;t)-\wp(b_j(t);t))^i}. \tag{3.1}$$

By assumption, the $b_j(t)$ are differentially algebraic. Recall, see Lemma 3.7, that for all j, we have $\wp(\omega;t)\in\mathfrak{D}((\omega+b_j(t)))$ (resp. $\wp(\omega;t)\in\omega^{-2}\mathfrak{D}[[\omega]]$). Then, for every i,j,

$$\frac{a_{i,j}(t)}{\Big(\wp(\omega;t)-\wp(b_j(t);t)\Big)^i}=\frac{a_{i,j}(t)}{\Big(\partial_\omega\wp(b_j(t);t)(\omega-b_j(t))\Big)^i}+O((\omega-b_j(t))^{-i+1}).$$

By the composition property of Lemma 3.2 and Proposition 3.4, for all k,ℓ the function $\Big(\partial_\omega^k\wp(b_j(t);t)\Big)^\ell$ is differentially algebraic. Let us write the Taylor expansion of the function

$$f(\omega;t)=\sum_{i=-n_j}^{\infty}\widetilde{a}_i(t)(\omega-b_j(t))^i.$$

Then, for $i<0$, one has

$$\widetilde{a}_i(t)=\frac{a_{i,j}(t)}{\partial_\omega\wp(b_j(t);t)^i}+f_{i,j},\ \text{ where } f_{i,j}\in\mathfrak{D}(a_{i+1,j}(t),\dots,a_{n_j,j}(t)).$$

Since the coefficients of the principal part at $b_j(t)$ are differentially algebraic we have $\widetilde{a}_i(t)\in\mathfrak{D}$. By Lemma 3.2, $\mathfrak{D}$ is a field, and we find by a decreasing induction that for all $1\leq i\leq n_j$, $a_{i,j}(t)\in\mathfrak{D}$. Similarly, for all i, we have

$$a_{i,\infty}(t)\wp(\omega;t)^i=\omega^{-2i}a_{i,\infty}(t)+O(\omega^{-2i+1}).$$

Then the coefficient of the term in ω^{-2i} with $i>0$ is of the form $a_{i,\infty}(t)+f_i$, where $f_i\in\mathfrak{D}(a_{i+1,\infty}(t),\dots,a_{n_\infty,\infty}(t))$. Since the coefficients of the principal part at 0 are differentially algebraic, we find $a_{i,\infty}(t)+f_i\in\mathfrak{D}$. By Lemma 3.2, $\mathfrak{D}$ is a field, and we find by a decreasing induction that for all $1\leq i\leq n_\infty$, $a_{i,\infty}(t)\in\mathfrak{D}$. Recall that by assumption, $f(a(t);t)$ is ∂_t-algebraic. By Lemma 3.2 and Proposition 3.4, we find

$$\widetilde{d}(t):=\sum_{i=1}^{n_\infty}a_{i,\infty}(t)\wp(a(t);t)^i+\sum_j\sum_{i=1}^{n_j}\frac{a_{i,j}(t)}{(\wp(a(t);t)-\wp(b_j(t);t))^i}\in\mathfrak{D}.$$

By the subtraction property of Lemma 3.2 we deduce that $\widetilde{c}(t)=f(a(t);t)-\widetilde{d}(t)$ is ∂_t-algebraic. In virtue of Lemma 3.2 and Proposition 3.4, every term in the right-hand side of (3.1) is differentially algebraic. With the field property of Lemma 3.2, this concludes the proof in the even case.

Assume that $\omega\mapsto f(\omega;t)$ is odd. The function $\partial_\omega\wp(\omega;t)^{-1}f(\omega;t)$ is even, and $\omega_1(t)\mathbb{Z}+\omega_2(t)\mathbb{Z}$, the poles of $\partial_\omega\wp(\omega;t)$, are ∂_t-algebraic; see Lemma 3.3. Then, we may apply the even case to deduce that $f(\omega;t)$ is of the form

$$\partial_\omega\wp(\omega;t)\widetilde{c}(t)+\sum_{i=1}^{n_\infty}a_{i,\infty}(t)\partial_\omega\wp(\omega;t)\wp(\omega;t)^i+\sum_j\sum_{i=1}^{n_j}\frac{a_{i,j}(t)\partial_\omega\wp(\omega;t)}{(\wp(\omega;t)-\wp(b_j(t);t))^i}.$$

Setting $a_{0,\infty}(t) := \widetilde{c}(t)$, we may rewrite the latter expression as

$$\sum_{i=0}^{n_\infty} a_{i,\infty}(t)\partial_\omega\wp(\omega;t)\wp(\omega;t)^i + \sum_j \sum_{i=1}^{n_j} \frac{a_{i,j}(t)\partial_\omega\wp(\omega;t)}{(\wp(\omega;t)-\wp(b_j(t);t))^i}.$$

By Proposition 3.4, for all j, we have $\partial_\omega\wp(\omega;t) \in \mathfrak{D}((\omega - b_j(t)))$ (resp. we have $\partial_\omega\wp(\omega;t) \in \mathfrak{D}((\omega)))$. The same reasoning as in the even case shows that for all $1 \leq i \leq n_j$, the functions $a_{i,j}(t)$ are differentially algebraic. Similarly, for all $0 \leq i \leq n_\infty$, the functions $a_{i,\infty}(t)$ are differentially algebraic. By Proposition 3.4 and Lemma 3.2, we find that $f(\omega;t)$ is differentially algebraic. This completes the proof in the odd case.

Let us consider the general case. Note that by Proposition 3.4, $\wp(\omega;t) - \wp(a(t);t)$ is differentially algebraic. So for all n, Lemma 3.2 ensures that $f(\omega;t)$ is differentially algebraic if and only if $(\wp(\omega;t) - \wp(a(t);t))^n f(\omega;t)$ is differentially algebraic. So without loss of generality, we may reduce to the case where $f(\pm a(t);t) = 0$. We write $f(\omega;t) = f_+(\omega;t) + f_-(\omega;t)$, where

$$f_+(\omega;t) := \frac{f(\omega;t) + f(-\omega;t)}{2},$$
$$f_-(\omega;t) := \frac{f(\omega;t) - f(-\omega;t)}{2}.$$

The poles of $\omega \mapsto f_\pm(\omega;t)$ are poles of f or opposite of the latter. By Lemma 3.2, they are ∂_t-algebraic and the coefficients of the principal parts are in $\mathfrak{D}$. Since $f(\pm a(t);t) = 0$ we find $f_\pm(a(t);t) = 0$. In particular it is differentially algebraic. From the even and the odd cases, $f_\pm(\omega;t)$ are differentially algebraic in their two variables. Since the sum of two differentially algebraic functions is differentially algebraic, see Lemma 3.2, we deduce that $f(\omega;t) = f_+(\omega;t) + f_-(\omega;t)$ is differentially algebraic. ☐

Remark 3.12.

- As in Remark 3.5, we may consider $\widetilde{\wp}(\omega;t)$, the Weierstrass functions associated to the lattice $\omega_1(t)\mathbb{Z} + \omega_3(t)\mathbb{Z}$, or the lattice $\omega_1(t)\mathbb{Z} + k\omega_2(t)\mathbb{Z}$, with $k \in \mathbb{N}^*$. Then, the proof of Theorem 3.10 can be straightforwardly adapted to this new lattice. We then deduce the following. If $\omega \mapsto f(\omega;t)$ is a function such that:
  (1) For all $t \in (0,1)$, $\omega \mapsto f(\omega;t) \in \mathbb{C}(\widetilde{\wp}(\omega;t), \partial_\omega\widetilde{\wp}(\omega;t))$.
  (2) There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega \mapsto f(\omega;t)$.
  (3) The coefficients of the principal parts of $\omega \mapsto f(\omega;t)$ are in $\mathfrak{D}$.
  (4) There exists $a(t) \in \mathfrak{D}$ such that $f(a(t);t) \in \mathfrak{D}$.

  Then, $f(\omega;t)$ is differentially algebraic in its two variables.
- Let us now just assume that $\omega \mapsto f(\omega;t)$ satisfies the above first three points and let $a(t) \in \mathfrak{D}$ that is not a pole. Then, $f(\omega;t) - f(a(t);t)$ satisfies the four points and is therefore differentially algebraic. By construction, the function $f(\omega;t) - f(a(t);t)$ has the same principal parts as $f(\omega;t)$.

Although r_x and r_y are not elliptic functions, we will see in the next section that it is sufficient to control the behavior of their poles and coefficients in order to apply Theorem 3.10.

Lemma 3.13. *The following holds:*

(A1) *The poles and coefficients of the principal parts of $\omega \mapsto r_x(\omega;t)$ belong to $\mathfrak{D}$.*
(A2) *There exists $a(t) \in \mathfrak{D}$ such that $r_x(a(t);t) \in \mathfrak{D}$.*

Similar statements hold for r_y.

Proof. Let us prove the result for r_x, the reasoning for r_y is similar. We refer to Section 2.6 for the notations used in this proof.

Recall that the series $Q(x,y;t)$ converges for $|x|,|y|,|t|<1$. Let us consider t in $(0,1)$. Take $\omega \in \mathcal{O}$ (note that $\mathcal{O}$ depends continuously on t), for each of the domains $|x(\omega;t)|<1$ and $|y(\omega;t)|<1$, one has the following equality of functions:

$$F_1(x(\omega;t);t) = r_x(\omega;t) \qquad \text{and} \qquad F_2(y(\omega;t);t) = r_y(\omega;t),$$

with no poles on these domains. Via the equality $0 = r_x(\omega;t)+r_y(\omega;t)-K(0,0;t)Q(0,0;t)+x(\omega;t)y(\omega;t)$, and Lemma 2.9 on the inclusion of poles, we deduce that the poles inside $\mathcal{O}$ of $\omega \mapsto r_x(\omega;t)$ are the poles inside $\mathcal{O}$ of $\omega \mapsto x(\omega;t)y(\omega;t)$ with $|y(\omega;t)|<1$. What is more, on that domain, $\omega \mapsto x(\omega;t)y(\omega;t)$ and $\omega \mapsto r_x(\omega;t)$ have the same principal parts. By Corollary 3.8, the poles of $\omega \mapsto r_x(\omega;t)$ inside $\mathcal{O}$ are differentially algebraic. Furthermore, the corresponding principal parts have differentially algebraic coefficients.

Recall, see (2.4), that $r_x(\omega+\omega_3(t);t) = r_x(\omega;t)+b_x(\omega;t)$. By Corollary 3.9, the poles and the coefficients of the principal parts of $\omega \mapsto b_x(\omega;t)$ are differentially algebraic. By Lemma 3.3, $\omega_3(t)$ is differentially algebraic. Recall that $\bigcup_{\ell\in\mathbb{Z}} \widetilde{\sigma}^{\ell}(\mathcal{O}) = \mathbb{C}$. With (2.4) and what precedes, we get assertion (A1).

It remains to prove assertion (A2). To lighten the notations we omit the dependence in t in what follows. Let us write $K(x,y;t) = \widetilde{B}_0(y)+x\widetilde{B}_1(y)+x^2\widetilde{B}_2(y)$. Let $\omega_0(t) \in \mathcal{O}$ such that

$$y(\omega_0) = 0 \qquad \text{and} \qquad x(\omega_0) = \frac{-\widetilde{B}_1(y(\omega_0)) + \sqrt{\widetilde{B}_1(y(\omega_0))^2 - 4\widetilde{B}_0(y(\omega_0))\widetilde{B}_2(y(\omega_0))}}{2\widetilde{B}_2(y(\omega_0))}.$$

The y-valuation of $\widetilde{B}_2(y)$ being at most two, we consider the following subcases.

- If it is 0 or 1, the valuation of the algebraic function $y \times \frac{-\widetilde{B}_1(y)+\sqrt{\widetilde{B}_1(y)^2-4\widetilde{B}_0(y)\widetilde{B}_2(y)}}{\widetilde{B}_2(y)}$ is nonnegative and then ω_0 is not a pole of $x(\omega;t)y(\omega;t)$.
- If it is 2, then $4\widetilde{B}_0(y)\widetilde{B}_2(y)$ converges to 0 when y goes to 0 and hence the same holds for $-\widetilde{B}_1(y)+\sqrt{\widetilde{B}_1(y)^2-4\widetilde{B}_0(y)\widetilde{B}_2(y)}$.

We further find that $y \times \left(-\widetilde{B}_1(y)+\sqrt{\widetilde{B}_1(y)^2-4\widetilde{B}_0(y)\widetilde{B}_2(y))}\right) \in O(y^2)$. In that case, we find that ω_0 is not a pole of $x(\omega;t)y(\omega;t)$ either. With $K(0,0;t)Q(0,0;t) = F_2(y(\omega_0;t);t) = r_y(\omega_0;t)$, and $0 = r_x(\omega;t)+r_y(\omega;t)-K(0,0;t)Q(0,0;t)+x(\omega;t)y(\omega;t)$, we then find $0 = r_x(\omega_0;t)+x(\omega_0;t)y(\omega_0;t)$. It then suffices to show that $x(\omega_0;t)y(\omega_0;t)$ is differentially algebraic. With the expression of $y(\omega_0;t)$ in Proposition 2.8, we find that ω_0 is solution of an equation of the form $\wp(\omega_0;t) = b(t)$ with $b(t) \in \mathfrak{D}$. With the same reasons as in the proof of Corollary 3.8, we find that ω_0 is differentially algebraic, and $x(\omega_0;t), y(\omega_0;t) \in \mathfrak{D}$. Then $r_x(\omega_0(t);t) \in \mathfrak{D}$. This concludes the proof. □

The following result relates the differential transcendence of $Q(x,y;t)$ and the differential transcendence of $r_x(\omega;t)$ and $r_y(\omega;t)$.

Proposition 3.14. *The following statements are equivalent.*

- *The generating function $Q(x,y;t)$ is differentially algebraic in its three variables.*
- *The series $F_1(x;t)$ and $F_2(y;t)$ are differentially algebraic in their two variables.*
- *The meromorphic continuations $r_x(\omega;t)$ and $r_y(\omega;t)$ are differentially algebraic in their two variables.*

Proof. If $Q(x,y;t)$ is differentially algebraic then $Q(x,0;t)$ is differentially algebraic. Since $K(x,0;t)$ is differentially algebraic, we use the ring property of Lemma 3.2 to deduce that $F_1(x;t) = K(x,0;t)Q(x,0;t)$ is differentially algebraic. (The reasoning is similar for the differential algebraicity of $F_2(y;t)$). Conversely, if $F_1(x;t)$ and $F_2(y;t)$ are differentially algebraic, then, by evaluation, so is $Q(0,0;t)$. As the right-hand side of the expression in Lemma 2.1 is a sum and product of elements that are differentially algebraic, it is differentially algebraic (by the field property in Lemma 3.2). Therefore, $K(x,y;t)Q(x,y;t)$ is differentially algebraic. Thus, $Q(x,y;t)$ is differentially algebraic. So the first two points are equivalent.

Assume that the series $F_1(x;t)$ is differentially algebraic in its two variables. Recall that $F_1(x(\omega;t);t) = r_x(\omega;t)$ where $x(\omega;t)$ is differentially algebraic; see Corollary 3.8. By composition of differentially algebraic functions, see Lemma 3.2, $r_x(\omega;t)$ is differentially algebraic. Conversely, on a domain where $x(\omega;t)$ is invertible, its inverse is also differentially algebraic; see Lemma 3.2. We conclude similarly that if $r_x(\omega;t)$ is differentially algebraic then $F_1(x;t)$ is differentially algebraic. A similar reasoning holds for the y variable and we find that $F_2(y;t)$ is differentially algebraic if and only if $r_y(\omega;t)$ is differentially algebraic. This proves the equivalence between the last two points. □

4. Differential algebraicity of the generating function

The goal of this section is to prove Theorem 1.1 (the ∂_x, ∂_y, and ∂_t differential algebraicity are equivalent). By Lemma 2.4, the result holds for all degenerate cases. By Lemma 2.6 and Proposition 2.5, it also holds when $\overline{E}_t$ *is not* an elliptic curve. So we now prove the case where $\overline{E}_t$ is an elliptic curve. Let G be the group of the walk (see Definition 2.7). Our proof handles separately the cases $|G| < \infty$ and $|G| = \infty$.

4.1. **Finite group case.**

Proposition 4.1. *Let us consider a nondegenerate model of walks, assume that $\overline{E}_t$ is an elliptic curve and $|G| < \infty$. Then, $Q(x,y;t)$ is ∂_x-algebraic, ∂_y-algebraic and ∂_t-algebraic.*

Proof. By Proposition 3.14, it suffices to show that $r_x(\omega;t)$ and $r_y(\omega;t)$ are differentially algebraic in their two variables. Let us only consider $r_x(\omega;t)$, the proof for $r_y(\omega;t)$ is similar. Recall that the $\omega_i(t)$ are continuous and that $\omega_3(t) \in (0, \omega_2(t))$ (see Equation (2.2)). Since $|G| < \infty$ and $\widetilde{\sigma}(\omega) = \omega + \omega_3(t)$, there exist $k, \ell \in \mathbb{N}^*$ with $\gcd(k,\ell) = 1$ such that $\omega_3(t)/\omega_2(t) = k/\ell$. By (2.4), we have $r_x(\omega + \omega_3(t);t) = r_x(\omega;t) + b_x(\omega;t)$, where $b_x(\omega;t) = y(-\omega;t)(x(\omega;t) - x(\omega + \omega_3(t);t))$. Let us recall some notations borrowed from the proof of [11, Theorem 4.1]. It is shown that we may write a decomposition of the form

$$r_x(\omega;t) = \psi(\omega;t) + \Phi(\omega;t)\phi(\omega;t). \tag{4.1}$$

More precisely,

- $\Phi(\omega;t) = \sum_{i=0}^{\ell-1} b_x(\omega + i\omega_3(t);t)$;
- $\phi(\omega;t) = \frac{\omega_1(t)}{2\mathbf{i}\pi}\zeta(\omega;t) - \frac{\omega}{\mathbf{i}\pi}\zeta(\omega_1(t)/2;t)$, where ζ is an opposite of the antiderivative of the Weierstrass function with periods $\omega_1(t)$ and $k\omega_2(t)$, that is

$$\zeta(\omega;t) = \frac{1}{\omega} + \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{\omega + \ell_1\omega_1(t) + \ell_2 k\omega_2(t)} - \frac{1}{\ell_1\omega_1(t) + \ell_2 k\omega_2(t)} + \frac{\omega}{(\ell_1\omega_1(t) + \ell_2 k\omega_2(t))^2} \right);$$

- for all $t \in (0,1)$, the function $\omega \mapsto \psi(\omega;t)$ is $(\omega_1(t), k\omega_2(t))$-periodic.

The idea is to prove successively that $\Phi(\omega;t)$, $\phi(\omega;t)$ and $\psi(\omega;t)$ are differentially algebraic. We will also see them as functions of ω and study their poles and principal parts.

Step 1: Study of $\Phi(\omega;t)$.

Lemma 4.2. *The following holds:*

- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega \mapsto \Phi(\omega;t)$.*
- *The coefficients of the principal parts of $\omega \mapsto \Phi(\omega;t)$ are in $\mathfrak{D}$.*
- *Φ is differentially algebraic in its two variables.*

Proof. Recall, see Lemma 3.3, that $\omega_3(t) \in \mathfrak{D}$. We may combine Corollary 3.9 and Lemma 3.2, to deduce that the poles and the coefficients of the principal parts of $\omega \mapsto \Phi(\omega;t)$ are ∂_t-algebraic. Furthermore by Lemma 3.2 and Proposition 3.4, Φ is differentially algebraic in its two variables. $\square$

Step 2: Study of $\phi(\omega;t)$.

Before proving that $\phi(\omega;t)$ is differentially algebraic, let us study $\zeta(\omega;t)$.

Lemma 4.3. *The following holds:*

- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega \mapsto \zeta(\omega;t)$.*
- *The coefficients of the principal parts of $\omega \mapsto \zeta(\omega;t)$ are in $\mathfrak{D}$.*
- *ζ is differentially algebraic in its two variables.*

Proof. In virtue of Lemma 3.3, the periods $\omega_1(t), \omega_2(t)$ are differentially algebraic. Then, the poles and the coefficients of the principal parts of $\omega \mapsto \zeta(\omega;t)$ belong to $\mathfrak{D}$.

Let $\widetilde{\wp}$ be the Weierstrass function with periods $\omega_1(t), k\omega_2(t)$ and let us write the classical differential equation

$$\partial_\omega\widetilde{\wp}(\omega;t)^2 = 4\widetilde{\wp}(\omega;t)^3 - \widetilde{g_2}(t)\widetilde{\wp}(\omega;t) - \widetilde{g_3}(t). \tag{4.2}$$

By Remark 3.5, $\widetilde{\wp}(\omega;t) = -\partial_\omega\zeta(\omega;t)$ is differentially algebraic in its two variables. Then, $\zeta(\omega;t)$ is ∂_ω-algebraic too. Let us prove the ∂_t-algebraicity. Let us differentiate (4.2) with respect to ∂_ω and simplify by $\partial_\omega\widetilde{\wp}(\omega;t)$, to find

$$2\partial_\omega^2\widetilde{\wp}(\omega;t) = 12\widetilde{\wp}(\omega;t)^2 - \widetilde{g_2}(t).$$

By Lemma 3.2, for all $k \geq 0$, the derivatives $\partial_\omega^k \widetilde{\wp}(\omega;t)$ are ∂_t-algebraic. Since the ∂_t-algebraic functions form a ring, see Lemma 3.2, we deduce that $\widetilde{g_2}(t)$ is ∂_t-algebraic. Using again the ring property of Lemma 3.2 in (4.2), we deduce that $\widetilde{g_3}(t)$ is ∂_t-algebraic too. We may see the elliptic functions as functions of ω and $\widetilde{g_2}, \widetilde{g_3}$. By [1, (18.6.19)],

$$(\widetilde{g_2}^3 - 27\widetilde{g_3}^2)\partial_{\widetilde{g_3}}\widetilde{\wp} = (3\widetilde{g_2}\zeta - \frac{9}{2}\widetilde{g_3}\omega)\partial_\omega \widetilde{\wp} + 6\widetilde{g_2}\widetilde{\wp}^2 - 9\widetilde{g_3}\widetilde{\wp} - \widetilde{g_2}^2. \tag{4.3}$$

We have $\partial_t \widetilde{\wp} = \partial_t \widetilde{g_3} \partial_{\widetilde{g_3}} \widetilde{\wp}$. If $\partial_t \widetilde{g_3} = 0$ then $\widetilde{\wp}$ does not depend on t. In particular, its poles are independent of t, which implies that the periods $\omega_1(t)$ and $k\omega_2(t)$ are independent of t. Then, $\zeta(\omega;t)$ is independent of t and therefore ∂_t-algebraic. We similarly deal with the case $\partial_t \widetilde{g_2} = 0$. So let us consider the situation where both functions $\partial_t \widetilde{g_2}$ and $\partial_t \widetilde{g_3}$ are not identically zero. By the derivation property of Lemma 3.2, we deduce that $\partial_t \widetilde{\wp}, \partial_t \widetilde{g_3}$ are ∂_t-algebraic. Since the ∂_t-algebraic functions form a field, see Lemma 3.2, we then find that $\partial_{\widetilde{g_3}} \widetilde{\wp} = \partial_t \widetilde{\wp} / \partial_t \widetilde{g_3}$ is ∂_t-algebraic. Since $\partial_t \widetilde{g_2} \neq 0$, we are in the situation where $\widetilde{g_2}$ is not identically zero. With (4.3), and the field property of Lemma 3.2, we deduce that $\zeta(\omega;t)$ is ∂_t-algebraic. This completes the proof of the lemma. □

Lemma 4.4. *The following holds:*

- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega \mapsto \phi(\omega;t)$.*
- *The coefficients of the principal parts of $\omega \mapsto \phi(\omega;t)$ are in $\mathfrak{D}$.*
- *ϕ is differentially algebraic in its two variables.*

Proof. In virtue of Lemma 3.3, the period $\omega_1(t)$ is differentially algebraic. By Lemma 3.2, and Lemma 4.3, we find that $\phi(\omega;t)$ is differentially algebraic in its two variables. Furthermore, the poles and the coefficients of the principal parts of $\omega \mapsto \phi(\omega;t)$ belong to $\mathfrak{D}$. □

Step 3: Study of $\psi(\omega;t)$.

Let us now study $\psi(\omega;t)$. By Lemma 3.13 there exists $a(t) \in \mathfrak{D}$ such that $r_x(a(t);t) \in \mathfrak{D}$. Furthermore, the poles and the coefficients of the principal parts of $\omega \mapsto r_x(\omega;t)$ are ∂_t-algebraic.

With (4.1), Lemma 4.2 and Lemma 4.4, we deduce that the poles of $\omega \mapsto \psi(\omega;t)$ are ∂_t-algebraic, and the coefficients of the principal parts are ∂_t-algebraic. Recall that for all t, $\omega \mapsto \psi(\omega;t)$ is $(\omega_1(t), k\omega_2(t))$-periodic. By Remark 3.12, we may build $\omega \mapsto \psi_0(\omega;t)$, that is differentially algebraic and $(\omega_1(t), k\omega_2(t))$-periodic, with same principal parts as $\omega \mapsto \psi(\omega;t)$. We have

$$r_x(\omega;t) = \psi(\omega;t) - \psi_0(\omega;t) + \Phi(\omega;t)\phi(\omega;t) + \psi_0(\omega;t). \tag{4.4}$$

Note that by construction $\omega \mapsto \psi(\omega;t) - \psi_0(\omega;t)$ has no poles. Since $\omega \mapsto r_x(\omega;t)$ has no poles at $a(t)$, we deduce with (4.4), that $\omega \mapsto \Phi(\omega;t)\phi(\omega;t) + \psi_0(\omega;t)$ has no poles at $a(t)$. Since $\Phi(\omega;t)\phi(\omega;t) + \psi_0(\omega;t)$ is differentially algebraic (as the sum of two differentially algebraic functions, see Lemma 3.2), with no poles at $a(t)$, we find that its evaluation at $a(t)$ is differentially algebraic. Since $r_x(a(t);t) \in \mathfrak{D}$ we use the ring property in Lemma 3.2 to deduce that $\psi(a(t);t) - \psi_0(a(t);t)$ is differentially algebraic.

Then, $\psi(\omega;t)-\psi_0(\omega;t)$ satisfies the assumptions of Theorem 3.10 (with $\omega_2(t)$ replaced by $k\omega_2(t)$) and we deduce that it is differentially algebraic by Remark 3.12. By Lemma 3.2, and the differential algebraicity of $\psi_0(\omega;t)$, we deduce that $\psi(\omega;t)$ is differentially algebraic.

Step 4: Study of $r_x(\omega;t)$.

Let us now finish the proof of Proposition 4.1. Since $\psi(\omega;t)$, $\Phi(\omega;t)$, and $\phi(\omega;t)$ are differentially algebraic in their two variables, we conclude that $r_x(\omega;t)=\psi(\omega;t)+\Phi(\omega;t)\phi(\omega;t)$ is differentially algebraic as the sum and product of differentially algebraic functions; see Lemma 3.2. □

4.2. **Infinite group case.** It now remains to handle the case where the group has infinite order. So let us consider a nondegenerate model of walks and assume that $\overline{E}_t$ is an elliptic curve and $|G|=\infty$. The equivalence between the ∂_x-algebraicity and the ∂_y-algebraicity can be straightforwardly deduced in this weighted context from the proof of [8, Proposition 3.10]. Let us see that the ∂_t-algebraicity implies the ∂_x-algebraicity. If $Q(x,y;t)$ is ∂_t-algebraic, then $Q(x,0;t)$ is ∂_t-algebraic. By [7, Theorem 3.12], if $Q(x,0;t)$ is ∂_t-algebraic, then it is ∂_x-algebraic. In virtue of Lemmas 2.1 and 3.2, we find that if $Q(x,0;t)$ is ∂_x-algebraic, then $Q(x,y;t)$ is ∂_x-algebraic. So to prove Theorem 1.1, it now suffices to show the following result.

Theorem 4.5. *Let us consider a nondegenerate model of walks, assume that $\overline{E}_t$ is an elliptic curve and $|G|=\infty$. If $Q(x,y;t)$ is ∂_x-algebraic, then it is ∂_t-algebraic.*

Proof. By Proposition 3.14, it suffices to show that $r_x(\omega;t)$ and $r_y(\omega;t)$ are differentially algebraic. Let us consider $r_x(\omega;t)$, the proof for $r_y(\omega;t)$ is similar. By Proposition 2.5, for all $t\in(0,1)$ fixed, $\overline{E}_t$ is an elliptic curve. Let G_t be the group G specialized at t. The order of the group G_t may depend upon t. However by [8, Proposition 2.6], see also [19, Proposition 14], which can be straightforwardly extended in the weighted framework, the set of $t\in(0,1)$ such that G_t has infinite order is dense. By assumption, for such t fixed, $x\mapsto F_1(x;t)$ is ∂_x-algebraic. By [16, Theorem 3.8], for all such t fixed there exists a $(\omega_1(t),\omega_2(t))$-periodic function $\widetilde{g}(\omega;t)$, such that

$$b_x(\omega;t)=\widetilde{g}(\omega+\omega_3(t);t)-\widetilde{g}(\omega;t). \tag{4.5}$$

By [16, Proposition 3.9], there exist $g(x;t)\in\mathbb{C}(x,t)$ and $h(y;t)\in\mathbb{C}(y,t)$ such that $g(x(\omega;t);t)=\widetilde{g}(\omega;t)$ and for all $(x,y)\in\overline{E}_t$,

$$xy=g(x;t)+h(y;t).$$

Since $g(x(\omega;t);t)=\widetilde{g}(\omega;t)$, we use Corollary 3.8 to deduce that we may continue $\widetilde{g}(\omega;t)$ in the t variable.

Step 1: Study of $\widetilde{g}(\omega;t)$.

Lemma 4.6. *The following holds:*

- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega\mapsto\widetilde{g}(\omega;t)$.*
- *The coefficients of the principal parts of $\omega\mapsto\widetilde{g}(\omega;t)$ are in $\mathfrak{D}$.*
- *$\widetilde{g}$ is differentially algebraic in its two variables.*

Proof. We claim that the poles of $\omega \mapsto \widetilde{g}(\omega;t)$ are of the form $\omega_0(t) + \ell\omega_3(t)$, where $\omega_0(t)$ is a pole of $\omega \mapsto b_x(\omega;t)$ and $\ell \in \mathbb{Z}$. To the contrary, assume that $a(t)$ is a pole that is not of this form. Then $a(t) - \omega_3(t)$ is a pole of $\omega \mapsto \widetilde{g}(\omega + \omega_3(t);t)$, that is not a pole of $\omega \mapsto b_x(\omega;t)$. With (4.5), we find that $a(t) - \omega_3(t)$ is a pole of $\omega \mapsto \widetilde{g}(\omega;t)$. We prove successively that for all $\ell \geq 0$, $a(t) - \ell\omega_3(t)$ is a pole of $\omega \mapsto \widetilde{g}(\omega;t)$. Since $\widetilde{g}(\omega;t)$ is $(\omega_1(t), \omega_2(t))$-periodic, $a(t) - \omega_3(t)\mathbb{N} + \omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z}$ are poles of $\omega \mapsto \widetilde{g}(\omega;t)$. Since $|G| = \infty$ and $\widetilde{\sigma}(\omega) = \omega + \omega_3(t)$, the sets $A_\ell := \{a(t) - \ell\omega_3(t) + \omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z}\}$, with $\ell \in \mathbb{N}$, are disjoint. Then, the set of poles of $\omega \mapsto \widetilde{g}(\omega;t)$ possesses an accumulation point which contradicts that the function is meromorphic. This proves the claim.

By Corollary 3.9, the poles of $\omega \mapsto b_x(\omega;t)$ are ∂_t-algebraic. By Lemma 3.3, $\omega_3(t)$ is ∂_t-algebraic too. With the claim, it follows that the poles of $\omega \mapsto \widetilde{g}(\omega;t)$ are ∂_t-algebraic. By Corollary 3.8, the coefficients of the principal parts of $\omega \mapsto x(\omega;t)$ are ∂_t-algebraic. With $g(x(\omega;t);t) = \widetilde{g}(\omega;t)$, and $g(x;t) \in \mathbb{C}(x,t)$, we deduce that the coefficients of the principal parts of $\omega \mapsto \widetilde{g}(\omega;t)$ are ∂_t-algebraic. Finally $\widetilde{g}(\omega;t)$ is differentially algebraic, as the composition of differentially algebraic functions; see Lemma 3.2. □

Step 2: Study of $\widetilde{f}(\omega;t) := r_x(\omega;t) - \widetilde{g}(\omega;t)$.

By (2.4) and (4.5), we find

$$\begin{aligned}\widetilde{f}(\omega + \omega_3(t);t) &= r_x(\omega + \omega_3(t);t) - \widetilde{g}(\omega + \omega_3(t);t) \\ &= r_x(\omega;t) + b_x(\omega;t) - (\widetilde{g}(\omega;t) + b_x(\omega;t)) = \widetilde{f}(\omega;t).\end{aligned}$$

Then, $\widetilde{f}(\omega;t)$ is $\omega_3(t)$-periodic. Recall that $\widetilde{g}(\omega;t)$ is $\omega_1(t)$-periodic. By (2.5), $r_x(\omega;t)$ is also $\omega_1(t)$-periodic. Therefore, $\omega \mapsto \widetilde{f}(\omega;t)$ is elliptic with periods $(\omega_1(t), \omega_3(t))$. Recall that the poles and the coefficients of the principal parts of $\omega \mapsto \widetilde{g}(\omega;t)$ are ∂_t-algebraic. By Lemma 3.13, the same holds for $r_x(\omega;t)$ and there exists $a(t) \in \mathfrak{D}$ such that $r_x(a(t);t)$ is differentially algebraic. By Remark 3.12, we may build $\omega \mapsto \widetilde{f}_0(\omega;t)$, that is differentially algebraic, $(\omega_1(t), \omega_3(t))$-periodic, and with same principal parts as $\omega \mapsto \widetilde{f}(\omega;t)$. Let us write

$$\widetilde{f}(\omega;t) - \widetilde{f}_0(\omega;t) = r_x(\omega;t) - \widetilde{g}(\omega;t) - \widetilde{f}_0(\omega;t).$$

The function $-\widetilde{g}(\omega;t) - \widetilde{f}_0(\omega;t)$ is differentially algebraic, as the sum of two differentially algebraic functions; see Lemma 3.2. Since $\widetilde{f}(\omega;t) - \widetilde{f}_0(\omega;t)$ has no poles and $a(t)$ is not a pole of $\omega \mapsto r_x(\omega;t)$, we deduce that $a(t)$ is not a pole of $\omega \mapsto -\widetilde{g}(\omega;t) - \widetilde{f}_0(\omega;t)$. Therefore, its evaluation at $a(t)$ is still differentially algebraic. Then, the same holds for $\omega \mapsto \widetilde{f}(\omega;t) - \widetilde{f}_0(\omega;t)$, which satisfies the assumptions of Theorem 3.10 (with $\omega_2(t)$ replaced with $\omega_3(t)$) and we deduce that it is differentially algebraic by Remark 3.12. Hence, $r_x(\omega;t) = \left(\widetilde{f}(\omega;t) - \widetilde{f}_0(\omega;t)\right) + \widetilde{g}(\omega;t) + \widetilde{f}_0(\omega;t)$ is differentially algebraic as the sum of differentially algebraic functions, see Lemma 3.2. This concludes the proof. □

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THE GENERATING FUNCTION OF THE SURVIVAL PROBABILITIES IN A CONE IS NOT RATIONAL

RODOLPHE GARBIT[1] AND KILIAN RASCHEL[2]

[1]Université d'Angers, France; https://www.math.univ-angers.fr/~garbit/
[2]CNRS, Université d'Angers, France; http://www.lmpt.univ-tours.fr/~raschel/

Abstract. We look at multidimensional random walks $(S_n)_{n\geqslant 0}$ in convex cones, and address the question of whether two naturally associated generating functions may define rational functions. The first series is the one of the survival probabilities $\mathbb{P}(\tau > n)$, where τ is the first exit time from a given cone; the second series is that of the excursion probabilities $\mathbb{P}(\tau > n, S_n = y)$. Our motivation to consider this question is twofold: first, it goes along with a global effort of the combinatorial community to classify the algebraic nature of the series counting random walks in cones; second, rationality questions of the generating functions are strongly associated with the asymptotic behaviors of the above probabilities, which have their own interest. Using well-known relations between rationality of a series and possible asymptotics of its coefficients, recent probabilistic estimates immediately imply that the excursion generating function is not rational. Regarding the survival probabilities generating function, we propose a short and self-contained proof that it cannot be rational neither.

Keywords: random walks in cones, survival probabilities, generating functions, rational functions, Laplace transform, univariate singularity analysis.

1. Introduction

Main result and our approach. For a d-dimensional random walk $(S_n)_{n\geqslant 0}$ with integrable and independent increments $X_n = S_n - S_{n-1}$ having common distribution μ, we consider the generating function

$$F(t) = \sum_{n\geqslant 0} a_n t^n = \sum_{n\geqslant 0} \mathbb{P}^x(\tau > n) t^n, \tag{1.1}$$

where $\mathbb{P}^x$ is a probability distribution under which the random walk starts at $S_0 = x$, and τ denotes the first exit time from a given cone K, i.e.,

$$\tau = \inf\{n > 0 : S_n \notin K\}.$$

See (1.7) for an explicit computation of (1.1) in a simple one-dimensional example. Our first main result can be stated as follows:

Theorem 1.1. *If the drift $m = \mathbb{E}X_1$ is not interior to the cone K, and if four further assumptions (A1)–(A4) (to be introduced below) are satisfied, then the generating function $F(t)$ in (1.1) is not a rational function.*

Let us emphasize that in Theorem 1.1, the distribution of X_1 is not assumed to be a discrete measure. This theorem covers, amongst other cases, the class of walks with small steps in orthants (with arbitrary weights on the steps), a model that has attracted a lot of attention from the combinatorial community, but is much more general.

The non-rationality of the generating function (1.1) is based on the fact that the numbers a_n don't have an asymptotic behavior that is compatible with the Taylor coefficients of a rational function. More precisely, we identify in Theorem 1.3 a rate $\rho \in (0,1]$ such that

$$a_n = \rho^n B_n, \tag{1.2}$$

with B_n satisfying

$$\text{(i)}\ \sqrt[n]{B_n} \to 1,$$

$$\text{(ii)}\ B_n \to 0.$$

Using then classical analytic combinatorics techniques (see in particular Theorem 4.1 and Lemma 4.2), one will directly deduce that the generating function (1.1) cannot be rational.

In this paper, we provide proofs of estimates (i) and (ii) which are self-contained, and as simple and elementary as possible. Only item (ii) is new. Item (i) (in particular the value of the rate ρ in (1.2)) is already obtained in [11], but for the reader's convenience, we shall give here a detailed proof in a simplified setting, that covers the cases which are relevant to combinatorialists. In a special case (when the drift is, in some sense, directed towards the vertex of the cone), the precise asymptotics of the survival probability (hence in particular items (i) and (ii)) is derived in [8].

Drift inside of the cone. In case of a drift $m = \mathbb{E}X_1$ interior to the cone, the probabilistic behavior is rather constrained as we have $\mathbb{P}^x(\tau > n) \to \mathbb{P}^x(\tau = \infty) > 0$. The positivity of the escape probability is intuitively clear, based on the law of large numbers and the fluctuations of the random walk; see Lemma 3.1 for a precise statement. Equivalently, in the neighborhood of $t = 1$, one has

$$F(t) \sim \frac{\mathbb{P}^x(\tau = \infty)}{1-t},$$

which contains no contradiction with F being a rational function. However, for one-dimensional walks with bounded jumps, it is proved in [1, Thm 4] that $\mathbb{P}^x(\tau > n) = \mathbb{P}^x(\tau = \infty) + \frac{c\rho^n}{n^{3/2}} + \cdots$, with $\rho \in (0,1)$, which is not compatible with F being rational.

One of the simplest examples for which the rationality of F in (1.1) was an open question before the present paper is the following: in the quarter plane $K = \mathbb{N}^2$, take a uniform distribution μ on $\{(1,0),(0,-1),(-1,0),(0,1),(1,1)\}$. Here, we answer this question and, more generally, solve the problem for the orthant $K = [0,\infty)^d$ and any (weighted) small step walk, i.e., random walk with increments X_k that belong to $\{-1,0,1\}^d$ almost surely. If $\mathbb{P}(X_k \in K) = 1$, then the random walk is *trapped* forever in K and $a_n = \mathbb{P}^x(\tau > n) = 1$ for all n, so that $F(t) = \frac{1}{1-t}$ is a rational function. Let us say the walk is *not trapped* if $\mathbb{P}(X_k \notin K) > 0$. Our second main result is the following:

Theorem 1.2. *For all d-dimensional weighted small step walks with a drift interior to the orthant $K = [0,\infty)^d$, not trapped and satisfying (A2), the generating function $F(t)$ in (1.1) is not rational.*

Here again, the non-rationality of the generating $F(t)$ is obtained as a consequence of estimates on $a_n = \mathbb{P}^x(\tau > n)$. More precisely, in Theorem 1.4, we prove that

$$\mathbb{P}^x(\tau > n) = \mathbb{P}^x(\tau = \infty) + \Theta(\rho^n B_n), \tag{1.3}$$

where $\rho \in (0, 1)$ and B_n satisfies $\sqrt[n]{B_n} \to 1$ and $B_n \to 0$, and the notation $f_n = \Theta(g_n)$ means that there exist constants $0 < c < C$ such that $cg_n \leqslant f_n \leqslant Cg_n$.

We could have unified the presentation of the interior and non-interior drift estimates, since $\mathbb{P}^x(\tau = \infty) = 0$ when the drift is not in K^o. However, we choose not to do so, because the last double-sided estimate (1.3) is obtained only in the small step walk setting. We leave open the general case of this interesting interior drift problem.

In the papers [12, 13] (see also [7]), the authors prove the non-rationality of $F(t)$, for certain models of singular walks in the quarter plane, by proving that $F(t)$ admits infinitely many poles.

Combinatorial motivations. Up to a scaling of the t-variable, our framework is equivalent to a more combinatorial question, related to the enumeration of walks. More precisely, in case μ is a uniform distribution on a finite set $\mathcal{S}$ (with cardinality $|\mathcal{S}|$), one has

$$F(|\mathcal{S}|t) = \sum_{n\geqslant 0} q_n t^n,$$

where q_n denotes the number of walks starting from x, having length n and staying in the cone K. More generally, when μ is any distribution, the series $F(t)$ counts the numbers of μ-weighted walks of length n staying in the cone K. Accordingly, all our results admit direct combinatorial interpretations.

Recently, in the combinatorial literature, the seminal paper [3] inspired the following question, which has attracted a lot of attention: given an orthant $K = \mathbb{N}^d = \{0, 1, \ldots\}^d$ and a distribution μ on $\mathbb{Z}^d$ (a step set in the combinatorial terminology), is the generating function (1.1), or its refined version

$$F(x_1, \ldots, x_d; t) = \sum_{n\geqslant 0} \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} \mathbb{P}^x(\tau > n, S_n = (n_1, \ldots, n_d)) x_1^{n_1} \cdots x_d^{n_d} t^n \tag{1.4}$$

a rational function? An algebraic function? A function satisfying a linear (or non-linear) differential equation? A hypertranscendental function, meaning that like Euler's Γ function it does not satisfy any differential equation? In the present article, we look at the possible rationality of the generating function.

Notice the following relation between (1.1) and (1.4):

$$F(1, \ldots, 1; t) = F(t).$$

On the other hand, $F(0, \ldots, 0; t)$ is the generating function of the excursion sequence

$$F(0, \ldots, 0; t) = \sum_{n\geqslant 0} \mathbb{P}^x(\tau > n, S_n = (0, \ldots, 0)) t^n,$$

which will be studied (based on earlier literature [5]) in Section 5.

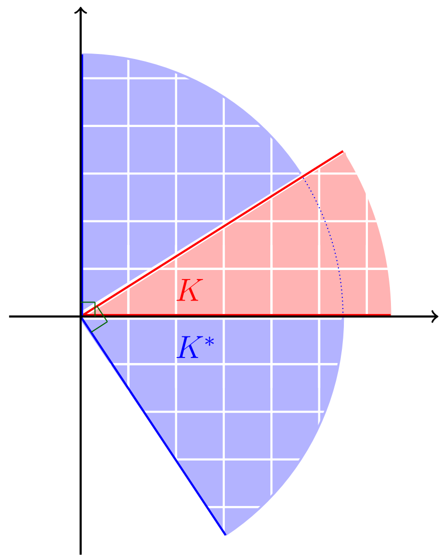

FIGURE 1. A cone K (in red) and its dual cone cone K^* (in blue)

Technical assumptions. In order to present the hypotheses in the statement of our main results, we need to introduce two objects, through which the exponential rate ρ in (1.2) will be determined:

- the Laplace transform L of the increment distribution μ:

$$L(t) = \mathbb{E}\Big(e^{\langle t, X_k\rangle}\Big) = \int_{\mathbb{R}^d} e^{\langle t,y\rangle}\mu(dy),$$

- the dual cone K^* associated with K (see Figure 1 for an example):

$$K^* = \{x \in \mathbb{R}^d : \langle x, y\rangle \geqslant 0 \text{ for all } y \in K\}. \tag{1.5}$$

  Obviously, K^* is a closed convex cone.

Throughout this paper, we make the following assumptions on the cone K and on the distribution μ of the random walk increments:

(A1) The cone K is closed, convex, with non-empty interior.

(A2) The random walk is truly d-dimensional, i.e., there is no $u \neq 0$ such that $\langle u, X_1\rangle = 0$ almost surely. Moreover, the random walk started at zero can reach the interior K^o of the cone: there exists $k > 0$ such that $\mathbb{P}^0(\tau > k, S_k \in K^o) > 0$.

(A3) The random walk increments are L^1. We call $m = \mathbb{E}X_1 = \int y\mu(dy)$ the drift.

(A4) There exists a point $t_0 \in K^*$ and a neighborhood V of t_0 such that the Laplace transform L of μ is finite in V and t_0 is a minimum point of L restricted to $K^* \cap V$.

We would like to give some intuition on the hypothesis (A4), which is designed to perform an exponential change of measure adapted to the geometry of the problem (see Subsection 2.1). First, (A4) implies the existence of *some* exponential moments, but not all; it is even not necessary that the increments have a moment of order one.

For example, consider a random variable X with probability density function

$$f(x) = c_\gamma \left(\frac{e^{-x^2}}{1+x^2} 1_{\{x>0\}} + \frac{e^{\gamma^2}}{1+x^2} 1_{\{x\leqslant 0\}} \right),$$

where $\gamma > 0$ is some fixed parameter and c_γ is the normalizing constant. Since the negative part of X has a half-Cauchy distribution, the variable X is not integrable. However, its Laplace transform

$$L(t) = \mathbb{E}(e^{tX}) = c_\gamma \int_0^\infty \frac{e^{-tx}e^{\gamma^2} + e^{tx}e^{-x^2}}{1+x^2} dx$$

is finite and differentiable for all $t > 0$, and its derivative

$$L'(t) = c_\gamma \int_0^\infty (e^{2tx-x^2} - e^{\gamma^2}) \frac{xe^{-tx}}{1+x^2} dx$$

is negative for $t \in (0, \gamma)$. Thus L reaches a minimum at some $t_0 > 0$ (clearly $L(t) \to \infty$ as $t \to \infty$). Now taking a random vector $Z = (X, Y)$ with density $f(x)f(y)$ gives a two-dimensional example where Z is not integrable but its Laplace transform is finite in the quadrant $[0, \infty)^2$ and reaches a global minimum inside the quadrant.

Regarding the existence of a local minimum in K^*, hypothesis (A4) is discussed in [11] (see its subsection 2.3; the condition is called (H2) there), where an equivalent geometric condition is given: in the presence of all exponential moments, condition (A4) is satisfied if and only if the support of the distribution μ is not included in any 'bad' half-space $\{x \in \mathbb{R}^d : \langle x, u\rangle \leqslant 0\}$, with u in the dual cone $K^* \setminus \{0\}$.

Under assumptions (A1)–(A4), we proved in [11] that the exponential rate ρ of the survival probability is equal to $L(t_0)$, meaning that for all $x \in K$,

$$\lim_{n\to\infty} \mathbb{P}^x(\tau > n)^{1/n} = L(t_0).$$

Furthermore, $L(t_0) < 1$ if and only if the drift m does not belong to the closed cone K. Here, we shall prove a little bit more:

Theorem 1.3. *Assume hypotheses (A1)–(A4) above. If $m \notin K^o$, then*

$$\mathbb{P}^x(\tau > n) = \rho^n B_n,$$

where $\rho = L(t_0) \in (0, 1]$, $\sqrt[n]{B_n} \to 1$ and $B_n \to 0$.

Regarding the interior drift case, we shall prove the following estimate, in the setting of small step walks in orthants:

Theorem 1.4. *For all d-dimensional weighted small step walks with a drift interior to the orthant $K = [0, \infty)^d$, not trapped and satisfying (A2), we have for all $x \in \mathbb{N}^d$*

$$\mathbb{P}^x(\tau > n) - \mathbb{P}^x(\tau = \infty) = \Theta(\rho^n B_n),$$

where $\rho \in (0, 1)$ and B_n satisfies $\sqrt[n]{B_n} \to 1$ and $B_n \to 0$.

A one-dimensional example. We now illustrate our previous results on the example of the simple random walk on $\mathbb{Z}$, with jump probabilities q to the left (-1) and $p = 1 - q$ to the right $(+1)$ and let $K = [0, \infty)$. In this setting,

$$\tau = \inf\{n > 0 : S_n < 0\} = \inf\{n > 0 : S_n = -1\}. \tag{1.6}$$

It is well known that, for any positive starting point $x \in \mathbb{N}$, the series (1.1) equals

$$F(t) = \frac{1 - \phi(t)^{x+1}}{1 - t}, \qquad \text{with } \phi(t) = \frac{1 - \sqrt{1 - 4pqt^2}}{2pt}. \tag{1.7}$$

We may directly observe on (1.7) that, as stated in our Theorems 1.1 and 1.2, the function $F(t)$ is not rational. Indeed $F(t) \in \mathbb{R}(t)$ if and only if $(1 - \sqrt{1 - 4pqt^2})^{x+1} \in \mathbb{R}(t)$. But an expression like $(1 - \sqrt{P})^n$ where P is a polynomial belongs to $\mathbb{R}(t)$ if and only if $\sqrt{P}$ belongs to $\mathbb{R}(t)$, as seen by a binomial expansion, and $\sqrt{P}$ is rational if and only if P is a square in $\mathbb{R}[t]$. This is not the case here with $P(t) = 1 - 4pqt^2$, unless p or q equals 0. However, $F(t)$ defines an algebraic function (as usual for one-dimensional random walks; see [1]).

In the zero drift case (meaning that $p = q = \frac{1}{2}$), expanding (1.7) at $t = 1$ and using singularity analysis, one finds

$$\mathbb{P}^x(\tau > n) \sim (x + 1)\sqrt{\frac{2}{\pi}}\frac{1}{n^{1/2}} \qquad \text{(in particular } \rho = 1\text{)}.$$

If the drift is negative $(q > p)$, the function F in (1.7) is analytic at 1 as $\phi(1) = 1$, and the singularities $t = \pm\frac{1}{2\sqrt{pq}}$ will both contribute to the asymptotics, which reads

$$\mathbb{P}^x(\tau > n) \sim (x + 1)\left(\frac{q}{p}\right)^{(x+1)/2}\left(\frac{1}{\frac{1}{2\sqrt{pq}} - 1} + \frac{(-1)^{x+n}}{\frac{1}{2\sqrt{pq}} + 1}\right)\frac{(2\sqrt{pq})^n}{\sqrt{2\pi}n^{3/2}}.$$

Finally, when the drift is positive $(p > q)$, the probability of survival admits the following two-term asymptotics (observe the similarity with the negative drift situation)

$$\mathbb{P}^x(\tau > n) = \left(1 - \left(\frac{q}{p}\right)^{x+1}\right) + (x+1)\left(\frac{q}{p}\right)^{(x+1)/2}\left(\frac{1}{\frac{1}{2\sqrt{pq}} - 1} + \frac{(-1)^{x+n}}{\frac{1}{2\sqrt{pq}} + 1}\right)\frac{(2\sqrt{pq})^n}{\sqrt{2\pi}n^{3/2}} + \cdots .$$

The three asymptotics above, which illustrate our Theorems 1.3 and 1.4, are obtained by studying the singularities of the generating function (1.7) and by using classical transfer theorems on the coefficients.

2. Survival probability in the non-interior drift case: proof of Theorem 1.3

2.1. **Basics on the Laplace transform.** Let us first recall some basic properties. The Laplace transform of a random vector $X = (X^{(1)}, \ldots, X^{(d)}) \in \mathbb{R}^d$ with probability distribution μ is the function L defined for $t \in \mathbb{R}^d$ by

$$L(t) = \mathbb{E}\Big(e^{\langle t, X\rangle}\Big) = \int_{\mathbb{R}^d} e^{\langle t, y\rangle}\mu(dy).$$

It is finite in some neighborhood of the origin if and only if $\mathbb{E}\Big(e^{\alpha\|X\|}\Big)$ is finite for some $\alpha > 0$. If L is finite in some neighborhood of the origin, say $\overline{B(0,r)}$, then L is infinitely differentiable in $B(0,r)$ and its partial derivatives are given there by

$$\frac{\partial L(t)}{\partial t_i} = \mathbb{E}\Big(X^{(i)} e^{\langle t,X\rangle}\Big).$$

Therefore, the expectation $\mathbb{E}X = (\mathbb{E}X^{(1)}, \dots, \mathbb{E}X^{(d)})$ of X is equal to the gradient of L at the origin $\nabla L(0)$. Notice that X is centered (i.e., $\mathbb{E}X = 0$) if and only if 0 is a critical point of L. Since L is a convex function, this means that 0 is a minimum point of L in $\overline{B(0,r)}$. Now suppose that L is finite in some ball $\overline{B(t_0,r)}$, and define a new probability measure μ_* by

$$\mu_*(dy) = \frac{e^{\langle t_0,y\rangle}}{L(t_0)}\mu(dy).$$

The Laplace transform L_* of μ_* is linked to that of μ by the relation $L_*(t) = L(t_0+t)/L(t_0)$, and therefore L_* is finite in some neighborhood of the origin. As a consequence, applying the results above shows that any random vector X_* with distribution μ_* satisfies:

- $\mathbb{E}\left(e^{\alpha\|X_*\|}\right) < \infty$ for some $\alpha > 0$;
- $\mathbb{E}X_* = \nabla L(t_0)/L(t_0)$.

As we shall see later, the relevant value of L for our problem is its minimum on the dual cone K^* defined by (1.5).

We now investigate further properties of $\mathbb{E}X_*$ when t_0 satisfies the assumption (A4), i.e., t_0 is a local minimum point of L restricted to K^*. By convexity of L, the point t_0 is necessarily a global minimum on K^*; we don't assume t_0 to be a global minimum on $\mathbb{R}^d$. Define the two sets

$$S = \Big\{u \in \mathbb{R}^d : \exists \varepsilon > 0, \forall s \in [-\varepsilon, \varepsilon], t_0 + su \in K^*\Big\},$$

$$S^+ = \Big\{u \in \mathbb{R}^d : \exists \varepsilon > 0, \forall s \in [0, \varepsilon], t_0 + su \in K^*\Big\}.$$

Of course $S \subset S^+$. Since K^* is a convex cone, the set S contains at least t_0, while the set S^+ contains at least K^*. Assuming (A4), we observe the following:

- if u belongs to S^+, then the function $\phi(s) = L(t_0 + su)$ defined on some small interval $[0,\varepsilon]$ reaches a minimum at $s = 0$, hence $\phi'(0) = \langle \nabla L(t_0), u\rangle \geqslant 0$. This holds for all $u \in K^*$ since $K^* \subset S^+$, therefore $\nabla L(t_0)$ belongs to the dual cone $(K^*)^*$ associated with K^*;
- if u belongs to S, the function $\phi(s)$ defined on some small interval $[-\varepsilon, \varepsilon]$ reaches its minimum at $s = 0$, hence $\phi'(0) = 0$. Therefore $\nabla L(t_0)$ is orthogonal to S (and so at least to t_0 itself).

Translating these observations in terms of the expectation of X_*, we obtain:

Lemma 2.1. *Assume* (A1) *and* (A4)*. The expectation $\mathbb{E}X_*$ of any random vector with distribution μ^* belongs to the cone K and is orthogonal to t_0.*

Proof. Since K is a closed convex cone, it is well known that $(K^*)^* = K$ (see Exercise 2.31 in [4] for example). Everything now follows from the relation $\mathbb{E}X_* = \nabla L(t_0)/L(t_0)$. □

2.2. **Proof of Theorem 1.3.** We shall use the preceding t_0 and μ_* in order to perform an exponential change of measure. For any non-negative and measurable function $f : \mathbb{R}^n \to [0, \infty)$, elementary algebraic manipulations give:

$$\begin{aligned}\mathbb{E}^x\Big(f(S_1, \ldots, S_n)\Big) &= \int_{\mathbb{R}^n} f\left(x + x_1, x + \sum_{i=1}^{2} x_i, \ldots, x + \sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} \mu(dx_i)\\ &= \rho^n \int_{\mathbb{R}^n} f\left(x + x_1, x + \sum_{i=1}^{2} x_i, \ldots, x + \sum_{i=1}^{n} x_i\right) e^{-\langle t_0, \sum_{i=1}^n x_i\rangle} \prod_{i=1}^{n} \mu_*(dx_i)\\ &= \rho^n e^{\langle t_0, x\rangle} \mathbb{E}_*^x\left(f(S_1, \ldots, S_n) e^{-\langle t_0, S_n\rangle}\right),\end{aligned}$$

where

- $\rho = L(t_0)$,
- $\mathbb{E}_*^x$ is the expectation with respect to $\mathbb{P}_*^x$, a probability distribution under which $(S_n)_{n\geqslant 0}$ is a random walk with increment distribution μ_* and started at $S_0 = x$.

Taking $f(s_1, \ldots, s_n) = \prod_{i=1}^n 1_K(s_i)$ leads to

$$\mathbb{P}^x(\tau > n) = \rho^n e^{\langle t_0, x\rangle} \mathbb{E}_*^x\Big(e^{-\langle t_0, S_n\rangle}, \tau > n\Big),$$

so that Theorem 1.3 will follow from the two lemmas below:

Lemma 2.2. *Assume* (A1)–(A4). *Then, for all* $x \in K$,

$$\lim_{n\to\infty} \sqrt[n]{\mathbb{E}_*^x\Big(e^{-\langle t_0, S_n\rangle}, \tau > n\Big)} = 1.$$

Lemma 2.3. *Assume* (A1)–(A4). *If the drift* $m = \mathbb{E}X_1$ *does not belong to* K^o, *then for all* $x \in K$,

$$\lim_{n\to\infty} \mathbb{E}_*^x\Big(e^{-\langle t_0, S_n\rangle}, \tau > n\Big) = 0.$$

Lemma 2.2 is fully proved in [11]. However, to make our paper self-contained, we propose here a short proof of it in a simplified setting. Instead of (A2) we will work under the following hypothesis: there exist $k > 0$ and $z \in K^o$ such that $\mathbb{P}(\tau > k, S_k = z) > 0$. In the majority of classical lattice random walks, the previous hypothesis is satisfied, as for instance for all 74 non-singular small step random walks considered in [3].

Proof of Lemma 2.2. First observe that on the event $\{\tau > n\}$, we have $S_n \in K$, hence $\langle t_0, S_n\rangle \geqslant 0$ since $t_0 \in K^*$. As a consequence $\mathbb{E}_*^x\Big(e^{-\langle t_0, S_n\rangle}, \tau > n\Big) \leqslant \mathbb{P}_*^x(\tau > n) \leqslant 1$, and what remains to prove is that

$$\liminf_{n\to\infty} \sqrt[n]{\mathbb{E}_*^x\Big(e^{-\langle t_0, S_n\rangle}, \tau > n\Big)} \geqslant 1.$$

By inclusion of events and basic properties of the n-th root limit, it suffices to prove the result for $x = 0$, in which case we get rid of the x superscript on $\mathbb{E}_*$ and $\mathbb{P}_*$. We compute a lower bound of the expectation as follows:

$$\mathbb{E}_*\Big(e^{-\langle t_0, S_n\rangle}, \tau > n\Big) \geqslant e^{-a_n} \mathbb{P}_*\Big(|\langle t_0, S_n\rangle| \leqslant a_n, \tau > n\Big),$$

with $a_n = n^{3/4}$. The e^{-a_n} term goes to 1 in the n-th root limit, thus we focus on the probability in the right-hand side.

Using our hypothesis, we can use the first $k\lfloor\sqrt{n}\rfloor$ steps to push the walk $\lfloor\sqrt{n}\rfloor$ times in the direction z without leaving the cone: by inclusion of events and the Markov property, we have

$$\mathbb{P}_*\Big(|\langle t_0, S_n\rangle| \leqslant a_n, \tau > n\Big) \geqslant \alpha^{b_n}\mathbb{P}_*^{b_n z}\Big(|\langle t_0, S_{n-kb_n}\rangle| \leqslant a_n, \tau > n - kb_n\Big),$$

where $\alpha = \mathbb{P}(\tau > k, S_k = z) > 0$ and $b_n = \lfloor\sqrt{n}\rfloor$. Here again, the α^{b_n} term will disappear in the n-th root limit, and the $-kb_n$ does not play any significant role in $n - kb_n$, so we are left to consider the probability

$$\mathbb{P}_*^{b_n z}\Big(|\langle t_0, S_n\rangle| \leqslant a_n, \tau > n\Big).$$

At this point, we take into account the "new drift" $d = \mathbb{E}_* X_1$ of the random walk under $\mathbb{P}_*$, and consider the centered random walk $\widetilde{S}_n = S_n - nd$. Lemma 2.1 asserts that:

- d is orthogonal to t_0, so that $\langle t_0, S_n\rangle = \langle t_0, \widetilde{S}_n\rangle$,
- d belongs to K, hence

$$\{\tau(\widetilde{S}_\ell) > n\} = \{\widetilde{S}_1, \ldots, \widetilde{S}_n \in K\} \subset \{S_1, \ldots, S_n \in K\} = \{\tau > n\}.$$

Due to these facts, our probability can be bounded from below by

$$\begin{aligned}\mathbb{P}_*^{b_n z}\Big(|\langle t_0, \widetilde{S}_n\rangle| \leqslant a_n, \tau(\widetilde{S}_\ell) > n\Big) &= \mathbb{P}_*\Big(|\langle t_0, b_n z + \widetilde{S}_n\rangle| \leqslant a_n, \tau(b_n z + \widetilde{S}_\ell) > n\Big)\\ &= \mathbb{P}_*\Big(|\langle t_0, z + \widetilde{S}_n b_n^{-1}\rangle| \leqslant a_n b_n^{-1}, \tau(z + \widetilde{S}_\ell b_n^{-1}) > n\Big)\\ &\geqslant \mathbb{P}_*\Big(\|\widetilde{S}_\ell b_n^{-1}\| < \varepsilon \text{ for all } \ell = 1, \ldots, n\Big),\end{aligned}$$

where we have used the homogeneity of the cone, namely $K/b_n = K$ on the second line, and then chosen $\varepsilon > 0$ so that the ball $B(z, \varepsilon) \subset K$. Now recall that, under $\mathbb{P}_*$, the increments X_n of the random walk S_n have a distribution μ^* with some exponential moments, hence the X_n's are in L^2, and so do the increments $X_n - d$ of the centered random walk $\widetilde{S}_n$. Therefore, the functional central limit theorem [2, Thm 8.2] is in force and, in conjunction with Portmanteau theorem [2, Thm 2.1], we obtain

$$\liminf_{n\to\infty} \mathbb{P}_*^{b_n z}\Big(|\langle t_0, \widetilde{S}_n\rangle| \leqslant a_n, \tau(\widetilde{S}_\ell) > n\Big) \geqslant \mathbb{P}_*\Big(\|B_t\| < \varepsilon \text{ for all } t \in [0,1]\Big) > 0,$$

where $(B_t)_{t\in[0,1]}$ is the image of a standard Brownian motion started at 0 under a (possibly degenerate) linear transformation. This concludes the proof of Lemma 2.2. □

Proof of Lemma 2.3. The proof will be done separately, according to whether t_0 is zero or not. First assume $t_0 \neq 0$. On the event $\{\tau > n\}$, for all $k = 1, \ldots, n$, we have that $S_k \in K$, hence $R_k = \langle t_0, S_k\rangle \geqslant 0$ since $t_0 \in K^*$. Therefore

$$\mathbb{E}_*^x\Big(e^{-\langle t_0, S_n\rangle}, \tau > n\Big) \leqslant \mathbb{P}_*^x\Big(R_k \geqslant 0 \text{ for all } k = 1, \ldots, n\Big).$$

Now, under $\mathbb{P}_*^x$, the process $R_k = \langle t_0, S_k\rangle$ is a random walk with increments $Y_k = \langle t_0, X_k\rangle$ having mean $\langle t_0, \mathbb{E}_* X_1\rangle = 0$ (see Lemma 2.1). Since the initial distribution μ is truly d-dimensional and μ_* is absolutely continuous with respect to μ, the new distribution μ_* is also truly d-dimensional.

Thus, under $\mathbb{P}_*^x$, the increments Y_k are non-degenerate (i.e., it does not hold that $Y_k = 0$ almost surely). It is well known (see [9, Thm 1 & 2 of XII,2]) that for such a one-dimensional random walk, almost surely,

$$-\infty = \liminf R_n < \limsup R_n = +\infty.$$

Accordingly,

$$\lim_{n\to\infty} \mathbb{P}_*^x\Big(R_k \geqslant 0 \text{ for all } k = 1, \ldots, n\Big) = \mathbb{P}_*^x\Big(R_k \geqslant 0 \text{ for all } k \geqslant 1\Big) = 0.$$

We now turn to the case $t_0 = 0$. This time $\langle t_0, S_n\rangle = 0$, so we don't learn anything by considering this specific one-dimensional random walk. The idea is to replace t_0 with an appropriate $\widetilde{t}_0$ and apply the same argument as before. To do this, observe that we know from Lemma 2.1 that $\mathbb{E}_* X_1$ belongs to the cone K, but when $t_0 = 0$ the change of measure has no effect: $\mu_* = \mu$. Hence the original drift $m = \mathbb{E}X_1$ belongs to K. Since we assumed $m \notin K^o$, we are left with a drift m on the boundary ∂K of the cone K.

If C is a closed cone, the interior of its dual cone has the following description:

$$(C^*)^o = \Big\{x \in \mathbb{R}^d : \langle x, y\rangle > 0 \text{ for all } y \in C \setminus \{0\}\Big\}$$

(see Exercise 2.31(d) in [4] for example). As a consequence, the boundary is given by

$$\partial C^* = \Big\{x \in C^* : \langle x, y\rangle = 0 \text{ for some } y \in C \setminus \{0\}\Big\},$$

and applying this to the closed convex cone $C = K^*$ gives

$$\partial K = \Big\{x \in K : \langle x, y\rangle = 0 \text{ for some } y \in K^* \setminus \{0\}\Big\},$$

since $(K^*)^* = K$. Going back to our drift $m \in \partial K$, there exists some $\widetilde{t}_0 \in K^* \setminus \{0\}$ such that $\langle \widetilde{t}_0, m\rangle = 0$. Setting $\widetilde{R}_k = \langle \widetilde{t}_0, S_k\rangle$, we obtain a centered and non-degenerate one-dimensional random walk such that $S_k \in K$ implies $\widetilde{R}_k \geqslant 0$. Therefore

$$\mathbb{E}_*^x\Big(e^{-\langle t_0, S_n\rangle}, \tau > n\Big) = \mathbb{P}^x(\tau > n) \leqslant \mathbb{P}_*^x\Big(\widetilde{R}_k \geqslant 0 \text{ for all } k = 1, \ldots, n\Big),$$

and the conclusion follows as in the first case. □

The proof of Theorem 1.3 is complete.

3. Survival probability in the interior drift case: proof of Theorem 1.4

In this section, we restrict our attention to the cone $K = [0, \infty)^d$ and small step walks, i.e., random walks on $\mathbb{Z}^d$ with increments X_k satisfying $X_k \in \{-1, 0, 1\}^d$ almost surely. For such walks, we investigate the case of a drift $m = \mathbb{E}X_k$ interior to the cone K, i.e., such that $\langle m, e_i\rangle > 0$ for $i = 1, \ldots, d$, where $(e_1, \ldots, e_d)$ denotes the standard basis of $\mathbb{R}^d$. We will use the notation $X_k^{(i)} = \langle X_k, e_i\rangle$. Since the drift is in the interior of K, we know that

$$\lim_{n\to\infty} \mathbb{P}^x(\tau > n) = \mathbb{P}^x(\tau = \infty) > 0$$

for all $x \in K$; see Lemma 3.1 for a precise statement and a proof.

Here we wish to estimate the error term $\delta_n = \mathbb{P}^x(\tau > n) - \mathbb{P}^x(\tau = \infty)$. We exclude the case where $\delta_n = 0$ for all n by assuming that the random walk is not trapped, i.e., the increments satisfy $\mathbb{P}(X_k \notin K) > 0$. Under this assumption we will prove Theorem 1.4, namely that

$$\mathbb{P}^x(\tau > n) - \mathbb{P}^x(\tau = \infty) = \Theta\left(\rho^n B_n\right).$$

Before going into the proof, we collect preliminary estimates on $\mathbb{P}^x(\tau = \infty)$.

3.1. **Exact formula for a one-dimensional small step walk.** First of all, we consider the one-dimensional setting with $p = \mathbb{P}(X_k = 1)$, $r = \mathbb{P}(X_k = 0)$, $q = \mathbb{P}(X_k = -1)$, $p + r + q = 1$. Let τ be as in (1.6) and assume $m = p - q > 0$. Then it is known that, for all $x \in \mathbb{N}$,

$$\mathbb{P}^x(\tau = \infty) = 1 - \left(\frac{q}{p}\right)^{x+1}.$$

If $q > 0$, this can be rewritten as

$$\mathbb{P}^x(\tau = \infty) = 1 - \gamma e^{-sx}, \tag{3.1}$$

where $\gamma = q/p$ and $s > 0$ is the unique solution to $e^{-s} = q/p$.

One way to obtain the formula above is to use the discrete harmonicity of the function $u_x = \mathbb{P}^x(\tau = \infty)$: by the Markov property, we have $u_x = qu_{x-1} + ru_x + pu_{x+1}$ for all $x \geqslant 1$, which is solved in $u_x = a + b\left(\frac{q}{p}\right)^x$. Then a and b are determined through initial and limit behaviors of u_x.

For future use, we notice the following fact: let

$$L(t) = \mathbb{E}\left(e^{tX_k}\right) = pe^t + r + qe^{-t}$$

be the Laplace transform associated with the random walk increments. Its derivative is given by $L'(t) = pe^t - qe^{-t}$. Evaluating at $t = -s$, where s is as above the solution to $e^{-s} = q/p$, leads to

$$L(-s) = 1 \quad \text{and} \quad L'(-s) = q - p = -m < 0. \tag{3.2}$$

The last value is exactly the opposite of the drift.

3.2. **Estimate for $\mathbb{P}^x(\tau < \infty)$ in the d-dimensional small step case.** Let us go back to our d-dimensional small step walk $(S_n)_n$ with drift m interior to the cone $K = [0, \infty)^d$ and such that $\mathbb{P}(X_k \notin K) > 0$. The simple inclusion of events

$$\Big\{\exists n > 0, \langle S_n, e_i\rangle < 0\Big\} \subset \Big\{\tau < \infty\Big\} \subset \bigcup_{i=1}^{d} \Big\{\exists n > 0, \langle S_n, e_i\rangle < 0\Big\}$$

leads to the bounds

$$\frac{g(x)}{d} \leqslant \mathbb{P}^x(\tau < \infty) \leqslant g(x), \tag{3.3}$$

where $g(x) = \sum_{i=1}^d \mathbb{P}^x(\exists n > 0, \langle S_n, e_i\rangle < 0)$. Now, for each i, the one-dimensional small step walk $(\langle S_n, e_i\rangle)_n$ with increments $X_k^{(i)}$ has a drift $\mathbb{E}X_k^{(i)} = \langle m, e_i\rangle > 0$. Since $\mathbb{P}(X_k \notin K) > 0$, the set I of indices i for which $\mathbb{P}(X_k^{(i)} = -1) > 0$ is non-empty, and applying the exact formula (3.1) of the preceding paragraph, we obtain:

$$g(x) = \sum_{i \in I} \gamma_i e^{-s_i\langle x, e_i\rangle},$$

where $\gamma_i = \mathbb{P}(X_k^{(i)} = -1)/\mathbb{P}(X_k^{(i)} = 1) \in (0, 1)$ and $s_i > 0$ is the unique solution to $e^{-s_i} = \gamma_i$.

3.3. **Proof of Theorem 1.4.** Fix $x \in \mathbb{N}^d$ and set

$$\delta_n = \mathbb{P}^x(\tau > n) - \mathbb{P}^x(\tau = \infty) = \mathbb{P}^x(\tau > n, \text{ but } S_m \notin K \text{ for some } m > n).$$

By the Markov property of the random walk, we can express δ_n as follows:

$$\delta_n = \mathbb{E}^x\left(\tau > n, \mathbb{P}^{S_n}(\tau < \infty)\right),$$

so that inequality (3.3) leads to $\delta_n = \Theta(g_n)$, where $g_n = \mathbb{E}^x(\tau > n, g(S_n))$. It remains to estimate

$$g_n = \sum_{i \in I} \gamma_i \mathbb{E}^x\left(\tau > n, e^{\langle S_n, -s_i e_i \rangle}\right).$$

To do this, we apply to each term in the sum a specific exponential change of measure. Set

$$\mu_{*i}(dy) = \frac{e^{\langle -s_i e_i, y \rangle}}{L(-s_i e_i)} \mu(dy),$$

where μ is the common distribution of the increments X_k of the random walk, and $L(t) = \mathbb{E}(e^{\langle t, X_k \rangle})$ is their Laplace transform. Then basic algebraic manipulations as in Section 2.2 lead to

$$\mathbb{E}^x\left(\tau > n, e^{\langle S_n, -s_i e_i \rangle}\right) = L(-s_i e_i)^n e^{\langle -s_i e_i, x \rangle} \mathbb{P}^x_{*i}\left(\tau > n\right).$$

Now observe that $t \mapsto L(te_i) = \mathbb{E}\left(e^{tX_k^{(i)}}\right)$ is the one-dimensional Laplace transform of the increments $X_k^{(i)}$. Since s_i is the solution to $e^{-s_i} = \gamma_i = \frac{\mathbb{P}(X_k^{(i)}=-1)}{\mathbb{P}(X_k^{(i)}=1)}$, we are in the same situation as in (3.2), so that

$$L(-s_i e_i) = 1 \quad \text{and} \quad \frac{\partial L}{\partial t_i}(-s_i e_i) = -\langle m, e_i \rangle < 0.$$

Therefore, equation (3.3) reads

$$\mathbb{E}^x\left(\tau > n, e^{\langle S_n, -s_i e_i \rangle}\right) = e^{\langle -s_i e_i, x \rangle} \mathbb{P}^x_{*i}\left(\tau > n\right),$$

and the new drift under $\mathbb{P}^x_{*i}$, which is given by the gradient of L at the point $-s_i e_i$, has a strictly negative i-th coordinate. As a consequence, this drift does not belong to the cone $K = [0, \infty)^d$, and it follows from Theorem 1.3 that

$$\mathbb{P}^x_{*i}(\tau > n) = \rho_i^n B_{i,n},$$

where $\rho_i \in (0,1)$, $\sqrt[n]{B_{i,n}} \to 1$ and $B_{i,n} \to 0$ as $n \to \infty$. Finally, we get

$$g_n = \sum_{i \in I} \gamma_i \rho_i^n B_{i,n},$$

which can be rewritten in the form $g_n = \rho^n B_n$, by selecting

$$\rho = \max\{\rho_i : i \in I\} < 1.$$

It is then clear that $\sqrt[n]{B_n} \to 1$ and $B_n \to 0$, and the proof is complete.

3.4. **Positivity of the escape probability.**

Lemma 3.1. *Assume* (A1) *and* (A2). *If the drift* $m = \mathbb{E}X_1$ *belongs to* K^o, *then the function* $h(x) = \mathbb{P}^x(\tau = \infty)$ *satisfies:*

(1) h is harmonic for the killed random walk, i.e.,

$$h(x) = \mathbb{E}^x(h(S_n), \tau > n).$$

(2) $h(x) > 0$ *for all* $x \in K$.

(3) $\lim_{t\to\infty} h(tu) = 1$ *for all* $u \in K^o$.

Proof. Item (1) is just the Markov property applied at time n. The relation is valid disregarding the position of the drift. We now prove (2).

First step. We begin with a simple geometric fact. For any $z \in K^o$, the non-decreasing sequence of sets $K - kz$ will ultimately cover the whole space, i.e., $\cup_{k\geqslant 0}(K - kz) = \mathbb{R}^d$. To see this, select $\varepsilon > 0$ such that $B(z, \varepsilon) \subset K$. For any $x \in \mathbb{R}^d$, there exists $k > 0$ such that $\|x/k\| < \varepsilon$, hence $z + \frac{x}{k}$ belongs to K. By homogeneity of K, it follows that $kz + x \in K$, i.e., $x \in K - kz$.

Second step. Let us consider the random walk (S_n) with drift $m \in K^o$ and select $\varepsilon > 0$ such that $B(m, \varepsilon) \subset K$. By the strong law of large numbers $S_n/n \to m$ almost surely, therefore, for almost all ω, there exists $n_0 = n_0(\omega)$ such that

$$n \geqslant n_0 \Rightarrow \left\| \frac{S_n(\omega)}{n} - m \right\| < \varepsilon \Rightarrow S_n(\omega) \in K.$$

Considering now the first positions $S_1(\omega), S_2(\omega), \ldots, S_{n_0-1}(\omega)$, the first step of the proof ensures that there exists $k \geqslant 0$ such that they all belong to $K - kz$, where $z \in K^0$ is to be fixed in the last step of the proof. Since one has

$$K \subset K - kz \text{ (recall that } K + K \subset K),$$

all positions $S_n(\omega), n \geqslant n_0$, also belong to $K - kz$ and we obtain the following:

$$\mathbb{P}\Big(\cup_{k\geqslant 0}\{S_n \in K - kz \text{ for all } n \geqslant 0\}\Big) = 1.$$

Since the events inside the probability above form a non-decreasing sequence, it follows that

$$\lim_{k\to\infty} \mathbb{P}\Big(S_n \in K - kz \text{ for all } n \geqslant 0\Big) = 1. \tag{3.4}$$

Last step. To conclude, we invoke hypothesis (A2), that claims the existence of an integer $\ell \geqslant 1$ such that

$$\mathbb{P}(\tau > \ell, S_\ell \in K^o) > 0.$$

Fix some $u \in K^o$. Since $K^o = \cup_{\lambda>0}(K + \lambda u)$, there is a $z = \lambda u \in K^o$ such that

$$\mathbb{P}(\tau > \ell, S_\ell \in K + z) = p > 0.$$

By the Markov property, a concatenation of m such ℓ-steps paths leads to

$$\mathbb{P}(\tau > m\ell, S_{m\ell} \in K + mz) \geqslant p^m > 0.$$

On the other hand, it follows from (3.4) that there exists $k \geqslant 0$ such that

$$\mathbb{P}(S_n \in K - kz \text{ for all } n \geqslant 0) \geqslant 1/2.$$

Now choose $m \geqslant k$. Since $S_{m\ell} \in K + mz$ and $S_n - S_{m\ell} \in K - kz$ imply $S_n \in K$, we obtain

$$\mathbb{P}(\tau = \infty) \geqslant \mathbb{P}(\tau > m\ell, S_{m\ell} \in K + mz) \times \mathbb{P}(S_n \in K - kz \text{ for all } n \geqslant 0) > 0.$$

We have just proved that $g(0) > 0$. The result follows since $g(x) \geqslant g(0)$ for all $x \in K$, by inclusion of events.

We conclude with the proof of (3). The limit (3.4) obtained in the *second step* of Item (2) can be recast as:

$$\lim_{k\to\infty} \mathbb{P}^{kz}(\tau = \infty) = 1,$$

where z is any vector in K^o. Since $g(x) = \mathbb{P}^x(\tau = \infty)$ is non-decreasing in every direction, the proof is completed. ☐

4. Proof of Theorems 1.1 and 1.2

In this section, we show that our estimates on $a_n = \mathbb{P}^x(\tau > n)$ given in Theorems 1.3 and 1.4 are not compatible with the generating function $F(t) = \sum_{n\geqslant 0} a_n t^n$ being rational, using classical singularity analysis for rational functions. The starting point is Theorem IV.9 in [10], which asserts the following:

Theorem 4.1. *If $F(z) = \sum_{n\geqslant 0} a_n z^n$ is a rational function that is analytic at 0 and has poles at points $\alpha_1, \alpha_2, \ldots, \alpha_k$, then its coefficients are sums of exponential-polynomials: there exist k polynomials P_j such that, for n larger than some fixed n_0,*

$$a_n = \sum_{j=1}^{k} P_j(n)\alpha_j^{-n}.$$

Both estimates in Theorems 1.3 and 1.4 have the following form:

$$a_n = a + \Theta(\rho^n B_n),$$

where $a \geqslant 0$, $\rho \in (0,1]$, $\sqrt[n]{B_n} \to 1$ and $B_n \to 0$. Therefore Theorems 1.1 and 1.2 asserting the non-rationality of F will follow in both cases from the following elementary lemma.

Lemma 4.2. *Let $c_1, \ldots, c_k$ be distinct non-zero complex numbers and $P_1, \ldots, P_k$ be non-zero complex polynomials. Set $a_n = \sum_{j=1}^{k} P_j(n) c_j^n$. If $a_n = a + \Theta(\rho^n B_n)$ for some $a \geqslant 0$, $\rho > 0$ and $B_n > 0$ such that $\sqrt[n]{B_n} \to 1$, then necessarily $B_n \not\to 0$.*

Proof. If $a_n = \sum_{j=1}^{k} P_j(n) c_j^n$, then $a_n - a$ has the same form, thus, without loss of generality, we can assume $a = 0$. Write $c_j = r_j z_j$ with $r_j > 0$ and $|z_j| = 1$. Let $r = \max\{r_j : j = 1, \ldots, k\}$ and let J be the subset of indices j such that $r_j = r$. Then

$$a_n = \sum_{j=1}^{k} P_j(n) c_j^n = r^n \left(\sum_{j\in J} P_j(n) z_j^n + o(t^n) \right),$$

where $0 < t < 1$. For future use, note that the numbers $z_j, j \in J$ are all distinct (this is so since we kept at most one c_j in any fixed "direction" z_j: the one with maximum modulus).

We first show that $r = \rho$. Since $a_n = \Theta(\rho^n B_n)$ and $\sqrt[n]{B_n} \to 1$, it follows that a_n/ρ^n goes to one in the n-th root limit. Thus, for any $\varepsilon > 0$,

$$\big((1-\varepsilon)\rho\big)^n \leqslant a_n \leqslant \big((1+\varepsilon)\rho\big)^n$$

for n large enough. Therefore

$$\left(\frac{(1-\varepsilon)\rho}{r}\right)^n \leqslant \left|\sum_{j\in J} P_j(n)z_j^n + o(t^n)\right| \leqslant \left(\frac{(1+\varepsilon)\rho}{r}\right)^n \tag{4.1}$$

for n large enough. If $\rho > r$ then we can choose $\varepsilon > 0$ such that the lower bound is A^n for some $A > 1$.
But then we would have

$$A^n \leqslant \left|\sum_{j\in J} P_j(n)z_j^n + o(t^n)\right| \leqslant \sum_{j\in J} |P_j(n)| + |o(t^n)|$$

and this is impossible since $\sum_{j\in J} |P_j(n)|$ grows polynomially. On the other hand, if $\rho < r$ then we can choose $\varepsilon > 0$ such that the upper bound in (4.1) is A^n for some $A < 1$. This implies

$$\sum_{j\in J} P_j(n)z_j^n \to 0.$$

Dividing this by n^p, where p stands for the maximum degree of polynomials P_j, leads to the convergence

$$\sum_{j\in J'} a_j z_j^n \to 0,$$

where $J' \subset J$ is a non-empty subset of indices (those j for which P_j has degree p) and the a_j's are non-zero complex numbers. Since the numbers z_j are distinct complex numbers with modulus 1, this contradicts Lemma 4.3 below. The assertion $r = \rho$ is now established, hence we have

$$\left|\sum_{j\in J} P_j(n)z_j^n + o(t^n)\right| = \frac{a_n}{\rho^n} = \Theta(B_n).$$

We have seen just before that this expression cannot go to zero as $n \to \infty$, thus $B_n \not\to 0$. $\square$

Lemma 4.3. *Let $z_1, \ldots, z_k$ be distinct complex numbers with modulus $\geqslant 1$. If*

$$\lim_{n\to\infty} \sum_{j=1}^k a_j z_j^n = 0,$$

then necessarily $a_1 = \cdots = a_k = 0$.

Proof. Denote by A_n the quantity $\sum_{j=1}^k a_j z_j^n$. Clearly, given any complex numbers $\alpha_0, \ldots, \alpha_{k-1}$,

$$\sum_{i=0}^{k-1} \alpha_i A_{n+i} = \sum_{j=1}^k a_j P(z_j) z_j^n, \tag{4.2}$$

where $P(z) = \sum_{i=0}^{k-1} \alpha_i z^i$. We can choose the polynomial P so as to have $P(z_1) = 1$ and all other $P(z_j) = 0$. We then take the limit of (4.2) as $n \to \infty$, using the assumption of Lemma 4.3. We find that the term $a_1 z_1^n$ should go to zero, which implies that $a_1 = 0$, since $|z_1| \geqslant 1$. A similar reasoning gives that all $a_j = 0$, and thus Lemma 4.3 is proved. $\square$

5. The excursion generating function

In this section, we look at lattice random walks in convex cones. Besides the generating function of the survival probabilities (1.1), it is natural to ask whether the excursion generating function

$$E(t) = \sum_{n\geqslant 0} \mathbb{P}^x(\tau > n, S_n = y)t^n \tag{5.1}$$

can be rational, for given starting and ending points $x, y \in K$. When the cone K is an orthant $\mathbb{N}^d$ and $x = y = (0, \dots, 0)$, the function $E(t)$ reduces to the series $F(0, \dots, 0; t)$ of (1.4). In order to state the result of this section, we introduce the following assumption:

(A4') There exists a point $\widetilde{t}_0 \in \mathbb{R}^d$ and a neighborhood V of $\widetilde{t}_0$ such that the Laplace transform L of μ is finite in V and $\widetilde{t}_0$ is a minimum point of L restricted to V.

Since L is a convex function, the point $\widetilde{t}_0$ above is necessarily a global minimum. If μ is truly d-dimensional (as assumed in (A2)), the function L is strictly convex and a necessary and sufficient condition for the existence of a global minimum is that the support of μ is not included in any closed half-space.

Theorem 5.1. *For any distribution satisfying to* (A1)–(A3), (A4') *and such that the random walk takes its values on a lattice, the generating function $E(t)$ in* (5.1) *is not a rational function.*

Contrary to our elementary and self-contained proof of Theorem 1.1, we don't have any elementary argument to prove Theorem 5.1. Instead, we may give a one-line proof based on earlier literature. Indeed, Denisov and Wachtel provide the following estimate in [5, Eq. (10)] (we use the generalization to convex cones as in [6, Cor. 1.3]): $\mathbb{P}^x(\tau > n, S_n = y)$ is either 0 (for periodicity reasons) or asymptotic to

$$C(x, y)\widetilde{\rho}^n n^{-p-d/2},$$

where $\widetilde{\rho} = L(\widetilde{t}_0)$ with $\widetilde{t}_0$ as in (A4'), d is the dimension and $p > 0$ is a geometric quantity related to the cone. One immediately concludes because the exponent of n is negative.

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HEIGHT OF WALKS WITH RESETS, THE MORAN MODEL, AND THE DISCRETE GUMBEL DISTRIBUTION

RAFIK AGUECH[1], ASMA ALTHAGAFI[2], AND CYRIL BANDERIER[3]

[1] Department of Statistics and Operations Research, King Saud Univ., Saudi Arabia, and Department of Mathematics, University of Monastir, Tunisia;
https://faculty.ksu.edu.sa/en/raguech

[2] Department of Statistics and Operations Research, King Saud Univ., Saudi Arabia;
https://www.researchgate.net/profile/Asma-Althagafi

[3] Laboratoire d'Informatique de Paris Nord, Univ. Sorbonne Paris Nord, France;
http://lipn.fr/~banderier

Abstract. In this article, we consider several models of random walks in one or several dimensions, additionally allowing, at any unit of time, a reset (or "catastrophe") of the walk with probability q. We establish the distribution of the final altitude. We prove algebraicity of the generating functions of walks of bounded height h (showing in passing the equivalence between Lagrange interpolation and the kernel method). To get these generating functions, our approach offers an algorithm of cost $O(1)$, instead of cost $O(h^3)$ if a Markov chain approach would be used. The simplest nontrivial model corresponds to famous dynamics in population genetics: the Moran model.

We prove that the height of these Moran walks asymptotically follows a discrete Gumbel distribution. For $q = 1/2$, this generalizes a model of carry propagation over binary numbers considered e.g. by von Neumann and Knuth. For generic q, using a Mellin transform approach, we show that the asymptotic height exhibits fluctuations for which we get an explicit description (and, in passing, new bounds for the digamma function). We end by showing how to solve multidimensional generalizations of these walks (where any subset of particles is attributed a different probability of dying) and we give an application to the soliton wave model.

Keywords: Random walks, renewal process, Moran model, analytic combinatorics, discrete Gumbel distribution, Mellin transform, kernel method, digamma function.

Contents

1. Introduction

The height of random walks is a fundamental parameter which occurs in many domains: in computer science (evolution of a stack, tree traversals, or cache algorithms [39]), in reliability or failure theory (maximal age of a component and inference statistics on the longevity before replacement [24]), in queueing theory (maximal length of the queue, with e.g. applications to traffic jam analysis [37]), in mathematical finance (e.g. in risk theory [28]), in bioinformatics (pattern matching and sequence alignment [2]), etc.

In combinatorics, random walks are studied via the corresponding notion of *lattice paths*, which play a central role, not only for intrinsic properties of such paths, but also as they are in bijection with many fundamental structures (trees, words, maps, ...). We refer to the nice *magnum opus* of Flajolet and Sedgewick on analytic combinatorics [22] for many enumerative and asymptotic examples.

While the behavior of an extremal parameter such as the height is well understood for walks corresponding to Brownian motion theory, it becomes more subtle when a notion of reset/renewal/resetting/catastrophe [8, 9, 14, 29, 33, 40, 42] is introduced in the model: indeed, typical behaviors in this model are often established by conditioning on events of probability zero in the model without reset, leading to possibly counterintuitive results.

In this article, we give several enumerative and asymptotic results on different statistics (final altitude, waiting time, height) of walks with resets, focusing on the so-called Moran walks (walks related to biological/population models considered by Moran in 1958; see Section 5 for more on this).

Plan of the article.
In Section 2, we consider a generic model of walks with resets (allowing any finite set of steps and a reset step). We describe the behavior of their final altitude (at finite time, and asymptotically). We obtain an algebraic closed form for the bivariate generating function (length/final altitude) for walks of bounded height h. Our approach uses a variant of the so-called kernel method, which has the advantage to avoid any case-by-case computation based on Markov chains/transfer matrices of size $h \times h$. In passing, we show the intimate link between Lagrange interpolation and the kernel method.

In Section 3, we consider Moran walks, a model described in Figure 1, for which we generalize an enumerative formula due to Pippenger [45]. We show that their height asymptotically follows a distribution which involves non-trivial fluctuations. We prove that this distribution is a *discrete* Gumbel distribution, and we clarify its links with the *continuous* Gumbel distribution. We give an application to the waiting time for reaching any given altitude.

In Section 4, we begin with a brief presentation of the Mellin transform method, and then use it to derive a precise analysis of the asymptotic average and variance of the height. The second asymptotic term involves some $O(1)$ fluctuations given by a Fourier series (which we prove to be infinitely differentiable, and for which we also derive generic bounds of independent interest). This extends (and fixes some error terms) in earlier analyses by von Neumann, Knuth, Flajolet and Sedgewick [13, 22, 38].

In Section 5, we tackle some multidimensional generalizations of Moran walks, with applications to a model in population genetics and to a wave propagation model (a soliton model), as considered by Itoh, Mahmoud, and Takahashi in [34, 35].

In Section 6, we conclude with a few possible extensions for future work.

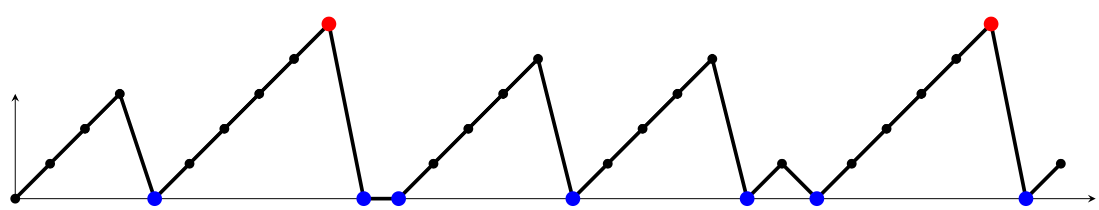


Figure 1. A Moran walk is a random walk which makes a jump $+1$ with probability p, and a reset (a jump to 0) with probability $1-p$. Above, one sees such a walk of length $n = 30$. Its final altitude is $Y_n = 1$, the height is $H_n = 5$ (reached twice, in red), having 7 resets (the 7 blue dots). In this article, we tackle the enumeration and asymptotics of such paths (and of generalizations involving more general step sets and higher dimension). We also prove that this simple model of walks leads to some noteworthy nontrivial asymptotic behavior of their height H_n.

2. Walks with resets: final altitude and height

We consider walks with steps in $\mathcal{S}$ (where $\mathcal{S}$ is a nonempty finite subset of $\mathbb{Z}$), which can additionally have a reset at any altitude. That is, we have the following process on $\mathbb{Z}$:

$$Y_0 = 0$$

$$Y_{n+1} = \begin{cases} Y_n + k, & \text{with probability } p_k \qquad (\text{for each } k \in \mathbb{Z}, \text{ with } p_k := 0 \text{ if } k \notin \mathcal{S}), \\ 0, & \text{with probability } q \qquad (\text{with } q + \sum_{k\in\mathcal{S}} p_k = 1). \end{cases}$$

(So if $Y_n = 0$ we have $Y_{n+1} = 0$ with probability $p_0 + q$.)
Thus, Y_n is the altitude of the process after n steps and $H_n := \max(Y_0, \dots, Y_n)$ is its height. It is convenient to encode the steps and their probabilities by the Laurent polynomial

$$P(u) := \sum_{k=c}^{d} p_k u^k \qquad (\text{with } c := \min \mathcal{S} \text{ and } d := \max \mathcal{S}). \tag{2.1}$$

We assume $0 < q < 1$ to avoid degenerate cases. We do not require that $c < 0$ or $d > 0$. Of course, if $c \geq 0$, the walk will live by design in $\mathbb{N}$ (it is e.g. the case for Moran walks of Figure 1). In Section 2.1, we determine the distribution of the final altitude (as illustrated in Figure 2 for different families of steps) and we investigate the height in Section 2.2.

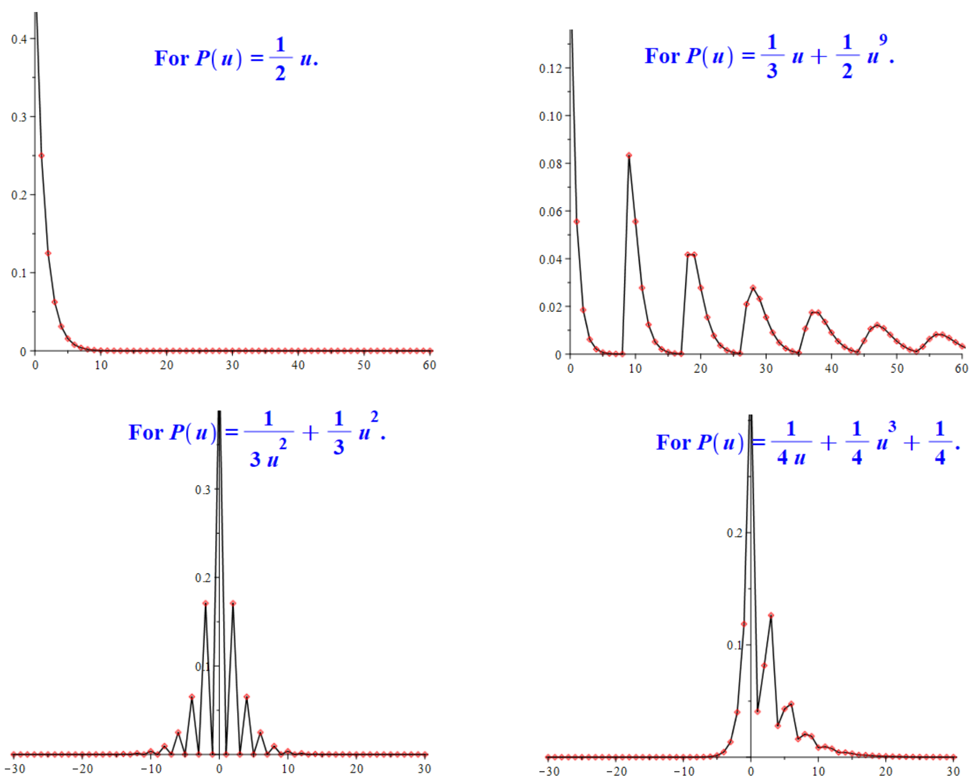


Figure 2. Plot of $\mathbb{P}(Y_n = k)$, the distribution of the altitudes of walks with resets, for $n = 100$ and different $P(u)$. It has its support in the $\mathbb{N}$-linear combinations of steps from $\mathcal{S}$. The final altitude is of order $O(1)$ and the probability to end at higher altitudes decreases exponentially fast (see Theorem 2.1 for closed-form expressions of the mean and the distribution).

2.1. **Final altitude Y_n.** Let us start with a simple result which paves the way for the more subtle generating function manipulations for the height that we tackle later in Section 2.2.

We use the classical convenient notations:

- $[z^n]G(z)$ stands for the coefficient of z^n in the power series $G(z)$,
- $\partial_u^j F(z,1)$ is the j-th derivative of $F(z,u)$ with respect to u, evaluated at $u=1$.

Theorem 2.1 (Final altitude at finite time)**.** *The final altitude of walks with resets follows a discrete law with probability generating function*

$$F(z,u) = \sum_{n\geq 0} \mathbb{E}[u^{Y_n}] z^n = \frac{1+qz/(1-z)}{1-zP(u)}, \tag{2.2}$$

where $P(u)$ is the Laurent polynomial encoding the allowed steps (a finite subset of $\mathbb{Z}$). Equivalently, for $k\in\mathbb{Z}$, we have

$$\mathbb{P}(Y_n = k) = [u^k]P(u)^n + q[u^k]\sum_{j=0}^{n-1} P(u)^j. \tag{2.3}$$

Let $\delta := P'(1)$ be the drift[1] *of the walk without reset, and $V := P''(1)$ its second factorial moment. The mean and the variance of the final altitude of the walk with resets are given by*

$$\mathbb{E}[Y_n] = \delta/q + (1-q)^{n-1}(\delta - \delta/q),$$

$$\mathbb{V}\mathrm{ar}[Y_n] = \frac{(V+\delta)\,q+\delta^2}{q^2} + (1-q)^n\left(2\,\frac{\delta^2 n}{(q-1)q} - \frac{V+\delta}{q}\right) - (1-q)^{2n}\frac{\delta^2}{q^2}.$$

For Moran walks (i.e., $P(u)=pu$ and $p=1-q$), the mean and the variance simplify to

$$\mathbb{E}[Y_n] = \frac{p}{q}\Big(1-p^n\Big) \quad \text{and} \quad \mathbb{V}\mathrm{ar}[Y_n] = \frac{p}{q^2}\Big(1-p^n\Big(p^{n+1}+(1+2n)q\Big)\Big).$$

Proof. The probability generating function can be written as

$$F(z,u) = \sum_{n\geq 0}\left(\sum_{k\in\mathbb{Z}}^{n} \mathbb{P}(Y_n=k)u^k\right) z^n = \sum_{n\geq 0} f_n(u) z^n,$$

where the $f_n(u)$'s are Laurent polynomials encoding the location of the walk at time n; thus we have $f_{n+1}(u) = P(u)f_n(u) + qf_n(1)$, with $f_0(u)=1$. Multiplying both sides of this recurrence by z^{n+1}, and summing over n, one gets

$$F(z,u)(1-zP(u)) = 1 + qzF(z,1).$$

As $F(z,1) = 1/(1-z)$, one obtains Formula (2.2). Note that the generating function can also be obtained by using a regular expression encoding these walks (by factorizing the walk in factors ending by a reset): $(\mathcal{S}^*q)^*(\mathcal{S})^*$, which translates to

$$F(z,u) = \frac{1}{1-qz\frac{1}{1-zP(1)}}\,\frac{1}{1-zP(u)},$$

where the occurrences of $P(1)$ and $P(u)$ reflect that only the altitudes after the last reset contribute to the final altitude of the full walk. Using $P(1)=1-q$, we get Formula (2.2).

The mean of Y_n is then obtained via $\mu_n := \mathbb{E}[Y_n] = [z^n]\partial_u F(z,1)$, while its variance is obtained via a second-order derivative: $\mathbb{V}\mathrm{ar}[Y_n] = [z^n]\partial_u^2 F(z,1) + \mu_n - \mu_n^2$. $\square$

[1]We recall that $P(1) = 1-q$, so another convention could have been to call drift the quantity $P'(1)/(1-q)$, i.e., we would then *condition* on having no reset (instead of considering walks without reset, weighted by the initial model (2.1)). This alternative convention does not simplify the subsequent formulas.

We can now establish the corresponding limit distribution.

Theorem 2.2 (Final altitude: asymptotics)**.** *Consider walks with $0 \notin \mathcal{S}$, $\gcd \mathcal{S} = 1$, and $d = \max \mathcal{S} > 0$ (these three constraints bring no loss of generality[2]). Therefore the support of the walk is either $\mathbb{Z}$ (with all altitudes being reachable), or $\mathbb{N}$ (with a finite set of altitudes impossible to reach, known as the unreachable set in the coin-exchange problem of Frobenius). The final altitude of these walks with resets behaves asymptotically according to these two cases.*

a) *For walks with $\min \mathcal{S} \geq 0$, we have for $k \in \mathbb{N}$ (not in the Frobenius unreachable set):*
$$q \cdot (\min_{i \in \mathcal{S}} p_i)^k \leq \lim_n \mathbb{P}(Y_n = k) \leq q \cdot (\max_{i \in \mathcal{S}} p_i)^{k/d}.$$
In particular, for Moran walks, we have $\mathbb{P}(Y_n = k) = qp^k$ for $0 \leq k < n$ and $\mathbb{P}(Y_n = n) = p^n$ so $\lim Y_n = \mathrm{Geom}(q) - 1$.

b) *For walks with $\min \mathcal{S} < 0$ and $\max \mathcal{S} > 0$, we have for $k \in \mathbb{Z}$:*
$$\mathbb{P}(Y_n = k) = qW_k(1-q) + (1-q)\frac{1}{\tau^{k+1}} \frac{1}{\sqrt{2\pi n P''(\tau)}} + O\left(\frac{1}{n}\right).$$

Moreover, both in Case a) and in Case b), $\mathbb{P}(Y_n = k)$ has a geometric decay for large k.

Proof. In Case a), we have $\min \mathcal{S} \geq 1$; the definition of $P(u)$ in (2.1) then entails $[u^k]P(u)^j = 0$ for large j. The limit of Equation (2.3) thus gives
$$\lim_{n \to +\infty} \mathbb{P}(Y_n = k) = q[u^k] \sum_{j=0}^{k} P(u)^j.$$
In particular, when it is not 0, this quantity is lower bounded by $q \cdot (\min_{i \in \mathcal{S}} p_i)^k$ and upper bounded by $q \cdot (\max_{i \in \mathcal{S}} p_i)^{k/d}$, and therefore decreases geometrically.

In Case b), the proof is more complicated and will recycle ingredients of the asymptotics of walks without reset. To this aim, first set $\widetilde{P}(u) := P(u)/P(1)$, i.e., the step set probabilities are renormalized to have global mass $\widetilde{P}(1) = 1$. Let $W_k(z)$ be the probability generating function of walks without reset, i.e., $W_k(z) = [u^k]\frac{1}{1 - z\widetilde{P}(u)} = \sum_{n \geq 0} w_{n,k} z^n$. We then rewrite Equation (2.3) as
$$\begin{aligned}
\mathbb{P}(Y_n = k) &= P(1)^n [u^k] \widetilde{P}(u)^n + q[u^k] \sum_{j=0}^{n-1} P(1)^j \widetilde{P}(u)^j \\
&= (1-q)P(1)^n w_{n,k} + q \sum_{j=0}^{n} P(1)^j w_{j,k} \\
&= (1-q)P(1)^n w_{n,k} + qP(1)^n [z^n] \frac{1}{1 - z/P(1)} W_k(z). \qquad (2.4)
\end{aligned}$$

If $\min \mathcal{S} < 0$ and $\max \mathcal{S} > 0$, then there is a unique real $\tau > 0$ such that $\widetilde{P}'(\tau) = 0$. It is proven in [5] that $\rho = 1/\widetilde{P}(\tau)$ is the radius of convergence of $W_k(z)$ and that $w_{n,k} \sim \tau^{-k} C \widetilde{P}(\tau)^n / \sqrt{2\pi n}$, where $C := \frac{1}{\tau}\sqrt{\widetilde{P}(\tau)/\widetilde{P}''(\tau)}$.

[2]There is no loss of generality. Indeed, if the walk as a periodic support (i.e., if $\gcd(\mathcal{S}) = g$ with $g > 1$) we rescale (without loss of generality) the step set $\mathcal{S}$ by dividing each step by g. Now, if $\max \mathcal{S} < 0$, then we multiply each step by -1. Last, if $0 \in \mathcal{S}$ we consider instead the equivalent model $\mathcal{S} := \mathcal{S} \setminus \{0\}$ and $q := q + p_0$.

Note that, as we have a probability generating function, we have $\rho = \widetilde{P}(\tau) = 1$. The asymptotics of (2.4) then follows by singularity analysis, as $1/(1 - z/P(1))$ is singular at $z = P(1) = 1 - q$, that is, before $W_k(z)$ which is singular at $z = 1$:

$$\mathbb{P}(Y_n = k) = qW_k(1-q) + (1-q)\tau^{-k} C \frac{P(\tau)^n}{\sqrt{2\pi n}} + O\left(\frac{1}{n}\right).$$

Note that Formulas (10) and (11) in [5, Theorem 1] give a closed form for $W_k(z)$. It implies in particular

$$0 < W_k(1-q) < (1-q)(c+d)C_1/C_2^{|k|+1},$$

where $C_1 > 0$ and $C_2 > 1$ are constants independent of k; thus $W_k(1-q)$ decays geometrically for $k \to \pm\infty$. This concludes our analysis of Case b) and gives the theorem. □

These limiting behaviors are thus in sharp contrast with the asymptotic behavior of the final altitude of walks on $\mathbb{Z}$ with no resets, which is $\delta n \pm O(\sqrt{n})$, with fluctuations given by a continuous distribution (Rayleigh or Gaussian; see [5]).

2.2. **The height H_n.** In order to study the height of these walks with resets, one considers the subset of them made of walks conditioned to have a height smaller than h. We want to obtain an explicit formula for their generating function

$$F^{\leq h}(z,u) := \sum_{n=0}^{+\infty} \mathbb{E}\Big(u^{Y_n} \mathbb{1}_{\{Y_1 \leq h, Y_2 \leq h, \ldots, Y_n \leq h\}}\Big) z^n.$$

If these walks are generated by a step set $\mathcal{S}$ having only positive jumps, a natural but naive approach to enumerate them would be to create a deterministic finite automaton (a finite discrete Markov chain) with h states encoding the possible altitudes of the process. It leads to a system of linear equations which would allow us to get the corresponding rational generating function. However, this approach to obtain the generating function (given h and the transition probabilities) suffers from three drawbacks:

- it would be of complexity h^3 (computing determinants of $h \times h$ matrices),
- it would be a case-by-case approach (new computations are needed for each h),
- it would fail if the step set $\mathcal{S}$ has some negative steps (then the support of the walk is $[-\infty, +h]$, and thus one would need an automaton with an infinite number of states).

So, we prefer here to use a more efficient approach, which relies on a powerful method (namely, the kernel method [7]): the complexity to obtain a closed-form formula for $F^{\leq h}(z,u)$ then drops[3] from $O(h^3)$ to $O(1)$ for any finite step set $\mathcal{S} \subset \mathbb{Z}$! This leads to the following theorem.

[3]The PhD thesis of Louis Dumont [17] compares the cost of different methods to compute the *coefficients* of such generating functions (which can be related to diagonals of rational functions); the full analysis has to take into account the space and time complexities, and some precomputation steps, of cost of course higher than $O(1)$, but in all cases it is more efficient than a Markov chain approach (see however Bacher [3] for a clever use of a transfer matrix point of view).

Theorem 2.3. *Let $F^{\le h}(z,u)$ be the probability generating function of walks on $\mathbb{Z}$ of height $\le h$ with resets, where the length and the final altitude of the walks are respectively encoded by the exponents of z and u. Let $P(u)$ encode the allowed jumps as in (2.1). One has*

$$F^{\le h}(z,u) = \sum_{n=0}^{+\infty} \mathbb{E}\Big(u^{Y_n} \mathbb{1}_{\{Y_1\le h, Y_2 \le h, \dots, Y_n \le h\}}\Big) z^n = \frac{W^{\le h}(z,u)}{1 - zqW^{\le h}(z,1)}, \tag{2.5}$$

where

$$W^{\le h}(z,u) := \frac{1 - \sum_{i=1}^{d} \left(\frac{u}{u_i}\right)^{h+1} \prod_{1\le j\le d, j\ne i} \frac{u_j - u}{u_j - u_i}}{1 - zP(u)} \tag{2.6}$$

is the generating function of walks of height $\le h$ without reset, and where $u_1, \dots, u_d$ are the roots of $1 - zP(u) = 0$ such that $\lim_{z\to 0} |u_i(z)| = +\infty$.

Remark 2.4 (A *rational* simplification). These generating functions are algebraic, as they rationally depends on the roots $u_i(z)$, which are themselves algebraic functions. Now, when the step set $\mathcal{S}$ has only *positive* steps, $W^{\le h}$ is a polynomial and $F^{\le h}$ simplifies to a rational function (despite the fact that their closed forms (2.6) and (2.5) involve algebraic functions!). This simplification can be seen either by the automaton point of view and the Kleene theorem, or by using the Vieta formulas on Newton sums (as, when one has only positive jumps, the u_i's are then *all* the roots of the kernel $1 - zP(u)$). For example, for $P(u) = u/3 + u^2/2$ and $h = 3$, we have

$$u_1(z) = \frac{-z + \sqrt{z^2 + 18z}}{3z} \qquad \text{and} \qquad u_2(z) = \frac{-z - \sqrt{z^2+18z}}{3z}$$

(the Vieta formulas are here: $u_1(z) + u_2(z) = -2/3$ and $u_1(z)u_2(z) = -2/z$); then, the quotient (2.5) involving these algebraic functions u_1 and u_2 simplifies, leading to

$$\begin{aligned}
W^{\le 3}(z,u) &= \frac{1}{1-zP(u)} \left(1 - \left(\frac{u}{u_1(z)}\right)^4 \frac{u_2(z) - u}{u_2(z) - u_1(z)} - \left(\frac{u}{u_2(z)}\right)^4 \frac{u_1(z) - u}{u_1(z) - u_2(z)}\right) \\
&= 1 + z\left(\frac{u^2}{2} + \frac{u}{3}\right) + z^2 \left(\frac{u^3}{3} + \frac{u^2}{9}\right) + \frac{z^3u^3}{27}, \\
F^{\le 3}(z,u) &= \frac{\left(1 + z\left(\frac{u^2}{2} + \frac{u}{3}\right) + z^2\left(\frac{u^3}{3} + \frac{u^2}{9}\right) + \frac{z^3u^3}{27}\right)}{1 - zq\left(1 + \frac{5z}{6} + \frac{4z^2}{9} + \frac{z^3}{27}\right)}.
\end{aligned}$$

Proof of Theorem 2.3. The probability generating function can be written as

$$F^{\le h}(z,u) = \sum_{n\ge 0} f_n^{\le h}(u) z^n = \sum_{k=0}^{h} F_k^{\le h}(z) u^k,$$

where $f_n^{\le h}(u)$ encodes the possible values of Y_n (constrained to be bounded by h over the full process), and where

$$F_k^{\le h}(z) = \sum_{n=0}^{+\infty} f_{n,k}^{\le h} z^n = \sum_{n=0}^{+\infty} \mathbb{P}\Big(Y_1 \le h,\, Y_2 \le h, \dots, Y_{n-1} \le h,\, Y_n = k \le h\Big) z^n$$

is the probability generating function of bounded walks ending at altitude k.

The dynamics of the process then entails the recurrence

$$f_{n+1}^{\leq h}(u) = P(u) f_n^{\leq h}(u) - \{u^{>h}\} P(u) f_{n,h}^{\leq h} u^h + q f_n^{\leq h}(1),$$

where $\{u^{>h}\}$ extracts monomials having a degree in u strictly larger than h. This mimics that at time $n+1$, either, with probability p_k, we increase by k the altitude of where we were at time n (that is, we multiply by u^k, and this is allowed as long as the walk stays at some altitude $\leq h$, thus we removed here the cases corresponding to the walks which would reach an altitude $> h$ at time $n+1$); or, with probability q, we have a reset to altitude 0 (i.e., all the mass of the walks at any altitude k, corresponding to the coefficient of u^k, is sent back to u^0; this is thus captured by the substitution $u = 1$).

This directly translates to the functional equation

$$F^{\leq h}(z,u) = 1 + zP(u) F^{\leq h}(z,u) - \sum_{k=0}^{d-1} F_{h-k}^{\leq h}(z) u^{h-k} \left(z \sum_{j=k+1}^{d} p_j u^j \right) + zq F^{\leq h}(z,1).$$

Setting $q = 0$, we get the functional equation for the generating function $W^{\leq h}$ of walks of height $\leq h$ without reset:

$$W^{\leq h}(z,u) = 1 + zP(u) W^{\leq h}(z,u) - \sum_{k=0}^{d-1} W_{h-k}^{\leq h}(z) u^{h-k} \left(z \sum_{j=k+1}^{d} p_j u^j \right). \tag{2.7}$$

Of course, the factorization of walks with resets into $(\mathcal{S}^* q)^* (\mathcal{S})^*$ entails $F^{\leq h}(z,u) = \mathrm{Seq}(W^{\leq h}(z,1) q) W^{\leq h}(z,u)$, which is Formula (2.5). So if we find a closed form for $W^{\leq h}$, we are happy as this also solves the initial problem for $F^{\leq h}$. Now, on the right-hand side of (2.7), the sum for k from 0 to $d-1$ is a polynomial in u, which we conveniently rewrite as

$$W^{\leq h}(z,u)(1 - zP(u)) = 1 - u^h \sum_{k=1}^{d} G_k(z) u^k. \tag{2.8}$$

It is possible to solve such an equation via the kernel method: the kernel is the factor $1 - zP(u)$ in (2.8), and if one considers the equation on the variety defined by $1 - zP(u) = 0$, this brings additional equations which will allow us to get a closed form for $W^{\leq h}(z,u)$. First, observe that this kernel is a (Laurent) polynomial in u of "positive" degree d. Then, from an analysis of its Newton polygon, one gets that it has d roots $u_1(z), \ldots, u_d(z)$ such that $u_i(z) \approx z^{-1/d}$ for $z \sim 0^+$ (the other roots being convergent at $z \sim 0^+$; see [5] for more on this issue). Thus, setting $u = u_i(z)$ (for $i = 1, \ldots, d$) in the functional equation (2.8) gives d new equations. Some care is required in this step: we have to check that one does not create series involving an infinite number of monomials with negative exponents[4].

[4]Let R be the ring of series $\sum_{n \in \mathbb{Z}} a_n z^n$. The Cauchy product of two series in R is well defined only with some additional convergence conditions, and, even if we restrict ourselves to series for which the product is well defined, we have to take care to the fact that they do not form an integral ring: indeed, we have many divisors of zero (e.g. for $S(z) := \sum_{n \in Z} z^n$, we have $zS = S$ and thus $(z-1)S = 0$). Most algebraic manipulations in this ring, if they are temporarily handling quantities which are not in the subring of power series (or Laurent/Puiseux/Fourier series), would lead to invalid identities in $\mathbb{C}[[z]]$.

In fact, in our case, the substitution $u = u_i$ is legitimate as $W^{\leq h}(z, u_i)$ becomes a well-defined Puiseux series in z: this follows from the fact that the coefficients $f_n^{\leq h}(u)$ are (Laurent) polynomials with "positive" degree bounded by h (and "negative" degree lower bounded by $-cn$), so $f_n^{\leq h}(u_i(z))$ is a Puiseux series with exponents from $-h/d$ to $+\infty$. Then, multiplying by z^n and summing over n, only a finite number of summands contribute to each monomial of $W^{\leq h}(z, u_i)$, which is thus well defined. Via these substitutions $u = u_i$, we obtain a linear system of d equations (which only contains the G_k's as unknowns). Then, by Cramer's rule, we get $G_k = \det(V_k)/\det(V)$, where

$$V = \begin{pmatrix} u_1^{h+1} & u_1^{h+2} & \dots & {u_1}^{h+d} \\ u_2^{h+1} & u_2^{h+2} & \dots & {u_2}^{h+d} \\ \vdots & \vdots & \vdots & \vdots \\ u_d^{h+1} & u_d^{h+2} & \dots & {u_d}^{h+d} \end{pmatrix} \quad \text{and} \quad V_k = \begin{pmatrix} u_1^{h+1} & \dots & u_1^{h+k-1} & 1 & u_1^{h+k+1} & \dots & {u_1}^{h+d} \\ u_2^{h+1} & \dots & u_2^{h+k-1} & 1 & u_2^{h+k+1} & \dots & {u_2}^{h+d} \\ \vdots & \vdots & \vdots & & \vdots & & \\ u_d^{h+1} & \dots & u_d^{h+k-1} & 1 & u_d^{h+k+1} & \dots & {u_d}^{h+d} \end{pmatrix},$$

that is, V_k is the matrix V with its k-th column entries replaced by 1. Thus, as V is a Vandermonde matrix, its determinant is

$$\det(V) = \left(\prod_{i=1}^{d} u_i^{h+1} \right) \prod_{1 \leq i < j \leq d} (u_j - u_i).$$

Now, to compute $\det(V_k)$, one first proves that

$$\Delta = \det \begin{pmatrix} u_1^1 & \dots & u_1^{k-1} & 1 & u_1^{k+1} & \dots & {u_1}^d \\ u_2^1 & \dots & u_2^{k-1} & 1 & u_2^{k+1} & \dots & {u_2}^d \\ \vdots & \vdots & \vdots & & \vdots & & \\ u_d^1 & \dots & u_d^{k-1} & 1 & u_d^{k+1} & \dots & {u_d}^d \end{pmatrix} = e_{d-k}(u_1, \dots, u_d) \prod_{1 \leq i < j \leq d} (u_j - u_i), \tag{2.9}$$

where we used the classical notation for the elementary symmetric polynomials:

$$e_k(x_1, \dots, x_d) := [t^k] \prod_{i=1}^{d} (1 + t x_i), \tag{2.10}$$

e.g., $e_3(x_1, \dots, x_5) = x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5$. Formula (2.9) follows from 2 facts:

- If $u_i = u_j$, then two rows of V_k are equal and thus the determinant is 0; this explains the Vandermonde product $\Pi := \prod_{1 \leq i<j \leq d}(u_j - u_i)$ on the right-hand side of Formula (2.9).
- Now writing the determinant as a sum over the $d!$ permutations of the entries gives a sum of monomials, each of total degree $(1 + 2 + ... + d) - k$ in the u_i's. Π being of total degree $\binom{d}{2} = d(d-1)/2$, it implies that Δ/Π is a polynomial which is symmetric and homogeneous of total degree $d - k$. Up to a constant factor (determined to be 1, by comparing any monomial), this polynomial has to be e_{d-k}, which captures exactly the missing u_i's in each of the $d!$ summands.

Then, performing a Laplace expansion of $\det(V_k)$ on its k-th column and using Formula (2.9), one gets (after simplification in the Cramer formula):

$$G_k(z) = \sum_{\ell=1}^{d} u_\ell^{-h-1} (-1)^{k+d} e_{d-k}(u_1, \dots, u_d)_{|u_\ell = 0} \prod_{\substack{1 \leq j \leq d \\ j \neq \ell}} \frac{1}{u_\ell - u_j}.$$

Now, using $\sum_{k=0}^{d}(-1)^{d-k}e_{d-k}(u_1,\ldots,u_d)u^k = \prod_{i=1}^{d}(u-u_i)$ (which is equivalent to the definition (2.10)), and regrouping the powers u_k^{-h-1}, we get

$$\sum_{k=1}^{d} G_k(z)u^{k-1} = \sum_{k=1}^{d} u_k^{-h-1} \prod_{1\le j\le d, j\ne k} \frac{u_j-u}{u_j-u_k}. \tag{2.11}$$

Combining Equations (2.11) and (2.7), we get Formula (2.6) for $W^{\le h}(z,u)$, and thus the closed form for $F^{\le h}(z,u)$. □

Remark 2.5 (Link with Lagrange interpolation). As we know the evaluation of the right-hand side of (2.8) in each of the u_k, Formula (2.11) is also equivalent to the Lagrange interpolation formula (which we thus reproved *en passant*). Moreover, this Lagrange interpolation approach offers a nice advantage: it is circumventing the fact that the factorization argument used to get the closed forms for the generating functions in [5, 12] works only if the walks start at altitude 0.

Now, if we go back to Moran walks (i.e., for $P(u) = pu$; see Figure 1), the generating function simplifies to the following noteworthy shape.

Corollary 2.6. *The probability generating function of Moran walks of height $\le h$ is*

$$F^{\le h}(z,u) = \frac{(1-pz)(1-(pzu)^{h+1})}{(1-puz)(1-z+(pz)^{h+1}zq)}, \tag{2.12}$$

where, in the power series, the length and the final altitude of the walks are respectively encoded by the exponents of z and u. Accordingly,

$$\mathbb{P}(H_n \le h) = [z^n]F^{\le h}(z,1) = [z^n]\frac{1-(pz)^{h+1}}{1-z+(pz)^{h+1}zq} \tag{2.13}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{h+1} \rfloor} (-qp^{h+1})^k \left(\binom{n-k(h+1)}{k} - p^{h+1}\binom{n-(k+1)(h+1)}{k} \right), \tag{2.14}$$

with the convention that $\binom{m}{k} = 0$ if $m < 0$.

Proof. The closed form (2.14) is obtained via the power series expansion $1/(1-T) = \sum T^j$ by applying the binomial theorem to each term T^j, with $T = z + (pz)^{h+1}zq$. □

The binomial sum (2.14) generalizes a formula obtained (for $p = 1/2$) by Pippenger in [45]. Therein, it is derived by an inclusion-exclusion principle (guided by the combinatorics of the carry propagation in binary words); for his problem, the generating function, and thus the corresponding binomial sum, are a little bit simpler than (2.13) and (2.14), and are then used to perform some real analysis for the asymptotics of the expected length.

In our case, equipped with this explicit expression for the probability generating function of Moran walks of bounded height, we can now tackle the question of the asymptotic distribution of this extremal parameter.

3. Asymptotic height of Moran walks

In this section, we establish a local limit law for the distribution of the height of Moran walks. One noteworthy consequence of the generating function explicit formula that we get in the previous section is that it allows us to have very efficient computations and simulations of the process at time n, for large n, as stressed by the following remark.

Remark 3.1 (Fast computation scheme for any given n and h). One does not need to run the process for n steps to have the exact distribution of H_n. Indeed, using the rational generating function from Corollary 2.6, for any p, h, and n, it is possible to get the exact value of $\mathbb{P}(H_n = h) = [z^n]\left(F^{\leq h}(z,1) - F^{\leq h-1}(z,1)\right)$ in time $O(\ln(n))$ via binary exponentiation.

This allows us to plot the distribution H_n, for quite large values of n (as an example, see Figure 3). Note that for our other generating functions, which are algebraic, there exists a fast algorithm of cost $\sqrt{n}\ln(n)$ to compute their n-th coefficient (this algorithm works more generally for all D-finite functions). This algorithm due to the brothers Chudnovsky is e.g. implemented in the Maple computer algebra system via the package Gfun; see [49]

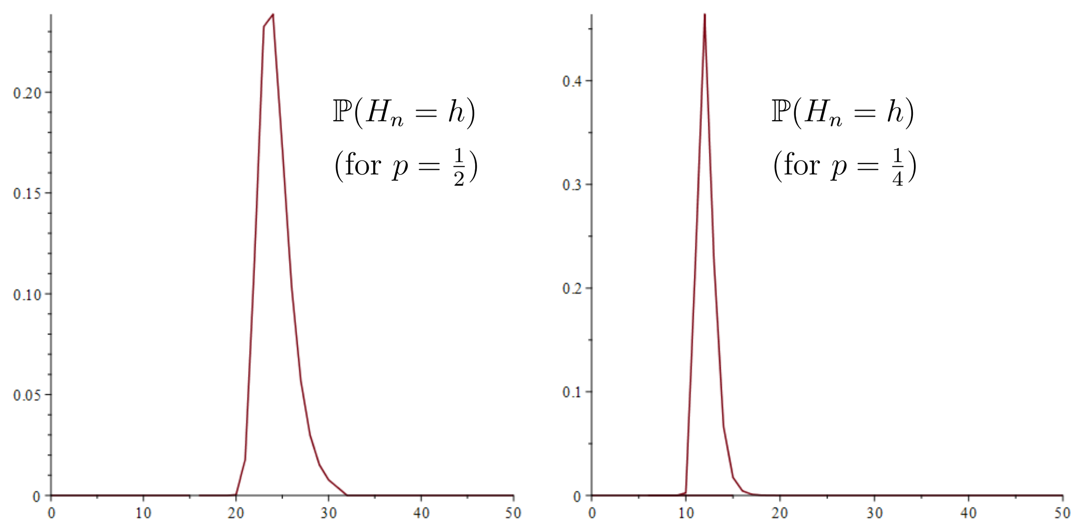


Figure 3. The distribution of H_n, for $n = 2^{25}$ (for $p = 1/2$ on the left and $p = 1/4$ on the right). One observes a sharp concentration around the height 25 for $p = 1/2$ and 12.5 for $p = 1/4$, suggesting a logarithmic link in base $1/p$ between n and H_n. We prove and refine this claim in the next pages.

3.1. **Localization of the dominant singularity.** As $F^{\leq h}(z,1)$ (as given by Equation (2.12)) is a rational function, all its singularities are poles. The asymptotic behavior of the coefficients of $F^{\leq h}(z,1)$ is governed by the closest pole(s) to zero (also called "dominant singularities" of $F^{\leq h}$). A natural candidate for being such a dominant singularity of $F^{\leq h}(z,1)$ would be $z = 1/p$, but it is in fact a removable singularity, as one has (e.g. via L'Hôpital's rule) $F^{\leq h}(1/p,1) = \frac{p(h+1)}{2p-1-qh}$. Thus, we can focus on the other roots of the denominator $D(z)$ of $F^{\leq h}(z,1)$.

Lemma 3.2 (Localization of the singularities of $F^{\leq h}$)**.** *For $p \in (0,1)$, the $h+2$ roots $z_1(h), \ldots, z_{h+2}(h)$ of $D(z) = 1 - z + qp^{h+1}z^{h+2}$ are such that we have for h large enough:*

(i) *$z_1(h)$ is the unique root strictly between 1 and $1/p$;*
(ii) *$z_2(h) = 1/p$ is the unique root of modulus $1/p$;*
(iii) *the remaining h roots $z_3(h), \ldots, z_{h+2}(h)$ are all of modulus $> 1/p$, and arbitrarily close (in modulus) to $1/p$ (for $h \to +\infty$);*
(iv) *all the roots are simple.*

Accordingly, $z_1(h)$ is the dominant singularity of $F^{\leq h}(z,1)$.

Proof. Let $z_*(h)$ be the unique positive zero of $D'(z) = -1 + (h+2)qp^{h+1}z^{h+1}$ given by

$$z_*(h) = \frac{1}{p}\left(\frac{1}{q(h+2)}\right)^{\frac{1}{h+1}}.$$

As $z_*(h)$ tends to $\frac{1}{p}$ from the left, we thus have $0 < z_*(h) < 1/p$ for h large enough. Moreover, $D(z)$ is decreasing for all z in the interval $[0, z_*(h)]$ and increasing in the interval $[z_*(h), +\infty]$. As $D(1/p) = 0$, one thus has $D(z_*(h)) < 0$. And since $D(1) > 0$, the intermediate value theorem implies the existence of (at least) one zero of D between 1 and $z_*(h)$. Combined with the (non)decreasing properties of D, this entails the unicity of this zero; let us call it $z_1(h)$. Then, Pringsheim's theorem (see e.g. [22]) asserts that $F^{\leq h}$ has a real positive dominant singularity which is thus $z_1(h)$, the first real positive zero of D. As $F^{\leq h}(z)$ is a probability generating function, all its singularities are of modulus ≥ 1. So we have $1 < z_1(h) < z_*(h) < 1/p$ and thus proved (i).

We now prove (ii). The fact that $z_2(h) = 1/p$ is a root follows from $1 - 1/p + q/p = 0$. Is there any other root of the same modulus? If $z = \exp(i\theta)/p$ (with $\theta \in [0, 2\pi]$) would be a root of $D(z)$, then this would imply $p = \exp(i\theta) - q\exp(i(h+2)\theta)$. By the reverse triangle inequality $\Big||x| - |y|\Big| \leq |x - y|$ (with equality only if $xy = 0$ or $x/y \in \mathbb{R}^+$), this would entail $\theta = 0$.

To prove (iii), we use the following version of Rouché's theorem: if $|D - g| < |g|$ on the boundary of a disk $\mathcal{D}$, then D and g have the same number of roots inside $\mathcal{D}$. We can apply this theorem to D with $g(z) := 1 - z$, for the disk $\mathcal{D}(0, \frac{1-\epsilon}{p})$: on its boundary, one indeed has $|D(z) - g(z)| = \frac{q}{p}|pz|^{h+2} \leq \frac{q}{p}|1-\epsilon|^{h+2} < \frac{q}{p}|1-\epsilon|^{2/q} < \frac{q-\epsilon}{p} \leq |g(z)|$, where the first strict inequality holds for $h \geq 2/q$ and the next strict inequality holds for any small enough ϵ (independently of h), as we have then $\frac{\ln(1-\epsilon/q)}{\ln(1-\epsilon)} < 2/q$. As the constraint on h is independent of ϵ, letting $\epsilon \to 0$, we infer that D has only one root strictly inside $\mathcal{D}(0, \frac{1}{p})$.

Now we can also apply this theorem to D with $g(z) := 1 + z^{h+2}$: on the boundary of the disk $\mathcal{D}(0, \frac{1+\epsilon}{p})$, one indeed has, for h large enough (depending on ϵ),

$$|D(z) - g(z)| \leq \left(\frac{1+\varepsilon}{p}\right)^{h+2}\left(1 - qp^{h+1}\right) + \frac{1+\varepsilon}{p} < \left(\frac{1+\varepsilon}{p}\right)^{h+2} - 1 \leq |g(z)|,$$

where the last -1 is just a crude bound of the term $-\frac{q}{p}(1+\varepsilon)^{h+2} + \frac{1+\varepsilon}{p}$ which converges to $-\infty$ for $h \to +\infty$. So D, like g, has $h+2$ roots inside this disk.
To prove (iv), note that the equation $D(z) = D'(z) = 0$ is forcing $z = 1 + \frac{1}{h+1}$, but $D'(1 + \frac{1}{h+1}) \to -1$ for $h \to +\infty$, therefore all the zeros are simple for h large enough. ☐

See Figure 5 on page 277 for an illustration of the location of the roots.

3.2. **Limit distribution of the height: the discrete Gumbel distribution.** The height distribution exhibits some a priori surprising asymptotic aspects, having a flavor of number theory/Diophantine approximation. Such phenomena, however, appear for a few other probabilistic processes where some statistics could have different asymptotic behaviors depending on some resonance between $\ln p$ and $\ln q$ (see e.g. Janson [36] or Flajolet, Vallée, and Roux [21] for some examples related to tries or binary search trees). In our case, it appears that a resonance between $\ln p$ and $\ln n$ plays a role.

Theorem 3.3 (Distribution of the height of Moran walks)**.** *We have*

$$\mathbb{P}\left(H_n \leq h\right) = \exp\left(-qnp^{h+1}\right)\left(1 + O\left(\frac{(\ln n)^3}{n}\right)\right), \tag{3.1}$$

where the error term is uniform for $h \in [0, n]$. Accordingly, $\mathbb{P}(H_n = h)$ is unimodal, with a peak at $h = h^*(n)$, the closest integer to $c^*(n)\frac{\ln(n)}{\ln(1/p)}$, where $c^*(n) := 1 - \frac{\ln(\ln(1/p)/q^2)}{\ln(n)}$, and we have

$$\mathbb{P}(H_n = h^*(n)) \sim p^{p/q} - p^{1/q}.$$

Moreover, the mass is sharply concentrated around $\frac{\ln n}{\ln(1/p)}$, as better seen by the following result, with a uniform error term in k:

$$\mathbb{P}\left(H_n \leq \left\lfloor \frac{\ln n}{\ln(1/p)} \right\rfloor + k\right) = \exp\left(-q\alpha(n)p^{k+1}\right)\left(1 + O\left(\frac{(\ln n)^3}{n}\right)\right),$$

with $\alpha(n) := p^{-\left\{\frac{\ln n}{-\ln p}\right\}}$ (where $\{x\}$ stands for the fractional part of x, and where $\lfloor x \rfloor$ stands for the floor function of x). [See Figure 3 on page 274 for an illustration of the distribution of H_n and Figure 4 for the behavior of the function $\alpha(n)$.]

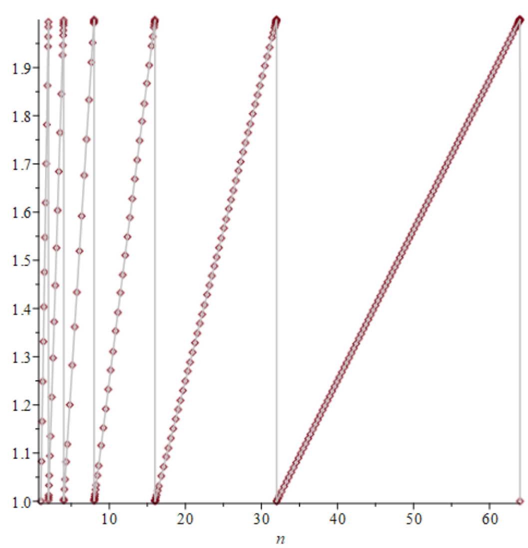


FIGURE 4. Plot of the function $\alpha(n) = p^{-\left\{\frac{\ln n}{-\ln p}\right\}}$ (for $p = 1/2$), which occurs in the fluctuations of the height of Moran walks (as stated in Theorem 3.3). The function $\alpha(n)$ is taking values in $[1, 1/p)$ for integers $n \geq 1$. It has a sawtooth wave shape, with frequencies getting larger and larger (with peaks at powers of $1/p$).

Proof. In the sequel, as the context is explicit, we simply denote by $z_1, \dots, z_{h+2}$ the zeros $z_1(h), \dots, z_{h+2}(h)$ of $D(z) = 1 - z + qp^{h+1}z^{h+2}$. From Lemma 3.2, for h large enough, all these zeros z_i are simple; the partial fraction decomposition of $1/D$ is then

$$\frac{1}{D(z)} = \sum_{i=1}^{h+2} \frac{1}{D'(z_i)\,(z - z_i)}$$

and as $D'(z_i) = -1 + (h+2)(z_i - 1)/z_i$, one thus gets

$$\begin{aligned}
F^{\leq h}(z,1) &= \frac{1-(pz)^{h+1}}{D(z)} = \sum_{i=1}^{h+2} \frac{1-(pz)^{h+1}}{D'(z_i)\,(z-z_i)} \\
&= \sum_{i=1}^{h+2} \left(\frac{1}{z_i - (z_i - 1)\,(h+2)} \left(\sum_{n=0}^{+\infty} z_i^{-n} z^n \right) - \frac{p^{h+1}}{z_i - (z_i - 1)\,(h+2)} \sum_{n=h+1}^{+\infty} z_i^{-n+h+1} z^n \right) \\
&= \sum_{i=1}^{h+2} \left(\frac{1}{z_i - (z_i - 1)\,(h+2)} \left(\sum_{n=0}^{h} z_i^{-n} z^n \right) + \frac{1-(pz_i)^{h+1}}{z_i - (z_i - 1)\,(h+2)} \sum_{n=h+1}^{+\infty} z_i^{-n} z^n \right).
\end{aligned}$$

It is combinatorially obvious that $\mathbb{P}\left(H_n \leq h\right) = 1$ for all $n \leq h$. So we now focus on $n > h$, for which we have, as $(pz_i)^{h+1} = \frac{z_i - 1}{qz_i}$ and $1 - \frac{z_i-1}{qz_i} = \frac{1-pz_i}{qz_i}$:

$$\begin{aligned}
\mathbb{P}\left(H_n \leq h\right) = [z^n]F^{\leq h}(z,1) &= \sum_{i=1}^{h+2} \frac{1-(pz_i)^{h+1}}{z_i - (z_i - 1)\,(h+2)} z_i^{-n} \\
&= \sum_{i=1}^{h+2} \frac{1 - pz_i}{q\,(1 + (1 - z_i)\,(h+1))} z_i^{-n-1} \\
&= Z_1(n,h) + O\left(hMp^{n+1}\right),
\end{aligned} \tag{3.2}$$

where $M = \max_{i=3,\dots,h+2} \left| \frac{1-pz_i}{q(1+(1-z_i)(h+1))} \right| = O(1)$ (note that the summand involving $z_2 = 1/p$ cancels), and where $Z_1(n,h) := \frac{1-pz_1}{q[1+(1-z_1)(h+1)]} z_1^{-n-1}$ is the contribution coming from the pole z_1.

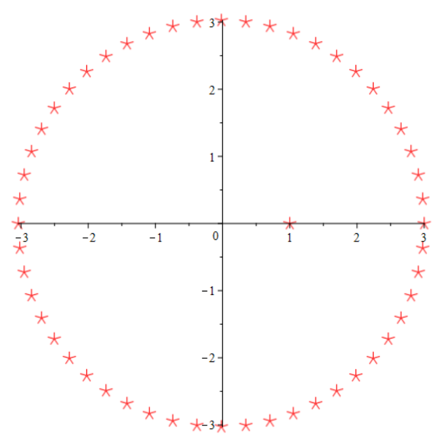


FIGURE 5. The roots of $D(z) = 1 - z + qp^{h+1}z^{h+2}$ (here, with $p = 1/3$ and $h = 51$). For large h, $D(z)$ has one dominant root z_1 just after 1, one root at $z = 1/p$, and the other roots have a slightly larger modulus, all asymptotically close to the circle $|z| = 1/p$; see Lemma 3.2.

Set $z_1 := 1 + \varepsilon_h$. Then $D(z_1) = 1 - (1+\varepsilon_h) + qp^{h+1}(1+\varepsilon_h)^{h+2} = 0$, thus this implies $\varepsilon_h = qp^{h+1}(1+\varepsilon_h)^{h+2}$; therefore we have $z_1 = 1 + \varepsilon_h = 1 + qp^{h+1} + O(hp^{2h})$. Now, for $h = h(n)$ tending to $+\infty$, this entails that the contribution $Z_1(n,h)$ of the pole z_1 (as given by Equation (3.2)) satisfies

$$\begin{aligned} Z_1(n,h) &= \frac{1 - p^{h+2} + O\left(hp^{2h}\right)}{1-(h+1)qp^{h+1} + O(h^2p^{2h})}(1+\varepsilon_h)^{-n-1} \\ &= \left(1 + q(h+1)p^{h+1} - p^{h+2} + O(h^2p^{2h})\right)\exp\left((n+1)\ln\left(\frac{1}{1+\varepsilon_h}\right)\right) \\ &= \left(1 + q(h+1)p^{h+1} - p^{h+2} + O(h^2p^{2h})\right)\exp\left(-(n+1)\varepsilon_h + \Theta((n+1)\varepsilon_h^2)\right). \end{aligned} \tag{3.3}$$

Observe that

$$\text{if} \qquad h = c\frac{\ln(n)}{\ln(1/p)} + c'\frac{\ln(\ln(n))}{\ln(1/p)} \qquad \text{then} \qquad p^h = \frac{1}{n^c \ln(n)^{c'}}. \tag{3.4}$$

(Here and in the sequel we always consider $c > 1/2$ and $c' \geq 0$. In fact, $c' > 0$ is not needed right now, but this will be required for the asymptotics of the mean of H_n in Section 4.)

For such values of h, the asymptotics of the first factor in Equation (3.3) is

$$1 + q(h+1)p^{h+1} - p^{h+2} + O(h^2p^{2h}) = 1 + O\left(\frac{1}{n^c\ln(n)^{c'-1}}\right), \tag{3.5}$$

and the asymptotics of the second factor in Equation (3.3) is

$$\begin{aligned} \exp\left(-(n+1)\varepsilon_h + O((n+1)\varepsilon_h^2)\right) &= \exp\left(-nqp^{h+1} + O(nhp^{2h}) - \varepsilon_h + \Theta(n^{1-2c}/\ln(n)^{2c'})\right) \\ &= \exp\left(-nqp^{h+1}\right)\left(1 + O(n^{1-2c}\ln(n)^{1-2c'}) - O(n^{-c}\ln(n)^{-c'}) + \Theta(n^{1-2c}/\ln(n)^{2c'}))\right). \end{aligned}$$

In this expansion, one now has to check which error term dominates. It is the big-oh term with n^{-c} if $c > 1$ and the big-oh with n^{1-2c} if $c \leq 1$. Multiplying with the asymptotic expansion from Equation (3.5) and using the approximation (3.2), we get the following result (in which we simplified the ln part of the error term in a non-optimal way which will be enough for our purpose):

$$\mathbb{P}(H_n \leq h) = \exp\left(-nqp^{h+1}\right)\left(1 + O\left(\frac{\ln n}{n^{\min(c,2c-1)}}\right)\right). \tag{3.6}$$

Moreover, this approximation holds for all $h \in [0,n]$: first, for $h \ll \frac{1}{2}\ln(n)/\ln(1/p)$ this follows from the fact that $\mathbb{P}(H_n \leq h)$ is increasing with respect to h, and then for $h \gg c\ln(n)$ this follows from the bound (4.8) hereafter.

In conclusion, for $h = \left\lfloor \frac{\ln n}{\ln(1/p)} \right\rfloor + k$, for any k such that $h \in \left[c_1\frac{\ln(n)}{\ln(1/p)}, c_2\frac{\ln(n)}{\ln(1/p)}\right]$ (with $1/2 < c_1 < c_2$), we have uniformly in k (when $n \to +\infty$):

$$\begin{aligned} \mathbb{P}(H_n \leq h) &= \exp\left(-nqp^{\lfloor \frac{\ln n}{\ln(1/p)} \rfloor + k + 1}\right)\left(1 + O\left(\frac{(\ln n)^3}{n}\right)\right) \\ &= \exp\left(-qp^{-\{\frac{\ln n}{-\ln p}\} + k + 1}\right)\left(1 + O\left(\frac{(\ln n)^3}{n}\right)\right), \end{aligned}$$

and we get Theorem 3.3 by setting $\alpha(n) := p^{-\{\frac{\ln n}{-\ln p}\}}$. □

If $p = q = 1/2$, we have $\alpha(n) = 2^{\{\lg(n)\}}$ (where the symbol lg stands for the binary logarithm, $\lg(x) = \log_2(x)$). This subcase of particular interest corresponds to a problem initially considered in 1946 by Burks, Goldstine, and von Neumann [13]: the study of carry propagation in computer binary arithmetic; it constitutes one of the first analyses of the cost of an algorithm! They gave crude bounds which were deeply improved by Knuth in 1978 [38]. This problem can also be seen as runs in binary words, and, as such, is analyzed by Flajolet and Sedgewick [22, Theorem V.1]. Therein, the analysis unfortunately contains a few typos which affect some of the error terms. Our proofs are incidentally fixing this issue.

These extremal parameters (runs, longest carry) are archetypal examples of problems leading to a Gumbel distribution (or a discrete version of it). This distribution indeed often appears in combinatorics as the distribution of parameters encoding a maximal value: e.g., maximum of i.i.d. geometric distributions [51], longest repetition of a pattern in lattice paths [46], runs in integer compositions [23], carry propagation in signed digit representations [30], largest part in some integer compositions, longest chain of nodes with a given arity in trees, maximum degree in some families of trees [47], the maximum protection number in simply generated trees [31]. For some of these examples, it was proven only for some specific families of structures, but there is no doubt that it holds generically. A general framework leading to such double exponential laws is given by Gourdon [26, Theorem 4] for the largest component in supercritical composition schemes (see also Bender and Gao [10]). We refer to Figure 6 for an illustration of some of these parameters.

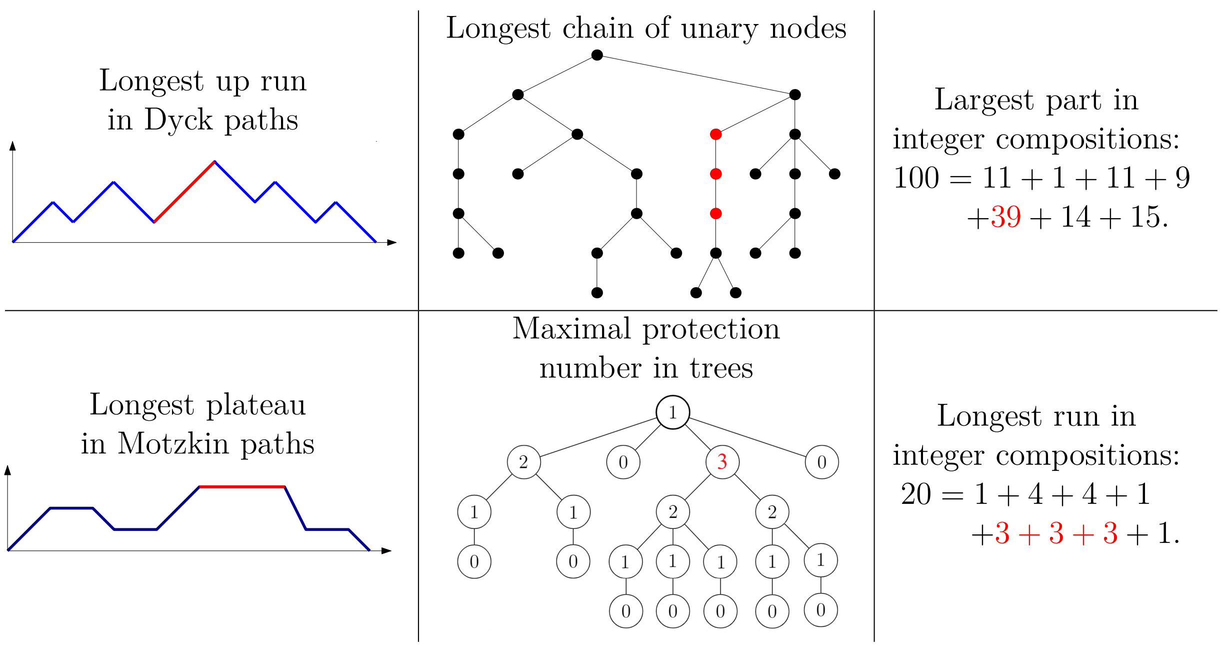


FIGURE 6. Many combinatorial structures have some parameters which asymptotically follow a discrete Gumbel distribution.

The Gumbel distribution is also called the “double exponential distribution”, or the “type-I generalized extreme value distribution”, and can also be expressed as a subcase of the Fisher–Tippett distribution. Let us give a formal definition.

Definition 3.4 (Gumbel distribution)**.** A continuous random variable X with support $[-\infty, +\infty]$ follows a *Gumbel distribution* (of parameters μ and β), denoted by Gumbel(μ, β), if

$$\mathbb{P}(X \leq x) = \exp\left(-\exp\left(-\frac{x-\mu}{\beta}\right)\right).$$

Its mean satisfies $\mathbb{E}[X] = \mu + \gamma\beta$ (where $\gamma = 0.5772\ldots$ is Euler's constant) and its variance satisfies $\mathbb{V}\mathrm{ar}[X] = \frac{\pi^2}{6}\beta^2$. It is unimodal with a peak at $x = \mu$ and its median is at $x = \mu - \beta\ln(\ln(2))$.

Definition 3.5 (Discrete Gumbel distribution)**.** A discrete random variable Y follows a *discrete Gumbel distribution* of parameters μ and β, which we denote Gumbel(μ, β)[5], if

$$\mathbb{P}(Y \leq h) = \exp\left(-\exp\left(-\frac{h-\mu}{\beta}\right)\right), \qquad \text{for all } h \in \mathbb{Z}. \tag{3.7}$$

In particular, one can always write $Y = \lceil X \rceil$, where X follows a continuous Gumbel(μ, β); note on the other side that $\lfloor X \rfloor$ follows a discrete Gumbel$(\mu - 1, \beta)$.

To obtain a nice formula for the mean and variance of a discrete Gumbel distribution remains an open problem: for example, for $Y \stackrel{d}{=}$ Gumbel$(0, 1)$, we have

$$\mathbb{E}[Y] = \sum_{h=-\infty}^{\infty} h\left(\exp(-\exp(-h)) - \exp(-\exp(-h+1)\right) = 1.077240905953631072609\ldots$$

(and it takes 5 seconds to get thousands of digits, as the terms decrease doubly exponentially fast), but will anybody find a closed form for this mysterious constant? Some insight on the variance of the discrete distribution Y can be obtained from the continuous distribution X via the following trivial but useful bounds which hold more generally as soon as $|X - Y| < 1$:

$$|\mathbb{E}[Y] - \mathbb{E}[X]| < 1 \qquad \text{and} \qquad |\mathbb{V}\mathrm{ar}[Y] - \mathbb{V}\mathrm{ar}[X]| < 2 + 4|\mathbb{E}[X]|. \tag{3.8}$$

We can now restate our previous theorem in terms of this discrete Gumbel distribution.

Corollary 3.6 (Gumbel limit law)**.** *The sequence of random variables* $\lceil H_n - \frac{\ln(pqn)}{\ln(1/p)} \rceil$ *converges for* $n \to +\infty$ *(in distribution and in moments) to the discrete* Gumbel$(0, \beta)$ *distribution with* $\beta = \frac{1}{\ln(1/p)}$*. Accordingly, it implies that*

$$\mathbb{E}[H_n] \sim \frac{\ln(pqn)}{\ln(1/p)} + \gamma\beta + \textit{an error smaller than } 1,$$

$$\mathbb{V}\mathrm{ar}[H_n] \sim \frac{\pi^2}{6\ln(p)^2} + \textit{ an error smaller than } 2 + 4\gamma\beta.$$

Proof. Consider the sequence of random variables $Y_n := \lceil H_n - \mu_n \rceil$. Then, the change of variable $h \mapsto h + \mu_n$ in Equation (3.1), with $\mu_n = \frac{\ln(pqn)}{\ln(1/p)}$ allows us to match $Y := \lim_n Y_n$ (where the limit is in distribution) with the discrete Gumbel defined in (3.7), for $\mu = 0$ and $\beta = \frac{1}{\ln(1/p)}$. Due to the exponentially small uniform error term in (3.1) on the support $[0, n]$ of H_n, we have a convergence in moments of Y_n to Y. Then, the asymptotics of the moments follow by applying the bounds (3.8) on the link between the mean/variance of the discrete and continuous Gumbel distribution. □

[5]With a slight abuse of notation, we use the same notation Gumbel(μ, β) for both the continuous distribution and the discrete distribution, adding the right adjective if needed to remove any ambiguity.

These moment asymptotics already constitute a notable result (falling as a good ripe fruit!), but a very interesting phenomenon is hidden in these imprecise errors terms: some bodacious fluctuations, that we fully describe in Section 4.

3.3. **Waiting time.** Let us end this section with an application to a natural statistic: the waiting time τ_h, i.e., the number of steps spent by the random walk when it reaches a given altitude h for the first time. There is an intimate relationship between height and waiting time (stated more formally in Equation (3.11) hereafter); it is thus natural that they have enumerative and asymptotic formulas of a similar nature, as better shown by the following corollary.

Corollary 3.7. *The waiting time τ_h for reaching height h satisfies*

$$\mathbb{P}(\tau_h = n) = [z^n]\frac{(1-pz)p^h z^h}{1 - z + qp^{h-1}z^h}. \tag{3.9}$$

The distribution function of τ_h satisfies

$$\mathbb{P}(\tau_h \le n) = 1 - \exp\left(-q\alpha(n)^2 np^h\right) + O\left(\frac{(\ln n)^3}{n}\right). \tag{3.10}$$

Proof. Consider a walk reaching for the first time altitude h at time n. Cut it after each reset. It gives a sequence of factors of length $k \le h$, followed by a last factor with h up steps. This translates into the combinatorial formula

$$\mathbb{P}(\tau_h = n) = [z^n]\frac{p^h z^h}{1 - \sum_{k=1}^{h-1} p^{k-1} q z^k},$$

which simplifies to Formula (3.9). Now, for the distribution function, instead of redoing a full analysis based on a partial fraction decomposition of this generating function, it is more convenient to use the relation

$$\mathbb{P}(\tau_h = n) = \mathbb{P}(H_n = h \text{ and } H_{n-1} < h), \tag{3.11}$$

thus this waiting time also satisfies

$$\mathbb{P}(\tau_h \le n) = \mathbb{P}(H_n \ge h) = 1 - \mathbb{P}(H_n \le h-1). \tag{3.12}$$

Then, using Theorem 3.3, we also have

$$\begin{aligned}
\mathbb{P}(H_n \le h-1) &= \mathbb{P}\left(H_n \le \left\lfloor \frac{\ln n}{\ln(1/p)} \right\rfloor + h - 1 - \left\lfloor \frac{\ln n}{\ln(1/p)} \right\rfloor\right) \\
&= \exp\left(-q\alpha(n)p^{h-\left\lfloor \frac{\ln n}{\ln(1/p)} \right\rfloor}\right) + O\left(\frac{(\ln n)^3}{n}\right) \\
&= \exp\left(-q\alpha(n)^2 p^{h+\frac{\ln n}{\ln p}}\right) + O\left(\frac{(\ln n)^3}{n}\right).
\end{aligned}$$

Via Formula (3.12) linking the waiting time τ_h and the height H_n, this entails (3.10). $\square$

We now turn to a finer analysis of the mean and variance of H_n.

4. Mean and variance of the height

4.1. **Fundamental properties of the Mellin transform.** In order to get a fine estimation of the average height, we use a Mellin transform, which, as we shall see, is the key tool to handle the corresponding asymptotics. We now present the needed definitions and formulas. We refer e.g. to Flajolet, Gourdon, and Dumas [19] or to the book *Analytic Combinatorics* [22, Appendix B.7] for more on the Mellin transform and numerous applications to asymptotics of harmonic sums, digital sums, and divide-and-conquer recurrences.

Definition 4.1 (Mellin transform)**.** Let $f(t)$ be a continuous function defined on the positive real axis $0 < t < +\infty$. The Mellin transform f^* of f is the function defined by

$$f^*(s) := \int_0^{+\infty} f(t) t^{s-1} dt.$$

This integral exists only for s such that the function $f(t)t^{s-1}$ is integrable on $(0, +\infty)$. Thus, if there exist two real numbers a and b, such that $a > b$ and

$$f(t) = \begin{cases} O(t^a), & \text{if } t \to 0 \\ O(t^b), & \text{if } t \to +\infty \end{cases}, \tag{4.1}$$

then the function f^* is well defined for any complex number s with real part such that $-a < \Re(s) < -b$; this domain is called the fundamental strip of f^*. Moreover, for all c in this domain, if $f^*(s)$ converges uniformly to 0 for $s = c \pm i\infty$, then the function f can be expressed for $t \in (0, +\infty)$ as the following inverse Mellin transform:

$$f(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} f^*(s) t^{-s} ds. \tag{4.2}$$

As an example, let us consider the gamma function, which illustrates well the role of the fundamental strip (and this example will also play a role in the next pages).

Example 4.2 (The gamma function as a Mellin transform)**.** The gamma function satisfies

$$\Gamma(s) = \int_0^{+\infty} \exp(-t) t^{s-1} dt \qquad (\text{for } 0 < \Re(s) < +\infty),$$

$$\Gamma(s) = \int_0^{+\infty} (1 - \exp(-t))\, t^{s-1} dt \qquad (\text{for } -1 < \Re(s) < 0). \tag{4.3}$$

An important consequence of Formula (4.2) is that, if f is a meromorphic function on $\mathbb{C}$, and if $\lim_{c\to+\infty} \int_{c-i\infty}^{c+i\infty} f^*(s)t^{-s} ds = 0$, then one can push the integration contour of Formula (4.2) to the right (taking $\lim_{c\to+\infty}$) and one then collects in passing the contributions from the residue at each pole s_k to the right of the fundamental strip. Now, for $t > 0$ and $a \in \mathbb{C}$, multiplying $t^{-s} = t^{-a} \sum_{\ell \geq 0} \ln(t)^\ell (a-s)^\ell / \ell!$ by the Laurent series of $f^*(s)$ at $s = s_k$, we see that $\mathrm{Res}[f^*(s)t^{-s}, s_k]$ can be expressed[6] as a sum of $\mathrm{order}(s_k)$ terms, and one gets

$$\begin{aligned} f(t) &= \sum_{\substack{s_k \text{ pole of } f^*(s)t^{-s} \\ \Re(s_k) \geq -b}} \mathrm{Res}[f^*(s)t^{-s}, s_k] \\ &= \sum_{\substack{s_k \text{ pole of } f^* \\ \Re(s_k) \geq -b}} \sum_{j=1}^{\mathrm{order}(s_k)} \mathrm{Res}[(s-s_k)^{j-1} f^*(s), s_k]\, t^{-s_k} \frac{(-1)^j}{(j-1)!} (\ln t)^{j-1}. \end{aligned} \tag{4.4}$$

[6] The notation $\mathrm{Res}[g(s), s_k]$ stands for the residue of $g(s)$ at $s = s_k$.

4.2. **Average height of Moran walks.** We now state the main result of this section.

Theorem 4.3 (Average height). *The average height of Moran walks of length n is given by*

$$\mathbb{E}[H_n] = \frac{\ln n}{\ln(1/p)} - \frac{\gamma}{\ln p} - \frac{1}{2} - \frac{\ln q}{\ln p} + \frac{Q(\ln(qn))}{\ln p} + O\left(\frac{(\ln n)^4}{n}\right), \tag{4.5}$$

where $\gamma = .57721\ldots$ is Euler's constant, and where Q is an oscillating function (a Fourier series of period $\ln(1/p)$) given by

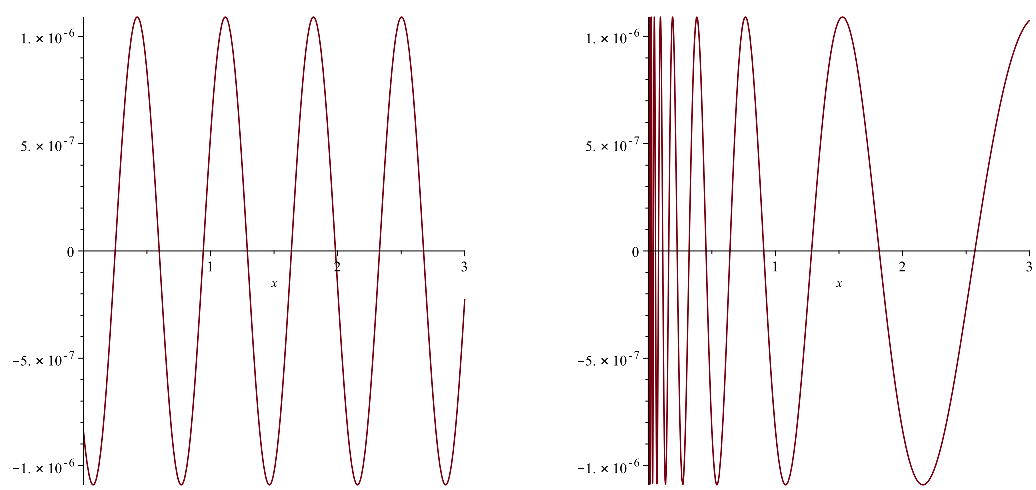
$$Q(x) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma(s_k)\exp(-s_k x) \quad \textit{where } s_k := \frac{2ik\pi}{\ln p}. \tag{4.6}$$

Remark 4.4 (Fourier series representation). The fact that Q is a Fourier series of period $\ln(1/p)$ and is real for $x \in \mathbb{R}$ is better seen via the alternative equivalent expression

$$Q(x) = 2\sum_{k\geq 1}\left(\Re(\Gamma(s_k))\cos\left(\frac{2k\pi x}{\ln(p)}\right) + \Im(\Gamma(s_k))\sin\left(\frac{2k\pi x}{\ln(p)}\right)\right),$$

where $\Re$ and $\Im$ stands for the real and imaginary parts. This is illustrated in Figure 7.

Remark 4.5 (Fourier series differentiability). Such asymptotics involving fluctuations dictated by a Fourier series are typical of results obtained via Mellin transforms. They often appear in the asymptotic cost of divide-and-conquer algorithms, or of expressions involving digital sums, harmonic sums, or finite differences (see the work of de Bruijn, Knuth, and Rice [15, 38], or Flajolet, Gourdon, and Dumas [19]). It is sometimes also possible to get them via some real analysis (like Pippenger did [45]), or like in the seminal work of Delange [16] on the sum of digits. Note that the Delange series is nowhere differentiable, while our Fourier series is infinitely differentiable, as proven in Theorem 4.10.



$Q(x)$ (for $p = 1/2$) $Q(\ln(px))$ (for $p = 1/2$)

FIGURE 7. The height of Moran walks involves asymptotic fluctuations encoded by a Fourier series $Q(x)$, of period $\ln(1/p)$, and weak amplitude. More precisely, it involves $Q(\ln(px))$ which thus oscillates an infinite number of times for $x \to 0^+$, and these oscillations get larger and larger for $x \to +\infty$. Moreover, Q oscillates faster when p tends to 1. We shall encounter later another Fourier series, $R(x)$, which shares all these properties.

Proof of Theorem 4.3. The proof exploits the fact that the mean $\mathbb{E}[H_n]$ asymptotically behaves like $\sum_{h=0}^{+\infty}\left(1-\exp(-nqp^{h+1})\right)$; this is proven by rewriting $\mathbb{E}[H_n]$ as follows:

$$\mathbb{E}[H_n] = \sum_{h=0}^{n} \left(1 - \mathbb{P}\left(H_n \le h\right)\right) = \Sigma_0 + \Sigma_1 + \Sigma_2 + \Sigma_3 - \Sigma_4 + \Sigma_\infty, \tag{4.7}$$

with

$$\begin{aligned}
\Sigma_0 &:= \sum_{0\le h<h_1} \left(\exp(-nqp^{h+1}) - \mathbb{P}\left(H_n \le h\right)\right),\\
\Sigma_1 &:= \sum_{h_1\le h<h_2} \left(\exp(-nqp^{h+1}) - \mathbb{P}\left(H_n \le h\right)\right),\\
\Sigma_2 &:= \sum_{h_2\le h<h_3} \left(\exp(-nqp^{h+1}) - \mathbb{P}\left(H_n \le h\right)\right),\\
\Sigma_3 &:= \sum_{h_3\le h\le n} \left(1 - \mathbb{P}\left(H_n \le h\right)\right),\\
\Sigma_4 &:= \sum_{h=h_3}^{+\infty} \left(1 - \exp(-nqp^{h+1})\right),\\
\Sigma_\infty &:= \sum_{h=0}^{+\infty} \left(1 - \exp(-nqp^{h+1})\right).
\end{aligned}$$

The key is to prove that, for some h_1, h_2, and h_3 adequately chosen, the sums $\Sigma_0, \Sigma_1, \Sigma_2$, Σ_3, and Σ_4 are asymptotically negligible, while the main contribution to $\mathbb{E}[H_n]$ comes from the last sum (namely, Σ_∞), which we will evaluate via a Mellin transform approach.

The reader not enjoying delta-epsilon proofs could have the feeling that "cutting epsilons into 5 parts" like above is a little bit discouraging but this is the price to pay to get the $O((\ln(n)^4/n)$ error term in Formula (4.5). In fact, in Equation (4.7) for $\mathbb{E}[H_n]$, it is possible to cut the sum into only 4 parts, but then this would lead to a final weaker $O(1/\sqrt{n})$ error term.

So let's be brave and begin with Σ_0. Here, for the range $0 \le h < h_1$, with $h_1 := \frac{3}{4}\frac{\ln(n)}{\ln(1/p)}$, we get

$$\begin{aligned}
|\Sigma_0| &\le h_1 \times \left(\max_{0\le h<h_1} \left(\exp(-nqp^{h+1}) + \max_{0\le h<h_1} \mathbb{P}\left(H_n \le h\right)\right)\right)\\
&= h_1 \times \left(\exp(-nqp^{h_1+1}) + \mathbb{P}\left(H_n \le h_1\right)\right)\\
&= h_1 \times \left(2\exp(-qpn^{1/4}) + O\left(\frac{(\ln n)^3}{n}\right)\right)\\
&= O\left(\frac{(\ln n)^4}{n}\right),
\end{aligned}$$

where, for the second line we used that the sequences are increasing with respect to h, and for the third line we used Formula (3.4) for p^h and the approximation of Theorem 3.3. Note that this bound for $|\Sigma_0|$ also implies the uniform bound

$$\mathbb{P}(H_n \le h) = O\left(\frac{(\ln n)^4}{n}\right) \qquad (\text{for } h < h_1).$$

Now, for Σ_1, in the range $h_1 \le h < h_2$, with $h_2 := \frac{\ln(n)}{\ln(1/p)} + \frac{\ln(\ln(n))}{\ln(1/p)}$, we rewrite h as $h := (1-t)h_1 + th_2$. Such values of h correspond to using $c = (t+3)/4$ and $c' = t$ in the Formula (3.4) for p^h.

Via the exponential bound on H_n from Formula (3.6), we get

$$|\Sigma_1| \le (h_2 - h_1) \times \left(\max_{h_1 \le h < h_2} \left(\exp(-nqp^{h+1}) + \max_{h_1 \le h < h_2} \mathbb{P}\left(H_n \le h\right) \right) \right)$$
$$\le h_2 \times \left(\exp(-nqp^{h_2+1}) + \mathbb{P}\left(H_n \le h_2\right) \right) = O((\ln n)^4/n).$$

Then, for Σ_2, in the range $h_2 \le h_3$, with $h_3 := \frac{4\ln(n)}{\ln(1/p)}$, we rewrite h as $h := (1-t)h_2 + th_3$. Such values of h correspond to using $c = 1+3t$ and $c' = 1-t$ in the Formula (3.4) for p^h. Via Formula (3.6), we get $|\Sigma_2| = O((\ln n)^3/n)$.

For the next sum, using the power series expansion of the exponential in Equation (3.3) (and keeping in mind that our choice of h_3 implies $p^{h_3} = 1/n^4$), we get

$$\Sigma_3 = \sum_{h=h_3}^{n} \left(1 - \mathbb{P}\left(H_n \le h\right)\right) \le (n+1-h_3)\left(1 - \mathbb{P}\left(H_n \le h_3\right)\right)$$
$$\le n(1 - \exp(-(n+1)qp^{h_3+1}))(1+o(1)) = O\left(\frac{1}{n^2}\right). \quad (4.8)$$

Finally, for the sum Σ_4, we use the power series expansions of $\exp(x)$ and of $1/(1-p)$ and we get:

$$\Sigma_4 = \sum_{h \ge h_3} (1 - \exp(-nqp^{h+1})) = \frac{nqp^{h_3+1}}{1-p} - \sum_{h\ge h_3}\sum_{k\ge 2} \frac{(-nqp^{h+1})^k}{k!} < np^{h_3+1} = O\left(\frac{1}{n^3}\right).$$

We got that Σ_0, Σ_1, Σ_2, Σ_3, and Σ_4 are $o(1)$. It remains to evaluate $\Sigma_\infty = \sum_{h\ge 0}(1 - e^{-nqp^{h+1}})$. Such a sum is typical of expressions which can be evaluated by Mellin transform techniques. To this aim, let $\phi(t) = \sum_{h\ge0}(1 - e^{-tqp^{h+1}})$ and set $f(t) := 1 - e^{-tpq}$ and $\mu_h := p^h$, then

$$\phi(t) = \sum_{h\ge 0} f(\mu_h t).$$

Let ϕ^* and f^* be, respectively, the Mellin transform of the functions ϕ and f. Using Identity (4.3) given in Example 4.2, we have $f^*(s) = -(pq)^{-s}\Gamma(s)$ on its fundamental strip $-1 < \Re(s) < 0$ and, as ϕ is a harmonic sum, its Mellin transform is

$$\phi^*(s) = f^*(s) \sum_{h\ge 0} \mu_h^{-s} = \frac{q^{-s}\Gamma(s)}{1-p^s}. \quad (4.9)$$

This function extends analytically to the full complex plane, with isolated poles at the negative integers (due to poles of $\Gamma(s)$ there), and with another set of isolated poles (the roots of $p^s = 1$). These two sets of poles have $s = 0$ in common. This implies that for $\Re(s) > -1$ the poles of ϕ^* are

$$\begin{cases} s_k = \frac{2ik\pi}{\ln p} \text{ for } k \in \mathbb{Z}, k \ne 0 & \text{(all are poles of order 1)}, \\ s_0 = 0 \quad \text{(the only pole of order 2)}. \end{cases}$$

Using Formula (4.4) for the inverse Mellin transform, we obtain

$$\phi(t) = \mathrm{Res}[s\phi^*, 0]\ \ln t - \mathrm{Res}[\phi^*, 0] - \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathrm{Res}[\phi^*, s_k]\ t^{-s_k}$$
$$= \frac{\ln t}{-\ln p} - \left(\frac{\gamma}{\ln p} + \frac{1}{2} + \frac{\ln q}{\ln p}\right) + \frac{1}{\ln p} \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma(s_k) q^{-s_k} t^{-s_k}.$$

We finally get the claim of the theorem by noting that $\mathbb{E}[H_n] = \phi(n) + O\left(\frac{(\ln n)^4}{n}\right)$. □

4.3. **Variance of the height of Moran walks.** We now prove that the height of Moran walks, despite a mean of order $O(\ln n)$ and a second moment of order $O((\ln n)^2)$, has a variance which involves surprising cancellations at these two orders, leading to an oscillating function of order $O(1)$ (in n), as implied by the following much more precise asymptotics.

Theorem 4.6. *The variance of the height of Moran walks satisfies*

$$\mathbb{V}\mathrm{ar}[H_n] = \frac{1}{\ln(p)^2}\left(Q^2(\ln(qn)) + 2\gamma Q(\ln(qn)) + 2R(\ln(qn)) + \frac{\pi^2}{6}\right) + \frac{1}{12} + O\left(\frac{(\ln n)^5}{n}\right),$$

where Q and R are Fourier series of small amplitudes given by Formulas (4.6) and (4.11).

Proof. To obtain the variance of H_n we first consider the second moment

$$\mathbb{E}[H_n^2] = \sum_{h\geq 0} \mathbb{P}(H_n = h)h^2 = \sum_{h\geq 0} \mathbb{P}(H_n^2 > h), \tag{4.10}$$

where we know from Theorem 3.3 that the summand can be approximated by

$$\mathbb{P}(H_n^2 > h) = 1 - \mathbb{P}\left(H_n \leq \sqrt{h}\right) = 1 - \exp\left(-nqp^{\lfloor\sqrt{h}\rfloor+1}\right) + O\left(\frac{(\ln n)^3}{n}\right).$$

Then, partitioning the last sum in (4.10) into the same intervals as in Formula (4.7), we get that $\mathbb{E}[H_n^2] = \phi_{\mathrm{var}}(n) + O\left(\frac{(\ln n)^4}{n}\right)$, where ϕ_{var} is the function defined by

$$\phi_{\mathrm{var}}(x) = \sum_{h\geq 0}\left(1 - \exp\left(-xqp^{\lfloor\sqrt{h}\rfloor+1}\right)\right).$$

From the behavior of $\phi_{\mathrm{var}}(x)$ at $x = 0$ and $x = +\infty$, using the property given in (4.1), we get that the Mellin transform of ϕ_{var} is defined on the fundamental strip $(-1, 0)$. Using the harmonic sum summation (4.9), one gets for s in this strip:

$$\phi^*_{\mathrm{var}}(s) = f^*(s)\sum_{h\geq 0}\left(p^{\lfloor\sqrt{h}\rfloor}\right)^{-s} = -\Gamma(s)(pq)^{-s}\sum_{h\geq 0}\left(p^{\lfloor\sqrt{h}\rfloor}\right)^{-s}.$$

Here, as we have

$$\sum_{h\geq 0}\left(p^{\lfloor\sqrt{h}\rfloor}\right)^{-s} = \sum_{n\geq 0}\sum_{h=n^2}^{(n+1)^2-1}\left(p^{-s}\right)^n = \sum_{n\geq 0}(2n+1)\left(p^{-s}\right)^n = \frac{1+p^{-s}}{(1-p^{-s})^2},$$

we finally get

$$\phi^*_{\mathrm{var}}(s) = \frac{-\Gamma(s)q^{-s}(1+p^s)}{(p^s-1)^2}.$$

What are the poles of $\phi^*_{\mathrm{var}}(s)$? These are $s = 0$ (a pole of order 3) and $s = s_k = 2ik\pi$ (for $k \in \mathbb{Z}, k \neq 0$, which are poles of order 2). Using Formula (4.4) for the inverse Mellin transform, one thus obtains

$$\begin{aligned}\phi_{\mathrm{var}}(t) =&\frac{\ln(t)^2}{\ln(p)^2} + \ln(t)\frac{\ln(p) + 2\ln(q) + 2\gamma - 2Q(\ln(qt))}{\ln(p)^2}\\ &- \frac{\ln(p)+2\ln(q)}{\ln(p)^2}Q(\ln(qt)) + \frac{2}{\ln(p)^2}R(\ln(qt))\\ &+ \frac{1}{3} + \frac{\gamma+\ln(q)}{\ln(p)} + \frac{\pi^2/6+\gamma^2}{\ln(p)^2} + \frac{2\gamma\ \ln(q) + \ln(q)^2}{\ln(p)^2},\end{aligned}$$

with the same $Q(x)$ as in (4.6), and where $R(x)$ is another Fourier series given by

$$R(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma'(s_k)\exp(-s_k x). \tag{4.11}$$

(Similarly to $Q(x)$, this Fourier series $R(x)$ is always real, as can be seen by replacing Γ by Γ' in Remark 4.4.)

Now that we obtained the asymptotic behavior of $\mathbb{E}[H_n^2]$, we conclude and obtain Theorem 4.6 via $\mathbb{V}\mathrm{ar}[H_n] = \mathbb{E}[H_n^2]-\mathbb{E}[H_n]^2$, where $\mathbb{E}[H_n]$ was computed in Theorem 4.3. ☐

4.4. **Height of excursions.** Excursions are walks in $\mathbb{N}^2$ ending at altitude 0 (where, as previously, time is encoded by the x-axis, and altitude by the y-axis). As in previous sections, let Y_n and H_n be the final altitude and height of a walk, and let the random variable $\widetilde{H}_n$ be the height of a walk of length n conditioned to be an excursion, that is, $\widetilde{H}_n = H_n|\{Y_n = 0\}$. For Moran walks, we get the following behavior.

Theorem 4.7 (Distribution and moments of the height of Moran excursions)**.** *The distribution of the height of excursions satisfies (for a uniform error term in k)*

$$\mathbb{P}\left(\widetilde{H}_n \le \left\lfloor \frac{\ln n}{\ln(1/p)} \right\rfloor + k\right) = \exp\left(-q\alpha(n-1)p^{k+1}\right) + O\left(\frac{(\ln n)^3}{n}\right),$$

with $\alpha(n) := p^{-\left\{\frac{\ln n}{\ln(1/p)}\right\}}$ *(where* $\{x\}$ *stands for the fractional part of* x*, and where* $\lfloor x \rfloor$ *stands for the floor function of* x*).*

Introducing temporarily the quantity $\ell_n := \ln(q(n-1))$*, and with the same Fourier series* Q *and* R *as in Theorems 4.3 and 4.6, the average and the variance are given by*

$$\mathbb{E}[\widetilde{H}_n] = \frac{\ln n}{\ln(1/p)} - \frac{\gamma}{\ln p} - \frac{1}{2} - \frac{\ln q}{\ln p} + \frac{Q(\ell_n)}{\ln p} + O\left(\frac{(\ln n)^4}{n}\right),$$

$$\mathbb{V}\mathrm{ar}[\widetilde{H}_n] = \frac{1}{\ln(p)^2}\left(Q^2(\ell_n) + 2\gamma Q(\ell_n) + 2R(\ell_n) + \frac{\pi^2}{6}\right) + \frac{1}{12} + O\left(\frac{(\ln n)^5}{n}\right).$$

Proof. As a Moran excursion necessarily ends by a reset, we have

$$\mathbb{P}(\widetilde{H}_n \le h) = \mathbb{P}\left(H_n \le h|\{Y_n = 0\}\right) = q\mathbb{P}(H_{n-1} \le h)/\mathbb{P}(Y_n = 0). \tag{4.12}$$

Thus, we have $\mathbb{P}(\widetilde{H}_n \le h) = \mathbb{P}(H_{n-1} \le h)$, $\mathbb{E}[\widetilde{H}_n] = \mathbb{E}[H_{n-1}]$, and $\mathbb{V}\mathrm{ar}[\widetilde{H}_n] = \mathbb{V}\mathrm{ar}[H_{n-1}]$, we can therefore directly recycle the results of Theorems 3.3, 4.3, and 4.6 to get the asymptotic distribution/mean/variance.

In this recycling, some care has to be brought while performing the substitution $n \to n-1$ in the asymptotic formulas for the walks: indeed, this could impact intermediate asymptotic terms (smaller than the main asymptotic term, but larger than the error term); however, in our case, all is safe as we have

$$\frac{(\ln(n\pm 1))^m}{(n\pm 1)^{m'}} = \frac{(\ln n)^m}{n^{m'}} + O\left(\frac{(\ln n)^m}{n^{m'+1}}\right).$$ ☐

This result is a simple consequence of the combinatorially obvious identity (4.12), so this direct link between the asymptotics of walks and excursions holds in wider generality for any model of walks with resets for which the step set $\mathcal{S}$ contains only positive steps.

4.5. **Fourier series: bounds and infinite differentiability.** In his seminal work [38], Knuth mentions at the end of his Section 3 that if one assumes that $\ln(qn)$ is equidistributed mod 1, then the sum $Q(\ln(qn))$ is of "average 0". Let us amend a little bit Knuth's assertion. Indeed, Weyl's criterion asserts that a sequence a_n is equidistributed mod 1 if and only if, for any positive integer ℓ, we have

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(2i\pi\ell a_n) = 0.$$

Considering this sum with $\ell = 1$ and $a_n = \ln(qn)$, and applying the Euler–Maclaurin formula to it, one gets that it does *not* converge to 0, and therefore $\ln(qn)$ is not equidistributed mod 1.

However, it is indeed true that the oscillating $Q(x)$ and $R(x)$ are of mean value zero over their period (i.e., $\int_0^{\ln(1/p)} Q(x)dx = 0$; see Figure 7 on page 283), and that $Q(\ln(qn))$ and $R(\ln(qn))$ are "almost" of mean value zero and that they possess small fluctuations. Let us give an explicit bound on their amplitude. To this aim, we first need to bound the digamma function[7], defined by

$$\psi(z) := \Gamma'(z)/\Gamma(z).$$

The function ψ can be seen as an analytic continuation of harmonic numbers and satisfies $\psi(t+1) = \psi(t) + 1/t$. While several bounds for $\psi(z)$ exist in the literature (see e.g. [52]), most of them are dedicated to $z \in \mathbb{R}$ (for example we have $\psi(t) < \ln(t) - 1/(2t)$ for $t > 0$), so we now establish a lemma for $z \in i\mathbb{R}$ (which we believe to be new, and which has its own interest beyond our application hereafter to bounds of Fourier series).

Lemma 4.8 (A bound for the digamma function on the imaginary axis)**.** *For $t > 0$, we have*

$$|\psi(it)| \leq \frac{1}{2} \ln\left(1+t^2\right) + \left(\frac{\pi}{2} + 1 - \gamma\right) + \frac{1}{t}, \tag{4.13}$$

which also implies the bound

$$|\psi(it)| \leq \left(\frac{\pi}{2} + 1 - \gamma + \frac{\ln 2}{2}\right) + \left(\ln(t)\mathbb{1}_{\{t\geq 1\}} + \frac{1}{t}\right).$$

Proof. Using Euler's representation of the gamma function as an infinite product, i.e.,

$$\Gamma(z) = \frac{1}{z} \prod_{k\geq 1} (1+1/k)^z/(1+z/k) = \frac{\exp(-\gamma z)}{z} \prod_{k\geq 1} \frac{\exp(z/k)}{1+z/k},$$

we get that its logarithmic derivative, $\psi(z) = \Gamma'(z)/\Gamma(z)$, satisfies, for $z \in \mathbb{C}, z \notin -\mathbb{N}$:

$$\psi(z) = -\frac{1}{z} - \gamma + \sum_{k=1}^{+\infty} \frac{z}{k(k+z)}.$$

[7] This is a rather misleading name: indeed, the digamma function is traditionally denoted by the letter psi (i.e., ψ), while it should logically be denoted by the Greek letter digamma (i.e., Ϝ, a letter which looks like a big Γ stack on a small Γ, which later gave birth to the more familiar letter F in the Latin alphabet). This paradox is due to the fact that Stirling, who introduced this function, did initially use the notation digamma Ϝ, but later authors switched the notation to ψ, while the initial name remained.

We refer to [18, Section 1.1] for more details on these formulas. Now, setting $z = it$ (with $t > 0$), and regrouping the imaginary and real parts gives

$$\psi(it) = i\left(\frac{1}{t} + \sum_{n=1}^{+\infty} \frac{t}{n^2+t^2}\right) + \left(\sum_{n=1}^{+\infty} \frac{t^2}{n\,(n^2+t^2)} - \gamma\right),$$

and thus, by the triangle inequality

$$|\psi(it)| \le \left(\frac{1}{t} + \sum_{n=1}^{+\infty} \frac{t}{n^2+t^2}\right) + \left(\sum_{n=1}^{+\infty} \frac{t^2}{n\,(n^2+t^2)} - \gamma\right). \tag{4.14}$$

Here, note that for all $n \le u < n+1$, we have $n^2 + t^2 \le u^2 + t^2 < (n+1)^2 + t^2$, and thus

$$\frac{t}{(n+1)^2+t^2} \le \int_n^{n+1} \frac{t}{u^2+t^2}du \le \frac{t}{n^2+t^2}.$$

Summing for n from 0 to $+\infty$, we obtain

$$\sum_{n=1}^{+\infty} \frac{t}{n^2+t^2} \le \sum_{n=0}^{+\infty} \int_n^{n+1} \frac{t}{u^2+t^2}du = \int_0^{+\infty} \frac{t}{u^2+t^2}du = \frac{\pi}{2}.$$

So the first infinite sum in (4.14) is bounded by $\pi/2$. For the second infinite sum, it is convenient to split it in the contribution from the summand for $n = 1$, which is bounded by

$$\max_{t \ge 0} \left(\frac{t^2}{1+t^2}\right) = 1,$$

plus the remaining part (i.e., the sum of the terms for $n \ge 2$):

$$\sum_{n=2}^{+\infty} \frac{t^2}{n\,(n^2+t^2)} \le \int_{t^{-1}}^{+\infty} \frac{1}{u(u^2+1)}du \;=\; \frac{1}{2}\ln(1+t^2).$$

Plugging these two bounds in (4.14) proves our lemma. □

Equipped with the previous lemma, we can now give our bounds for $Q(x)$ and $R(x)$.

Proposition 4.9 (Uniform bounds for the oscillations)**.** *The oscillating functions $Q(x)$ and $R(x)$ are uniformly bounded by*

$$\sup_{x\in\mathbb{R}^+} |Q(x)| \le \frac{\ln(p)}{\pi} \operatorname{lnexp}\left(p, \frac{4}{5}\pi^2\right),$$

$$\sup_{x\in\mathbb{R}^+} |R(x)| \le \frac{\ln(p)}{\pi} \left[\operatorname{lnexp}\left(p, \frac{4}{5}\pi^2\right) + \left(\frac{\pi}{2} + 1 - \gamma - \frac{\ln(p)}{2\pi}\right) \operatorname{lnexp}\left(p, \frac{114}{155}\pi^2\right)\right], \tag{4.15}$$

where

$$\operatorname{lnexp}(p,\beta) := \ln\left(1 - \exp\left(\frac{\beta}{\ln(p)}\right)\right).$$

For $p = 1/2$, we have more precisely

$$\sup_{x\in\mathbb{R}^+} |Q(x)| = 1.090430\cdots \times 10^{-6} \qquad \textit{and} \qquad \sup_{x\in\mathbb{R}^+} |R(x)| = 2.987768\cdots \times 10^{-6}.$$

Proof. Applying the triangle inequality on the definition of $Q(x)$ in (4.6), we get

$$|Q(x)| \le \sum_{k\in\mathbb{Z}\setminus\{0\}} |\Gamma(s_k)| \times |\exp(-s_k x)| \le 2\sum_{k\ge 1} |\Gamma(s_k)|$$

(a quantity independent of x, as $|\exp(-s_k x)| = 1$). Then, using the complement formula for the gamma function, we have $\Gamma(-z)\Gamma(z) = \frac{\pi}{z\sin(\pi(z+1))}$ (for $z \notin \mathbb{Z}$). Using this relation for $z = it$ (with $t \in \mathbb{R}$) together with the relation $\overline{\Gamma(z)} = \Gamma(\bar{z})$, we infer that

$$|\Gamma(it)|^2 = \Gamma(it)\Gamma(-it) = \frac{\pi}{t\sinh(\pi t)}. \tag{4.16}$$

Thus, for $t = \frac{2\pi}{-\ln p}$, this gives

$$\sup_{x\in\mathbb{R}^+} |Q(x)| \le 2\sum_{k\ge 1}\sqrt{\frac{\pi}{kt\sinh(\pi kt)}} = \sqrt{\frac{\ln(1/p)}{2}}\sum_{k\ge 1}\sqrt{\frac{1}{k\sinh(\pi kt)}}. \tag{4.17}$$

As, for $x \ge 0$, we have $\sinh(x) \ge (1/4)x\exp(4x/5)$, we get

$$\begin{aligned}
\sup_{x\in\mathbb{R}^+} |Q(x)| &\le \sqrt{\frac{\ln(1/p)}{2}}\sum_{k\ge 1}\sqrt{\frac{1}{(1/4)\pi k^2 t\exp(4\pi kt/5)}} \\
&= \ln(1/p)\sum_{k\ge 1}\frac{1}{\pi k\exp\left(\frac{2}{5}\pi kt\right)} \qquad (4.18)\\
&= \frac{\ln(p)}{\pi}\ln\left(1-\exp\left(\frac{4\pi^2}{5\ln(p)}\right)\right). \qquad (4.19)
\end{aligned}$$

Note that the more relaxed bound (4.19) is quite close to the stricter bound (4.17): e.g. for $p = 1/2$ the bound (4.17) gives the upper bound $1.090430\cdots \times 10^{-6}$ (and one can numerically check that these first digits also constitute a *lower* bound), while the bound (4.19) gives the upper bound 2.49×10^{-6}.

Now, for bounding $R(x)$, we use the identity $\Gamma'(z) = \psi(z)\Gamma(z)$, with the bound (4.13) from Lemma 4.8 for $|\psi(it)|$, and the bound (4.18) for $|\Gamma(it)|$:

$$\begin{aligned}
|R(x)| &\le 2\sum_{k\ge 1}|\Gamma'(s_k)| = 2\sum_{k\ge 1}|\psi(s_k)|\,|\Gamma(s_k)| \\
&\le \sum_{k\ge 1}\left(\frac{1}{2}\ln\left(1+\left(\frac{2\pi k}{\ln(p)}\right)^2\right)+\frac{\pi}{2}+1-\gamma-\frac{\ln p}{2\pi k}\right)\frac{\ln(1/p)}{\pi k\exp\left(-\frac{4}{5}\pi^2 k/\ln(p)\right)}. \qquad (4.20)
\end{aligned}$$

Now, it is easy to check that we have $\frac{1}{2}\ln(1+x^2) \le \exp\left(\frac{1}{31}\pi x\right)$ for all $x > 0$. Then, noting $t = -2\pi/\ln(p)$, we get

$$\begin{aligned}
\sum_{k\ge 1}\frac{1}{2}\ln\left(1+(kt)^2\right)\frac{\ln(1/p)}{\pi k\exp\left(\frac{2}{5}\pi kt\right)} &\le \sum_{k\ge 1}\frac{\ln(1/p)}{\pi k\exp\left(\frac{57}{155}\pi kt\right)} \\
&= \frac{\ln(p)}{\pi}\ln\left(1-\exp\left(\frac{114\pi^2}{155\ln(p)}\right)\right).
\end{aligned}$$

Together with the contribution of the remaining summands in (4.20), this gives the bound (4.15) for $|R(x)|$. □

From this, we can establish the infinite differentiability of our fluctuations.

Theorem 4.10 (Fourier series infinite differentiability)**.** *The Fourier series*

$$Q(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma(s_k)\exp(-s_k x) \qquad \textit{and} \qquad R(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma'(s_k)\exp(-s_k x)$$

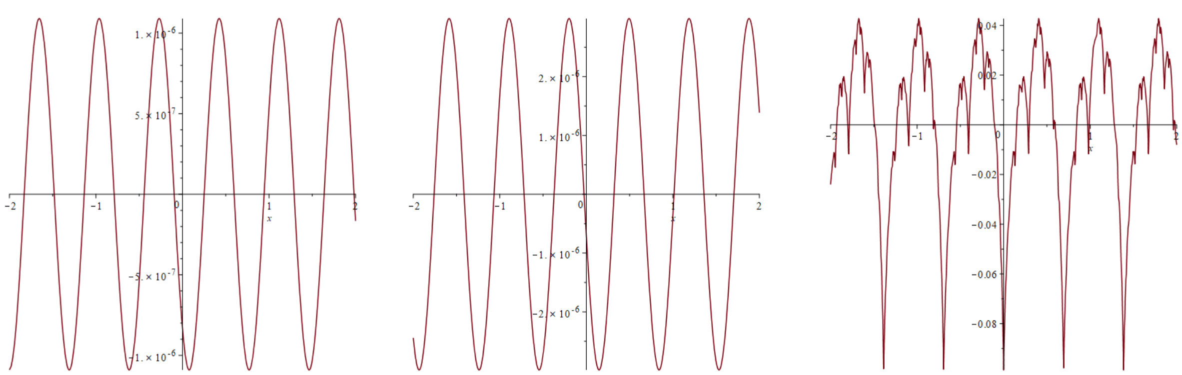
(where $s_k = \frac{2ik\pi}{\ln p}$) are infinitely differentiable on $\mathbb{R}$.

Proof. A Fourier series $f(x) = \sum_{k\in\mathbb{Z}} c_k \exp(-ikx)$ satisfies the Weierstrass M-test if there exists a sequence M_n such that $|c_k \exp(-ikx)| + |c_{-k}\exp(ikx)| < M_k$ (for all $x \in \mathbb{R}$) and $\sum_{k\geq 0} M_k$ converges. If $f(x)$ and $g(x) := -i\sum_{k\in\mathbb{Z}} kc_k \exp(-ikx)$ both satisfy the Weierstrass M-test, then they converge absolutely and uniformly in $\mathbb{R}$, and $f' = g$.

Thus, by successive application of this M-test, if the coefficients decay polynomially, i.e., we have $|c_{-k}| + |c_k| = O(|k|^{-d-1})$, then $f(x)$ is in $\mathcal{C}^d$ (that is, d times differentiable) and $f(x)$ is in $\mathcal{C}^\infty$ (that is, infinitely differentiable) if its coefficients decay faster than any polynomial rate. By Equation (4.16), the coefficients $\Gamma(s_k)$ decay like $\approx \exp(-k\pi/\ln(p))$, so $Q(x)$ is in $\mathcal{C}^\infty$. By Equation (4.20), the coefficients $\Gamma'(s_k)$ also decay like an exponential, so $R(x)$ is in $\mathcal{C}^\infty$. $\square$

It is interesting to compare this smoothness result with the situation observed by Delange [16] in his seminal work on the sum of digits of n in base $1/p$ (when $1/p$ is an integer). Therein, he proved an asymptotic behavior involving fluctuations dictated by a Fourier series, which can also be obtained by a Mellin transform approach, quite similarly to the road followed in our article. It appears that his Fourier series (already mentioned in Remark 4.5) has coefficients $\zeta(s_k)/((1+s_k)s_k) \approx k^{-1.5}$; it is thus not surprising that the Delange series is nowhere differentiable, in sharp contrast with the smoothness of our Fourier series (see Figure 8).

This concludes our analysis of the height and the corresponding fluctuations.



$Q(x)$ (for $p = 1/2$) $\qquad$ $R(x)$ (for $p = 1/2$) $\qquad$ $\mathit{Delange}(x)$ (for $p = 1/2$)

Figure 8. Our Fourier series Q and R are infinitely differentiable, while the Fourier series obtained by Delange is nowhere differentiable. This follows from the asymptotics of their coefficients, as explained in the proof of Theorem 4.10.

5. Some results for the Moran model in dimension $m > 1$

5.1. **Joint distribution of ages for the Moran model with $m > 1$.** Moran processes are models of population evolution (or mutation transmission) where the population is of constant size (some individuals could die but are then immediately replaced by a new individual). Depending on the applications, several variants were considered in the literature starting with the seminal work of Moran himself [43, 44], up to more recent extensions (for example to spatially structured population [41].

Motivated by the model with resets of Itoh, Mahmoud, and Takahashi [34, 35], we now define the *Moran model with m individuals*. It is a process parametrized by some probabilities p and p_i's such that $p + \sum_{i=0}^{m} p_i = 1$, and which starts at time 0 with m individuals of age 0. Then, at each new unit of time,

- either, with probability p, all survive (their age increases by 1),
- either, with probability p_i (for $1 \le i \le m$), the i-th individual dies (it is then replaced by a new i-th individual of age 0), while the age of the $m-1$ surviving individuals increases by 1,
- either, with probability p_0, all die and are replaced by m new individuals of age 0.

Now, we define the sequence of multivariate polynomials $f_n(x_1, \dots, x_m)$ (for $n \in \mathbb{N}$) by the fact that the coefficient of $x_1^{k_1} \cdots x_m^{k_m}$ in $f_n(x_1, \dots, x_m)$ is the probability that, at time n, the i-th individual has age k_i (for $i = 1, \dots, m$). Accordingly, $F(t, x_1, \dots, x_m) := \sum_{n \ge 0} f_n(x_1, \dots, x_m) t^n$ is the probability generating function associated to the above Moran model, where the time is encoded by the exponent of t.

Theorem 5.1. *The probability generating function of the Moran model is a rational function, and it admits the closed form*

$$F(t, x_1, \dots, x_m) = \frac{\sum_{k=0}^{2^m-1} (-1)^k P_k t^k}{\Delta}, \tag{5.1}$$

where the P_k's are polynomials (given in the proof) in the x_i's, p, p_i's, and where Δ is the following polynomial of degree 2^m in t:

$$\Delta = \prod_{I \subseteq \{1,\dots,m\}} \left(1 - t \left(p + p_0 [\![I = \{1, \dots, m\}]\!] + \sum_{i \in I} p_i \right) \prod_{i \notin I} x_i \right).$$

Proof. The Moran model evolution is encoded by the following functional equation for the probability generating function F:

$$\begin{aligned} F(t, x_1, \dots, x_m) = 1 &+ t p x_1 \cdots x_m F(t, x_1, \dots, x_m) + t p_0 F(t, 1, \dots, 1) \\ &+ t \left(\sum_{i=1}^{m} p_i \frac{x_1 \cdots x_m}{x_i} F(t, x_1, \dots, x_m)_{|x_i=1} \right), \end{aligned} \tag{5.2}$$

where $F_{|x_i=1}$ means F evaluated at $x_i = 1$.

To solve this single functional equation (which has $m+2$ unknowns[8]), the trick is to transform it into a linear system of equations with... 2^m unknowns! Indeed, by substituting $x_i = 1$ (in all the possible ways) in the functional equation (5.2), we get a system of 2^m equations.

[8]We temporarily count $F(t, 1, \dots, 1)$ as unknown, even if it is obviously equal to $1/(1-t)$, as F is a probability generating function.

Then, we encode this system by a matrix M, where we cleverly (sic!) choose the order in which unknowns are associated to the lines/columns of M. Let us define this order; to this aim consider the Cartesian product $\mathcal{X} := \{1, x_1\} \times \cdots \times \{1, x_m\}$. For any pair of m-tuples $\mathbf{X}$ and $\mathbf{Y}$ from $\mathcal{X}$, one writes $\mathbf{X} \prec \mathbf{Y}$ if the number of 1's in $\mathbf{X}$ is less than the number of 1's in $\mathbf{Y}$, or, when they have the same number of 1's, if $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the lexicographical order induced by $x_1 \prec \cdots \prec x_m \prec 1$. For example, we have $(x_1, x_2) \prec (x_1, 1) \prec (1, x_2) \prec (1, 1)$. Listing all the elements of $\mathcal{X}$ in increasing order, we get a list of 2^m tuples $X_1, \ldots, X_{2^m}$. The matrix M encoding the aforementioned system of equations is constructed such that the i-th line of the matrix M corresponds to the unknown $F(t, X_i)$ and the j-th column corresponds to the unknown $F(t, X_j)$.

With this order, the matrix M is an upper triangular matrix (as each of the substitution of some x_i's by some 1's in Equation (5.2) leads from some tuple $\mathbf{X}$ to $m+2$ larger tuples $\mathbf{Y}$), and thus the determinant of M is the product of its diagonal terms:

$$\det M = \prod_{I \subseteq \{1,\ldots,m\}} \left(1 - t\left(p + p_0 [\![I = \{1, \ldots, m\}]\!] + \sum_{i \in I} p_i\right) \prod_{i \notin I} x_i\right),$$

where we use Iverson's bracket notation[9].

As this determinant $\Delta := \det M$ is not zero, this entails by Cramer's rule that $F(t, x_1, \ldots, x_m)$ can be written as a rational function with denominator Δ (note that, for some specific real values of p and the p_i's, it is not excluded that the numerator could have a shared factor with Δ). Of course, computing the determinant of each comatrix, and using the relation $p_0 = 1 - (p + p_1 + \cdots + p_m)$, we get symmetric polynomial expressions for the P_k's occurring in (5.1), e.g.:

$$\begin{aligned}
P_0 &= 1,\\
P_1 &= p\left(\prod_{i=1}^{m}(1 + x_i) - \prod_{i=1}^{m} x_i\right) + \sum_{i=1}^{m} x_i \sum_{\substack{j=1,\ldots,m\\ j \neq i}} p_j,\\
&\ \vdots\\
P_{2^m-1} &= \left(\prod_{i=1}^{m} x_i^m\right) \prod_{I \subsetneq \{1,\ldots,m\}} \left(p + \sum_{i \in I} p_i\right). \qquad \square
\end{aligned}$$

Note that the case $p_0 = 0$, $p_i = 1/m$ for $i = 1, \ldots, m$ (with $m \geq 2$) was analyzed by Itoh and Mahmoud [34]: they proved that the age of each individual converges to a shifted geometric distribution, namely $\mathrm{Geom}(1/m) - 1$. They also show that the number of individuals of age k at time n converges to a Bernoulli distribution, namely $\mathrm{Ber}((m/(m-1))^k)$. Our Theorem 5.1 constitutes a joint law version of these results, at discrete times, for generic p_i's. For example, introducing $G(t, v) := \sum_{j=1}^{m} \binom{m}{j} v^j [x_1^k \ldots x_j^k] F(t, x_1, \ldots, x_m)$, the coefficient $[t^n]\partial_v G(t, 1)$ gives the average number of individuals of age k at time n. (Note that the sum with the binomial coefficients $\binom{m}{j}$ has to be replaced by a sum over the subsets of $\{1, \ldots, m\}$ if the p_i's and the initial conditions for the $x_i's$ are not symmetric.)

[9] This notation, $[\![\text{assertion}]\!]$, is 1 if the assertion is true, and 0 if not. It was introduced in the semantics of the language APL by its founder, Kenneth Iverson. It was later popularized in mathematics by Graham, Knuth, and Patashnik [27].

5.2. **A multidimensional generalization of the Moran model.** Interestingly, the same strategy of proof allows us to solve a wide generalization of the Moran model, where

- with probability p_I, all the individuals from the subset I of $\{1,\dots,m\}$ die (they are then replaced by new individuals of age 0), while the age of each surviving individual increases by 1.
- the process starts with m individuals of any (possibly distinct) ages, encoded by a monomial $f_0(x_1,\dots,x_m)$.

This translates to the following single functional equation, involving 2^m unknowns:

$$F(t,x_1,\dots,x_m) = f_0(x_1,\dots,x_m) + t \sum_{I\in\{1,\dots,m\}} p_I F(t,\mathbf{X}_I) \prod_{i\notin I} x_i,$$

where $\mathbf{X}_I = (x_1,\dots,x_m)_{|x_i\,=\,1 \text{ for all } i\,\in\, I}$.

Obviously, by taking $f_0 = 1$, $p_\varnothing = p$, $p_{\{1,\dots,m\}} = p_0$, $p_{\{i\}} = p_i$, and all other $p_I = 0$, the generalized model simplifies to the classical Moran model of Theorem 5.1. Another natural set of probabilities is $p_I = q^k(1-q)^{m-k}$, where k is the number of elements in I. It encodes the model where, at each unit of time, each individual dies with probability q.

More generally, for any set of p_I's, one gets the following result.

Theorem 5.2. *The probability generating function of the generalized Moran model is a rational function:*

$$F(t,x_1,\dots,x_m) = \frac{\sum_{k=0}^{2^m-1}(-1)^k Q_k t^k}{\Delta}, \tag{5.3}$$

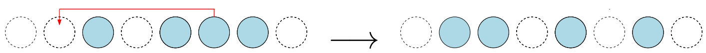
where the Q_k's are polynomials in the x_i's and p_I's for $I\subset\{1,\dots,m\}$, and where Δ is the following polynomial of degree 2^m in t:

$$\Delta = \prod_{I\subseteq\{1,\dots,m\}} \left(1 - t\left(p_\varnothing + p_{\{1,\dots,m\}}[\![I=\{1,\dots,m\}]\!] + \sum_{i\in I} p_{\{i\}}\right)\prod_{i\notin I} x_i\right).$$

Note that, for this generalized model, the denominator Δ is the same as in Theorem 5.1, and the Q_k's are a lifting of the P_k's from Theorem 5.1, involving more terms and variables (namely, all the p_I's). For these two models, these polynomials P_k and Q_k are variants of symmetric functions. We comment more on this fact now.

Remark 5.3 (Links with bi-indexed families of symmetric functions)*.* Many problems related to lattice paths lead to generating functions expressible in terms of symmetric functions; this results from the kernel method, which involves a Vandermonde-like determinant, and thus leads to variants of Schur functions [4,6,11]. For the generalized Moran model we also get symmetric expressions, as the problem is by design symmetric, but in a more subtle way: one does not get formulas nicely expressible in terms of classical symmetric functions. This is due to the fact that we have to play with two distinct sets of variables (the p_i's and the x_i's), the occurrences of which are not fully independent. It appears that these subtle dependencies are well encoded by the *MacMahon elementary symmetric functions*, defined by $e_{j,k} := [t_x^j t_p^k]\prod_{i=1}^m (1+t_x x_i + t_p p_i)$. For example, we have $e_{2,1} = x_1x_2p_3 + x_2x_3p_1 + x_3x_1p_2$. They allow us to provide more compact formulas for our generating functions, like $P_1 = e_{1,1} + p\sum_{j=1}^m e_{j,0}$. We plan to study these aspects in a forthcoming work. Note that these MacMahon symmetric functions also appear in problems a priori unrelated to our multidimensional Moran walks, see e.g. the articles of Gessel [25] and Rosas [48].

5.3. **Application to the soliton wave model.** The soliton wave model (as considered by Itoh, Mahmoud, and Takahashi [35]) is a stochastic system of particles encoding a unidirectional wave. The number of particles is constant during the full process: we have m particles on $\mathbb{Z}$ which can only moves to the left as follows. At time $n = 0$, the initial configuration consists of m particles, at x-coordinates $1, \dots, m$. Then, at each unit of time $n = 1, 2, \dots,$ uniformly at random, one of the m particles jumps just to the left of the first particle (the wave front), thus leaving an empty space at its starting position:



Note that at time n the location of the leftmost particle has thus x-coordinate $1 - n$. See Figure 9 for an illustration of 6 iterations of this process, where, for drawing convenience, we shift the origin of the x-axis after each step, so that the first particle is always at x-coordinate 1.

Then, applying Theorem 5.2 to this model, we get the following proposition.

Proposition 5.4. *The joint distribution $F(t, x_1, \dots, x_m)$ of the time/positions of the particles in the soliton wave model is given by Formula (5.3), by taking as initial condition $f_0 = x_1^1 x_2^2 \dots x_m^m$, and, as probabilities of transition, $p_{\{i\}} = 1/m$ and all other $p_I = 0$; what is more, the denominator of $F(t, x_1, \dots, x_m)$ thus simplifies to*

$$\Delta = \prod_{I \subseteq \{1,\dots,m\}} \left(1 - t\frac{|I|}{m}\right) \prod_{i \notin I} x_i,$$

where $|I|$ stands for the number of elements of the set I.

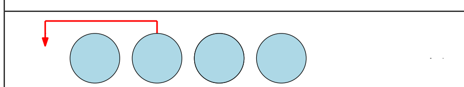
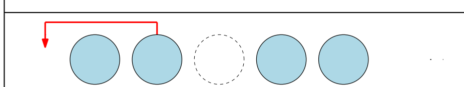
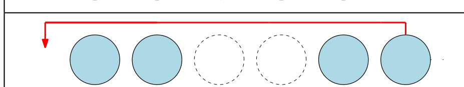
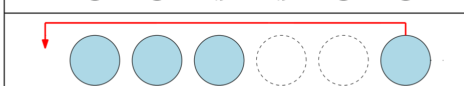
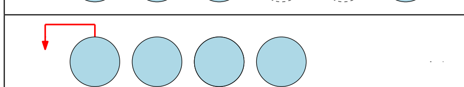
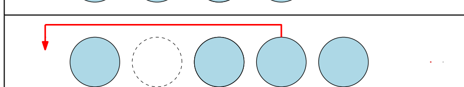
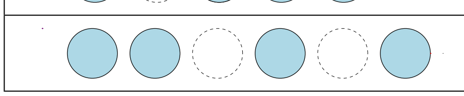
Wave	Time n	Length L_n
	0	4
	1	5
	2	6
	3	6
	4	4
	5	5
	6	6

FIGURE 9. The soliton wave model: a wave is a sequence of particles (the sequence may have some inner holes), and at each unit of time, one particle is selected and jumps at the very start of the wave (and thus leaves an empty slot where it was). Trailing empty slots are ignored (this occurs when the last particle is selected, e.g. from step 3 to 4 above).

Figure 9 also shows that this model has one degree of freedom, that is, the soliton wave model with m particles can be modeled as $m-1$ interactive urns $U_1, \ldots, U_{m-1}$: the urn U_k contains the number of white cells between the k-th and $(k+1)$-th blue particle. Accordingly, this interactive urn process starts with $U_k(0) = 0$ for all k, and then, at each unit of time, we have one of the following m events (with probability $1/m$):

- $U_1(n+1) = U_1(n) + 1$ and other urns are unchanged.
- for $k = 2, \ldots, m-1$: $U_1(n+1) = 0$, $U_j(n+1) := U_{j-1}(n)$ (for $j = 2, \ldots, k-1$), $U_k(n+1) := U_{k-1}(n) + U_k(n) + 1$, and remaining urns are unchanged.
- $U_1(n+1) = 0$ and, for $k \geq 2$, $U_k(n+1) := U_{k-1}(n)$.

The length of the soliton is then given by $L_n = m + U_1(n) + \cdots + U_{m-1}(n)$; it can equivalently be viewed as the maximum of the x-coordinates (at time n) of each particle.

6. Conclusion and future works

In this article, we considered several statistics (final altitude, waiting time, height) associated to walks with resets, for any given finite step set. For the case of the simplest non-trivial model (namely, for Moran walks), we prove that the asymptotic height exhibits some subtle behavior related to the discrete Gumbel distribution. In a forthcoming article, we plan to consider the asymptotic analysis of the height for more general walks.

In our formulas for walks of length n, taking $q' := q/n$ (and more generally $q' = q(n)$) as the probability of reset leads to models which can counterbalance the infinite negative drift of the initial model, and thus present a different type of asymptotic behavior. Studying these models and their phase transitions in more detail would be interesting.

In Section 5, we considered several multidimensional extensions of such walks, with applications to the soliton wave model, or to models in genetics. More multidimensional variants of Moran models allowing both positive and negative jumps (and with or without resets) can be handled using the approach presented in this article (see [1]). One interesting example is the one where each dimension evolves like a Motzkin path, this model was e.g. considered in the haploid Moran model [32], where the authors use a Markov chain approach, using duality/reversibility to establish links with Ornstein–Uhlenbeck processes. Note that even if one adds resets to such Motzkin-like models, one keeps nice links with continuous fractions associated to birth and death processes; see [20]. The analysis becomes much more complicated as soon as jumps of amplitude ≥ 2 are allowed; in such cases, our approach based on the kernel method strikes again.

Another natural extension is to consider walks in the quarter plane with resets (a natural model of two queues evolving in parallel); even for walks with jumps of amplitude 1, the exact enumeration and the asymptotic behavior of the (maximal) height remain open. Other more ad hoc extensions consider some age-dependent probabilities p_i's, then leading to partial differential equations for the corresponding generating functions. Some specific cases lead to closed-form solutions.

All these variants of Moran models are parametrized by the p_i's. One can then turn to the tuning of several statistical tests: having some experimental data, it is natural to look for maximum likelihood estimators of the p_i's, and to study if they are unbiased, sufficient, and consistent (for more on these notions, see e.g. [50]). In conclusion, the Moran model offers a large variety of interesting models, with many aspects to explore!

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Cyril Banderier, Christian Krattenthaler, Michael Wallner, *Editors.*

Lattice Path Combinatorics and Interactions

Lattice paths are a fundamental concept in combinatorics, providing a geometric perspective on many counting problems. By mapping problems onto a lattice, mathematicians can exploit symmetry, recursion, and combinatorial identities to derive elegant solutions. Lattice paths are also interesting *per se*, serving as a powerful tool to model scenarios ranging from simple random walks to more intricate probability processes. Their study opens up new avenues for understanding many combinatorial structures, capturing universal phenomena such as limit laws and limiting objects.

This volume contains twelve contributions of experts on different aspects of lattice paths. It covers the analysis of several parameters (height, area, peaks, ascents, ...), paths with different step sets, paths constrained to be in some domains. All these contributions illustrate a variety of methods, such as approaches from bijective and enumerative combinatorics, the kernel method for solving functional equations, analytic combinatorics to establish asymptotic behaviors, differential Galois theory to prove the transcendence of generating functions, non-commutative operators to enumerate walk parameters, q-analogues, special functions, and continued fractions.

This volume also provides links to probability theory, queueing theory, computer algebra, and illustrates interactions with models coming from physics (statistical mechanics), biology (population genetics), or computer science (analysis of algorithms).

Contributors:

R. Aguech
A. Althagafi
D. W. Ash
C. Banderier
A. Blecher
T. Dreyfus
S. Elizalde
R. Flórez
R. Garbit
A. J. Guttmann
J. S. Kim
A. Knopfmacher
V. Kotěšovec
C. Krattenthaler
T. Mansour
S. Ouvry
A. Polychronakos
H. Prodinger
J. L. Ramírez
K. Raschel
D. Stanton
M. Vamvakari
F. A. Velandia
D. Villamizar
M. Wallner

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