COUNTING NONINTERSECTING LATTICE PATHS WITH TURNS

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Abstract. We derive enumeration formulas for families of nonintersecting lattice paths with given starting and end points and a given total number of North-East turns. These formulas are important for the computation of Hilbert series for determinantal and pfaffian rings.

1. Introduction. Recent work of Abhyankar and Kulkarni [1, 2] and of Conca, Herzog and Trung [4, 5, 8] showed that the computation of Hilbert series for determinantal and pfaffian rings boils down to counting families of n nonintersecting lattice paths with given starting and end points and a given total number of turns in certain regions. If one forgets about the number of turns, i.e., if one is interested in the plain enumeration of nonintersecting lattice paths with given starting and end points, then the solution is a certain determinant. This is classical now (cf. [6; 7, Cor. 2; 17, Theorem 1.2]). However, the method that is used for the plain enumeration (the "Gessel–Viennot involution", which actually can be traced back to Lindström [15] and Karlin and McGregor [9]), is not appropriate to keep track of turns. Still, the answers to "turn enumeration" are determinants. But new methods are needed now.

In this note we develop the basic theory of turn enumeration of nonintersecting lattice paths. Theorem 1 solves the turn enumeration of (unrestricted) nonintersecting lattice paths with given starting and end points, Theorem 4 provides a generalization. Theorem 2 solves the turn enumeration of nonintersecting lattice paths that stay below a diagonal line with given starting and end points, Theorem 5 provides a generalization. Finally, Theorem 3 solves a problem that is equivalent to the turn enumeration of nonintersecting lattice paths that stay above a diagonal line with given

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starting and end points, Theorem 6 provides a generalization. We also briefly indicate how these theorems are related to the computation of Hilbert series for determinantal and pfaffian rings.

What concerns the proofs of these results, it turns out that lattice paths are not the right objects to play with. The objects that are natural in the context of turn enumeration are *two-rowed arrays*. To prove the determinant formulas we construct operations on two-rowed arrays (the operations (24)/(25) → (28)/(29), (37a)/(37b) → (38a)/(38b), (32) → (33), (39) → (40)) that are in some sense analogous to the Gessel–Viennot involution for paths and the reflection principle for paths. These operations are inspired by operations from [10, 13].

A full account of this theory will be the subject of forthcoming papers [11, 12]. In particular, these papers contain more enumeration results concerning nonintersecting lattice paths with a given total number of turns. Besides, the impact of our results to the theory of determinantal and pfaffian rings is explained in detail there. Also, more general enumeration results for non-crossing two-rowed arrays are presented there that lead to summation theorems for Schur functions. An interesting feature is that it is the Robinson–Schensted–Knuth correspondence and its properties that constitute the link between "turn enumeration" of nonintersecting lattice paths and the above mentioned applications.

The paper is organized as follows. In the next section we explain our terminology and state our results (Theorems 1–6). Then, in section 3, we provide the proofs of Theorems 4–6. Theorems 1–3 follow as special cases.

2. The results. We consider lattice paths in the plane consisting of unit horizontal and vertical steps in the positive direction. In the sequel we shall call them shortly *paths*. Let P be a path from $\mathcal{A} = (A_1, A_2)$ to $\mathcal{E} = (E_1, E_2)$. Later we frequently abbreviate the fact that a path P goes from $\mathcal{A}$ to $\mathcal{E}$ by $P : \mathcal{A} \to \mathcal{E}$.

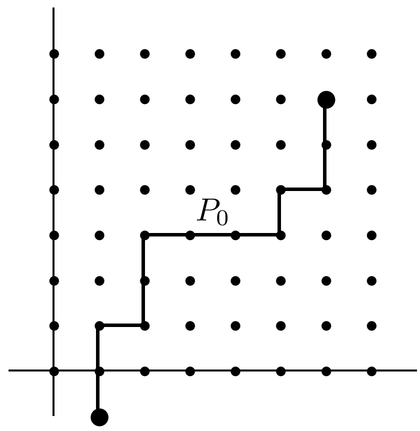


Figure 1

A point in a path P which is the end point of a vertical step and at the same time the starting point of a horizontal step will be called a *North-East turn* (*NE-turn* for short) of the path P. The NE-turns of the path in Figure 1 are $(1, 1)$, $(2, 3)$, and $(5, 4)$. Similarly, a point in a path P which is the end point of a horizontal step and at the same time the starting point of a vertical step will be called an *East-North turn* (*EN-turn* for short) of the path P. The EN-turns of the path in Figure 1 are $(2,1)$, $(5,3)$, and $(6,4)$.

The aim of this note is to prove the following three theorems about the enumeration of nonintersecting lattice paths with a given total number of turns. Here, as usual, paths are called nonintersecting if no two of them have a point in common.

Theorem 1. *Let $\mathcal{A}_i=(A_1^{(i)},A_2^{(i)})$ and $\mathcal{E}_i=(E_1^{(i)},E_2^{(i)})$ be lattice points satisfying*

$$A_1^{(1)}\le A_1^{(2)}\le\cdots\le A_1^{(r)},\quad A_2^{(1)}>A_2^{(2)}>\cdots>A_2^{(r)},$$

and

$$E_1^{(1)}<E_1^{(2)}<\cdots<E_1^{(r)},\quad E_2^{(1)}\ge E_2^{(2)}\ge\cdots\ge E_2^{(r)}.$$

The number of all families $\mathcal{P}=(P_1,\dots,P_r)$ of nonintersecting lattice paths $P_i:\mathcal{A}_i\to\mathcal{E}_i$, such that the paths of $\mathcal{P}$ altogether contain exactly K NE-turns, is

$$\sum_{k_1+\dots+k_r=K}\det_{1\le s,t\le r}\left(\binom{E_1^{(t)}-A_1^{(s)}+s-t}{k_s+s-t}\binom{E_2^{(t)}-A_2^{(s)}-s+t}{k_s}\right).\qquad\square\qquad(1)$$

Remark. A special case of Theorem 1 is of relevance in the computation of Hilbert series for determinantal rings. This was shown by several authors [5, 14, 16]. In fact, Kulkarni [14, Main Theorem 5] derived this special case ($r=p$, $K=E$, $\mathcal{A}_i=(0,a_{p-i+1})$, $\mathcal{E}_i=(m(2)-b_{p-i+1},m(1))$) from Abhyankar's formula [1, (20.14.4), p. 484] for the Hilbert series for certain determinantal rings, while Conca and Herzog [5] used it to give an alternative proof of Abhyankar's formula, see also [11]. On the other hand, Modak [16] gave an independent (manipulative) proof of this special case. Slight variations of Theorem 1 solve the computation of Hilbert series for rings generated by minors of a symmetric matrix as considered by Conca [4], see [11].

Theorem 2. *Let $\mathcal{A}_i=(A_1^{(i)},A_2^{(i)})$ and $\mathcal{E}_i=(E_1^{(i)},E_2^{(i)})$ be lattice points satisfying*

$$A_1^{(1)}\le A_1^{(2)}\le\cdots\le A_1^{(r)},\quad A_2^{(1)}>A_2^{(2)}>\cdots>A_2^{(r)},$$
$$E_1^{(1)}<E_1^{(2)}<\cdots<E_1^{(r)},\quad E_2^{(1)}\ge E_2^{(2)}\ge\cdots\ge E_2^{(r)},$$

and $A_1^{(i)}\ge A_2^{(i)}$, $E_1^{(i)}\ge E_2^{(i)}$, $i=1,\dots,r$. The number of all families $\mathcal{P}=(P_1,\dots,P_r)$ of nonintersecting lattice paths $P_i:\mathcal{A}_i\to\mathcal{E}_i$, which do not cross the line $x=y$, and where the paths of $\mathcal{P}$ altogether contain exactly K NE-turns, is

$$\sum_{k_1+\dots+k_r=K}\det_{1\le s,t\le r}\left(\binom{E_1^{(t)}-A_1^{(s)}+s-t}{k_s+s-t}\binom{E_2^{(t)}-A_2^{(s)}-s+t}{k_s}\right.$$
$$\left.-\binom{E_1^{(t)}-A_2^{(s)}-s-t+1}{k_s-t}\binom{E_2^{(t)}-A_1^{(s)}+s+t-1}{k_s+s}\right).\qquad\square\qquad(2)$$

Remark. Theorem 2 can be applied for the computation of the Hilbert series for certain ladder determinantal rings (one sided, with a diagonal upper bound) and also for pfaffian rings, see [11]. For arbitrary one-sided ladders see [11, 12].

Theorem 3. *Let $\mathcal{A}_i=(A_1^{(i)},A_2^{(i)})$ and $\mathcal{E}_i=(E_1^{(i)},E_2^{(i)})$ be lattice points satisfying*

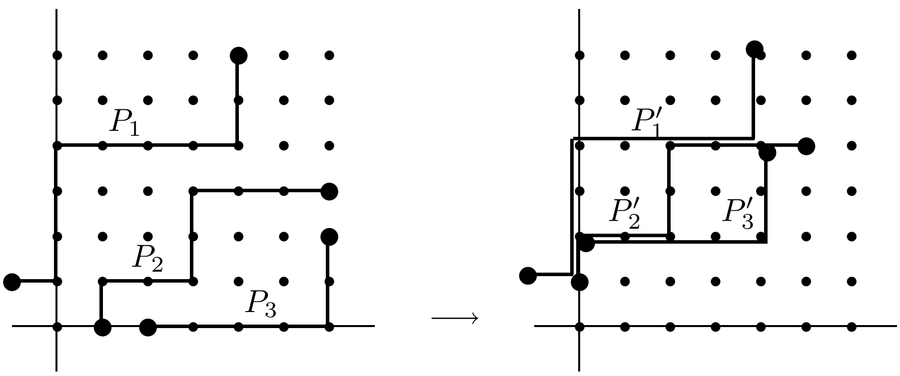
$$A_1^{(1)}<A_1^{(2)}<\dots<A_1^{(r)},\quad A_2^{(1)}\ge A_2^{(2)}\ge\dots\ge A_2^{(r)},$$
$$E_1^{(1)}\le E_1^{(2)}\le\dots\le E_1^{(r)},\quad E_2^{(1)}>E_2^{(2)}>\dots>E_2^{(r)},$$

and $A_1^{(i)}\ge A_2^{(i)}$, $E_1^{(i)}\ge E_2^{(i)}$, $i=1,\dots,r$. The number of all families $\mathcal{P}=(P_1,\dots,P_r)$ of nonintersecting lattice paths $P_i:\mathcal{A}_i\to\mathcal{E}_i$, $P_i:\mathcal{A}_i\to\mathcal{E}_i$, which do not cross the line $x=y$, and where the paths of $\mathcal{P}$ altogether contain exactly K EN-turns, is

$$\sum_{k_1+\dots+k_r=K}\det_{1\le s,t\le r}\left(\binom{E_1^{(t)}-A_1^{(s)}+s-t}{k_s+s-t}\binom{E_2^{(t)}-A_2^{(s)}-s+t}{k_s}\right.$$
$$\left.-\binom{E_1^{(t)}-A_2^{(s)}-s-t+3}{k_s-t+1}\binom{E_2^{(t)}-A_1^{(s)}+s+t-3}{k_s+s-1}\right).\qquad\square\quad(3)$$

Actually, more general results can be shown (see Theorems 4,5,6 below). However, they are more conveniently formulated after before having modified the problem.

Suppose, $\mathcal{P}=(P_1,\dots,P_r)$ is a family of nonintersecting lattice paths $P_i:\mathcal{A}_i\to\mathcal{E}_i$. Now we shift the i-th path P_i in the direction $(-i+1,i-1)$ thus obtaining the new path P_i', $i=1,\dots,r$. The new family $\mathcal{P}'=(P_1',\dots,P_r')$ might be intersecting, however it is *non-crossing* (see Figure 2).



nonintersecting lattice paths non-crossing lattice paths

Figure 2

Before we make precise what the exact meaning of non-crossing in this context is (see (10) below), we introduce some notation.

Obviously, given the starting and the final point of a path, the North-East turns uniquely determine the path. Suppose that P is a path from $\mathcal{A}=(A_1,A_2)$ to $\mathcal{E}=(E_1,E_2)$ and let the North-East turns of P be (a_1,b_1), (a_2,b_2), $\dots$, (a_k,b_k), where we assume that the (a_i,b_i) are ordered from left to right, which is equivalent with $A_1\le a_1<a_2<\dots<a_k\le E_1-1$, and $A_2+1\le b_1<b_2<\dots<b_k\le E_2$. Then P can be represented by the two-rowed array

$$\begin{matrix}a_1 & a_2 & \dots & a_k\\ b_1 & b_2 & \dots & b_k\end{matrix}\ ,\qquad(4)$$

or, if we wish to make the bounds which are caused by the starting and the final point transparent,

$$\begin{array}{rccl} A_1 \le & a_1\ a_2\ \dots\ a_k & \le E_1 - 1 \\ A_2 + 1 \le & b_1\ b_2\ \dots\ b_k & \le E_2 \end{array} . \tag{5}$$

For a given starting point and a given final point, by definition the empty array is the representation for the only path that has no North-East turn. For the path in Figure 1 we obtain the array representation

$$\begin{array}{c} 1\ 2\ 5 \\ 1\ 3\ 4 \end{array} ,$$

or with bounds included,

$$\begin{array}{rcl} 1 \le & 1\ 2\ 5 & \le 5 \\ 0 \le & 1\ 3\ 4 & \le 6 \end{array} .$$

Later, also two-rowed arrays with rows of unequal length will be considered. But these arrays also will have the property that the rows are strictly increasing. So by convention, whenever we speak of two-rowed arrays we mean two-rowed arrays with strictly increasing rows. We shall frequently use the short notation $(a \mid b)$ for two-rowed arrays, where a denotes the sequence (a_i) of elements of the first row, and b denotes the sequence (b_i) of elements of the second row.

Let P_1, P_2 be two paths, $P_1 : \mathcal{A} \to \mathcal{E}$, $P_2 : \mathcal{B} \to \mathcal{F}$, where $\mathcal{A} = (A_1, A_2)$, $\mathcal{B} = (B_1, B_2)$, $\mathcal{E} = (E_1, E_2)$, $\mathcal{F} = (F_1, F_2)$ with

$$A_1 \le B_1,\ A_2 > B_2,\ E_1 < F_1,\ E_2 \ge F_2.$$

Roughly speaking, these inequalities mean that $\mathcal{A}$ is located in the North-West of $\mathcal{B}$ (strictly in direction North and weakly in direction West), and $\mathcal{E}$ is located in the North-West of $\mathcal{F}$ (weakly in direction North and strictly in direction West). Let the array representations of P_1 and P_2 be

$$P_1 : \quad \begin{array}{rcccl} A_1 \le & a_1 & \dots & a_k & \le E_1 - 1 \\ A_2 + 1 \le & b_1 & \dots & b_k & \le E_2 \end{array} \tag{6}$$

and

$$P_2 : \quad \begin{array}{rcccl} B_1 \le & c_1 & \dots & c_l & \le F_1 - 1 \\ B_2 + 1 \le & d_1 & \dots & d_l & \le F_2 \end{array} , \tag{7}$$

respectively.

Suppose that P_1 and P_2 intersect, i.e. have a point in common. Let $\mathcal{S}$ be a meeting point of P_1 and P_2. By definition set $a_{k+1} := E_1$ and $b_0 := A_2$. (Note that the thereby augmented sequences a and b remain strictly increasing.)

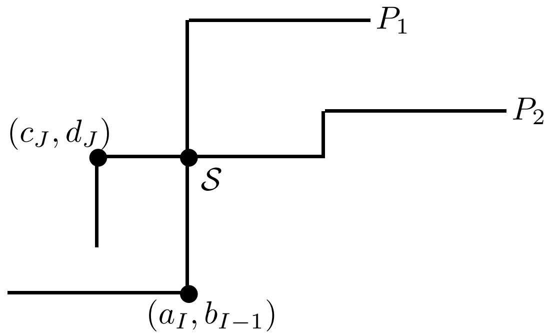


Figure 3

Considering the East-North turn (a_I, b_{I-1}) in P_1 immediately preceding $\mathcal{S}$ (and being allowed to be equal to $\mathcal{S}$) and the North-East turn (c_J, d_J) in P_2 immediately preceding $\mathcal{S}$ (and being allowed to be equal to $\mathcal{S}$), we get the inequalities (cf. Figure 3)

$$c_J \le a_I, \tag{8a}$$

$$b_{I-1} \le d_J, \tag{8b}$$

where

$$1 \le I \le k+1, \quad 1 \le J \le l. \tag{8c}$$

Of course, k, l, a_I, b_I, c_J, d_J, etc., refer to the array representations of P_1 and P_2. It now becomes apparent that the above assignments for a_{k+1} and b_0 are needed for the inequalities (8a,b) to make sense for $I = 1$ or $I = k+1$. Note that $S = (a_I, d_J)$. Vice versa, if (8a,b,c) is satisfied then there must be a meeting point between P_1 and P_2 (because of the particular location of the starting and end points $\mathcal{A}, \mathcal{B}, \mathcal{E}, \mathcal{F}$).

Summarizing, the existence of I, J satisfying (8a,b,c) characterize the array representations of *intersecting* pairs of paths.

Now, if P_2 is shifted in direction $(-1, 1)$, we obtain the path P_2' with array representation

$$\begin{matrix} B_1' \le & c_1' & \dots & c_l' & \le F_1' - 1 \\ B_2' + 1 \le & d_1' & \dots & d_l' & \le F_2' \end{matrix}\ , \tag{9}$$

where $c_i' = c_i - 1$, $d_i' = d_i + 1$, $B_1' = B_1 - 1$, $B_2' = B_2 + 1$, $F_1' = F_1 - 1$, $F_2' = F_2 + 1$. The conditions (8a,b,c) become

$$c_J' < a_I, \tag{10a}$$

$$b_{I-1} < d_J', \tag{10b}$$

where

$$1 \le I \le k+1, \quad 1 \le J \le l. \tag{10c}$$

We take (10a,b,c) as definition of two paths P_1 and P_2 with array representations (6) and (9), respectively, being non-crossing. We call the point (a_I, d_J') *crossing point* of P_1 and P_2.

Let $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ be sequences of indeterminates. Given a path P_1 with array representation (6) we define a *weight* for P_1 by

$$w_{\mathbf{x},\mathbf{y}}(P_1) = \prod_{i=1}^{k} x_{a_i} y_{b_i}. \tag{11}$$

This weight is extended to families $\mathcal{P} = (P_1, \dots, P_r)$ of lattice paths by

$$w_{\mathbf{x},\mathbf{y}}(\mathcal{P}) = \prod_{j=1}^{r} w_{\mathbf{x},\mathbf{y}}(P_j). \tag{12}$$

Now we are prepared to formulate the promised generalizations of Theorems 1,2,3.

Theorem 4. *Let $\mathcal{A}_i=(A_1^{(i)},A_2^{(i)})$ and $\mathcal{E}_i=(E_1^{(i)},E_2^{(i)})$ be lattice points satisfying*

$$A_1^{(1)}+1\le A_1^{(2)}+2\le\dots\le A_1^{(r)}+r,\quad A_2^{(1)}\ge A_2^{(2)}\ge\dots\ge A_2^{(r)}, \tag{13}$$

and

$$E_1^{(1)}\le E_1^{(2)}\le\dots\le E_1^{(r)},\quad E_2^{(1)}-1\ge E_2^{(2)}-2\ge\dots\ge E_2^{(r)}-r. \tag{14}$$

The generating function $\sum_{\mathcal{P}} w_{\mathbf{x},\mathbf{y}}(\mathcal{P})$ where the sum is over all families $\mathcal{P}=(P_1,\dots,P_r)$ of non-crossing lattice paths $P_i:\mathcal{A}_i\to\mathcal{E}_i$, is

$$\det_{1\le s,t\le r}\left(f_{s-t}(\mathbf{x},A_1^{(s)},E_1^{(t)}-1,\mathbf{y},A_2^{(s)}+1,E_2^{(t)})\right), \tag{15}$$

where $f_m(\mathbf{x},a,b,\mathbf{y},c,d)=\sum_k e_{k+m}(x_a,\dots,x_b)e_k(y_c,\dots,y_d)$ with $e_n(z_1,\dots,z_h)$ denoting the elementary symmetric function in the variables $z_1,\dots,z_h$.

Theorem 5. *Let $\mathcal{A}_i=(A_1^{(i)},A_2^{(i)})$ and $\mathcal{E}_i=(E_1^{(i)},E_2^{(i)})$ be lattice points satisfying (13) and (14) and*

$$A_1^{(1)}\ge A_2^{(1)}\quad\text{and}\quad E_1^{(1)}\ge E_2^{(1)}. \tag{16}$$

The generating function $\sum_{\mathcal{P}} w_{\mathbf{x},\mathbf{x}}(\mathcal{P})$ where the sum is over all families $\mathcal{P}=(P_1,\dots,P_r)$ of non-crossing lattice paths $P_i:\mathcal{A}_i\to\mathcal{E}_i$, such that P_1 does not cross the line $x=y$, is

$$\det_{1\le s,t\le r}\Big(f_{s-t}(\mathbf{x},A_1^{(s)},E_1^{(t)}-1,\mathbf{x},A_2^{(s)}+1,E_2^{(t)}) - f_{-s-t}(\mathbf{x},A_2^{(s)}+1,E_1^{(t)}-1,\mathbf{x},A_1^{(s)},E_2^{(t)})\Big). \tag{17}$$

Theorem 6. *Let $\mathcal{A}_i=(A_1^{(i)},A_2^{(i)})$ and $\mathcal{E}_i=(E_1^{(i)},E_2^{(i)})$ be lattice points satisfying*

$$A_1^{(1)}\le A_1^{(2)}\le\dots\le A_1^{(r)},\quad A_2^{(1)}-1\ge A_2^{(2)}-2\ge\dots\ge A_2^{(r)}-r, \tag{18}$$

$$E_1^{(1)}+1\le E_1^{(2)}+2\le\dots\le E_1^{(r)}+r,\quad E_2^{(1)}\ge E_2^{(2)}\ge\dots\ge E_2^{(r)}, \tag{19}$$

and (16). The generating function $\sum_{\mathcal{P}} w_{\mathbf{x},\mathbf{x}}(\mathcal{P})$ where the sum is over all families $\mathcal{P}=(P_1,\dots,P_r)$ of non-crossing lattice paths $P_i:\mathcal{A}_i\to\mathcal{E}_i$, such that P_1 does not cross the line $x=y$, is

$$\det_{1\le s,t\le r}\Big(f_{s-t}(\mathbf{x},A_1^{(s)}+1,E_1^{(t)},\mathbf{x},A_2^{(s)},E_2^{(t)}-1) - f_{-s-t+2}(\mathbf{x},A_2^{(s)},E_1^{(t)},\mathbf{x},A_1^{(s)}+1,E_2^{(t)}-1)\Big). \tag{20}$$

Clearly, Theorems 1,2,3 result from Theorems 4,5,6, respectively, by “unshifting”, i.e. by replacing $A_1^{(i)}$ by $A_1^{(i)}-i+1$, $A_2^{(i)}$ by $A_2^{(i)}+i-1$, etc., setting $x_i=y_i=z$ and extracting the coefficient of z^{2K} in (15), (17), and (20), respectively.

3. The proofs.

Proof of Theorem 4. In the proof we are also considering *skew* two-rowed arrays. Let $j>0$. We say that the two-rowed array P *is of the type* j if P has the form

$$\begin{matrix} a_{-j+1} & a_{-j+2} & \dots & a_{-1} & a_0 & a_1 & \dots & a_k \\ & & & & & b_1 & \dots & b_k \end{matrix}$$

for some $k\ge 0$. We say that P *is of the type* $-j$ if P has the form

$$\begin{matrix} & & & & & a_1 & \dots & a_k \\ b_{-j+1} & b_{-j+2} & \dots & b_{-1} & b_0 & b_1 & \dots & b_k \end{matrix}$$

for some $k\ge 0$. Note that the placement of indices is chosen such that non-positive indices can occur only in *one* row of P, while the positive indices occur in both rows of P. We extend the weight function $w_{\mathbf{x},\mathbf{y}}$ to skew arrays in the obvious way,

$$w_{\mathbf{x},\mathbf{y}}(P)=\prod_i x_{a_i}\prod_j y_{b_j}.$$

First we give the combinatorial interpretation of the determinant (15) in terms of two-rowed arrays. It is easy to see that (15) is the generating function

$$\sum_{(\mathcal{P},\sigma)}\operatorname{sgn}\sigma\, w_{\mathbf{x},\mathbf{y}}(\mathcal{P}), \tag{21}$$

where the sum is over all pairs $(\mathcal{P},\sigma)$ of permutations σ in $\mathfrak{S}_r$, the symmetric group of order r, and families $\mathcal{P}=(P_1,\dots,P_r)$ of two-rowed arrays, P_i being of type $\sigma(i)-i$ and the bounds for the entries of P_i being as follows,

$$\begin{matrix} A_1^{(\sigma(i))}\le & \dots\ a_{k_i}^{(i)} & \le E_1^{(i)}-1 \\ A_2^{(\sigma(i))}+1\le & \dots\ b_{k_i}^{(i)} & \le E_2^{(i)} \end{matrix}, \tag{22}$$

$i=1,\dots,r$.

The outline of the proof is as follows. Next we extend the notion of being non-crossing to two-rowed arrays. We then show that in the sum (21) all contributions corresponding to pairs $(\mathcal{P},\sigma)$, where $\mathcal{P}$ is a crossing family (to be explained below) of two-rowed arrays, cancel. This is done by constructing a weight-preserving, sign-reversing involution on those pairs. Finally we show that in a pair $(\mathcal{P},\sigma)$ with $\sigma\ne\mathrm{id}$ the family $\mathcal{P}$ must be crossing. This establishes that only pairs $(\mathcal{P},\mathrm{id})$ where $\mathcal{P}$ is a non-crossing family of two-rowed arrays contribute to the sum (21). But these pairs exactly correspond to the families of non-crossing paths under consideration, hence Theorem 4 would be proved.

Let M_1 and M_2 be two-rowed arrays, given by

$$M_1:\qquad \begin{matrix} A_1\le & \dots\ a_k & \le E_1-1 \\ A_2+1\le & \dots\ b_k & \le E_2 \end{matrix}$$

and

$$M_2: \qquad \begin{array}{rcl} B_1 \le & \dots\ c_l & \le F_1 - 1 \\ B_2 + 1 \le & \dots\ d_l & \le F_2 \end{array},$$

respectively. By definition we set $a_{k+1} := E_1$, we set $b_0 := A_2$ in case that the sequence $\dots\ b_0$ is empty, and we set $c_0 := B_1 - 1$ in case that the sequence $\dots\ c_0$ is empty. We say that (a_I, d_J) is a *crossing point* of M_1 and M_2 if

$$c_J < a_I \tag{23a}$$

$$b_{I-1} < d_J \tag{23b}$$

and

$$1 \le I \le k+1, \quad 0 \le J \le l. \tag{23c}$$

These inequalities should be understood to hold only if all variables are defined. In particular, if the sequence $\dots\ d_0$ is empty the inequality (23b) does not make sense for $J = 0$. However, if the sequence $\dots\ c_0$ is empty the inequality (23a) makes sense because of the conventional assignment for c_0 above.

Let $(\mathcal{P}, \sigma)$ be a pair under consideration for the sum (21). Besides, we assume that $\mathcal{P}$ contains two two-rowed arrays P_i and P_{i+1} with consecutive indices that have a crossing point. In the sequel two-rowed arrays with consecutive indices will be called *neighbouring* two-rowed arrays. A pair $(\mathcal{P}, \sigma)$ where $\mathcal{P}$ contains neighbouring two-rowed arrays with a crossing point will be called *crossing*. Otherwise it will be called *non-crossing*. We are going to construct a weight-preserving (with respect to the weight function $w_{\mathbf{x},\mathbf{y}}$) and sign-reversing (with respect to $\operatorname{sgn}\sigma$) involution on crossing pairs $(\mathcal{P}, \sigma)$. Consider all crossing points of neighbouring arrays. Among these points choose those with maximal x-coordinate, and among all those choose the crossing point with maximal y-coordinate. Denote this crossing point by S. Let i be minimal such that S is a crossing point of P_i and P_{i+1}. Let $P_i = (a \mid b) = (\dots a_{k_i} \mid \dots b_{k_i})$ and $P_{i+1} = (c \mid d) = (\dots c_{k_{i+1}} \mid \dots d_{k_{i+1}})$. Recall that P_i is of type $\sigma(i) - i$ and P_{i+1} is of type $\sigma(i+1) - i - 1$ and that the bounds of the entries in P_i and P_{i+1} are determined by (22). By (23), S being a crossing point of P_i and P_{i+1} means that there exist I and J such that P_i looks like

$$\begin{array}{rcl} A_1^{(\sigma(i))} \le & \dots\ a_{I-1}\ a_I\ \dots\ a_{k_i} & \le E_1^{(i)} - 1 \\ A_2^{(\sigma(i))} + 1 \le & \dots\ b_{I-1}\ b_I\ \dots\ b_{k_i} & \le E_2^{(i)} \end{array}, \tag{24}$$

P_{i+1} looks like

$$\begin{array}{rcl} A_1^{(\sigma(i+1))} \le & \dots\dots\dots\ c_J\ c_{J+1}\ \dots\ c_{k_{i+1}} & \le E_1^{(i+1)} - 1 \\ A_2^{(\sigma(i+1))} + 1 \le & \dots\ d_{J-1}\ d_J\ \dots\dots\dots\ d_{k_{i+1}} & \le E_2^{(i+1)} \end{array}, \tag{25}$$

$S = (a_I, d_J)$,

$$c_J < a_I \tag{26a}$$

$$b_{I-1} < d_J \tag{26b}$$

and

$$1 \le I \le k_i + 1, \quad 0 \le J \le k_{i+1}. \tag{26c}$$

Because of the construction of S the indices I *and* J are maximal with respect to (26a,b,c).

We map $(\mathcal{P}, \sigma)$ to the pair $(\bar{\mathcal{P}}, \sigma \circ (i, i+1))$ ($(i, i+1)$ denotes the transposition exchanging i and $i+1$), where $\bar{\mathcal{P}} = (P_1, \dots, P_{i-1}, \bar{P}_i, \bar{P}_{i+1}, P_{i+2}, \dots, P_r)$ with $\bar{P}_i$ being given by

$$\begin{matrix} \dots & c_J & a_I & \dots & a_{k_i} \\ \dots & d_{J-1} & b_I & \dots & b_{k_i} \end{matrix}, \tag{27a}$$

$\bar{P}_{i+1}$ being given by

$$\begin{matrix} \dots\dots\dots & a_{I-1} & c_{J+1} & \dots & c_{k_{i+1}} \\ \dots & b_{I-1} & d_J & \dots\dots\dots & d_{k_{i+1}} \end{matrix}. \tag{27b}$$

First of all, this operation is well-defined, i.e., all the rows in (27a) and (27b) are strictly increasing. To see this we have to check $c_J < a_I$, $d_{J-1} < b_I$, $a_{I-1} < c_{J+1}$, and $b_{I-1} < d_J$. This is obvious for the first and last inequality, because of (26a) and (26b). As for the second inequality, let us suppose $d_{J-1} \ge b_I$. Then, by (26a), we have $c_J < a_I < a_{I+1}$ and $b_I \le d_{J-1} < d_J$. This means that (a_{I+1}, d_J) is a crossing point of P_i and P_{i+1}, with an x-coordinate larger than that of $\mathcal{S} = (a_I, d_J)$, contradicting the "maximality" of $\mathcal{S}$. Similarly, if we assume $a_{I-1} \ge c_{J+1}$, we have $c_{J+1} \le a_{I-1} < a_I$ and, by (26b), $b_{I-1} < d_J < d_{J+1}$. This means that (a_I, d_{J+1}) is a crossing point of P_i and P_{i+1}, with a y-coordinate larger than that of $\mathcal{S} = (a_I, d_J)$, again contradicting the "maximality" of $\mathcal{S}$.

We claim that $(\bar{\mathcal{P}}, \sigma \circ (i, i+1))$ is again a pair under consideration for the generating function (21). That is, we claim that $\bar{P}_i$ is of type $(\sigma \circ (i, i+1))(i) - i = \sigma(i+1) - i$, that $\bar{P}_{i+1}$ is of type $(\sigma \circ (i, i+1))(i+1) - i - 1 = \sigma(i) - i - 1$, and that the bounds for the entries of $\bar{P}_i$ are given by

$$\begin{matrix} A_1^{(\sigma(i+1))} \le & \dots & c_J & a_I & \dots & a_{k_i} & \le E_1^{(i)} - 1 \\ A_2^{(\sigma(i+1))} + 1 \le & \dots & d_{J-1} & b_I & \dots & b_{k_i} & \le E_2^{(i)} \end{matrix}, \tag{28}$$

and that those for $\bar{P}_{i+1}$ are given by

$$\begin{matrix} A_1^{(\sigma(i))} \le & \dots\dots\dots & a_{I-1} & c_{J+1} & \dots & c_{k_{i+1}} & \le E_1^{(i+1)} - 1 \\ A_2^{(\sigma(i))} + 1 \le & \dots & b_{I-1} & d_J & \dots\dots\dots & d_{k_{i+1}} & \le E_2^{(i+1)} \end{matrix}. \tag{29}$$

The claims concerning the types of $\bar{P}_i$ and $\bar{P}_{i+1}$ are trivial. Therefore let us consider the bounds. We distinguish between several cases.

(a) If $2 \le I \le k_i$ and $2 \le J \le k_{i+1} - 1$ there is no problem, since then the sequences $\dots a_{I-1}$, $a_I \dots a_{k_i}$, $\dots b_{I-1}$, $b_I \dots b_{k_i}$, $\dots c_J$, $c_{J+1} \dots c_{k_{i+1}}$, $\dots d_{J-1}$, $d_J \dots d_{k_{i+1}}$ are all nonempty and therefore the constraints in (28) and (29) obviously hold.

(b) If $I = 1$ and the sequence $\dots a_0$ is empty we have to prove $c_{J+1} \ge A_1^{(\sigma(i))}$. Suppose $a_1 > c_{J+1}$. The inequality (26b) implies $b_0 < d_J < d_{J+1}$ therefore the point (a_1, d_{J+1}) would also be a crossing point of P_i and P_{i+1} which violates the maximality of J with respect to (26a,b,c). Hence, by (24) we have $c_{J+1} \geq a_1 \geq A_1^{(\sigma(i))}$.

(c) If $I = 1$ and the sequence $\dots b_0$ is empty we have to prove $d_J \geq A_2^{(\sigma(i))} + 1$. In this case the assignment $b_0 := A_2^{(\sigma(i))}$ applies. Because of (26b) we have $A_2^{(\sigma(i))} = b_0 < d_J$.

(d) If $J = 0$ and the sequence $\dots c_0$ is empty we have to prove $a_I \geq A_1^{(\sigma(i+1))}$. In this case the assignment $c_0 := A_1^{(\sigma(i+1))} - 1$ applies. Because of (26a) we have $A_1^{(\sigma(i+1))} - 1 = c_0 < a_I$.

(e) If $J = 1$ and the sequence $\dots d_0$ is nonempty nothing has to be proved.

(f) If $J = 1$ and the sequence $\dots d_0$ is empty we have to prove $b_I \geq A_2^{(\sigma(i+1))} + 1$. Suppose $d_1 > b_I$. The inequality (26a) implies $c_J < a_I < a_{I+1}$ therefore the point (a_{I+1}, d_J) would also be a crossing point of P_i and P_{i+1} which violates the maximality of I with respect to (26a,b,c). Hence, by (25) we have $b_I \geq d_1 \geq A_2^{(\sigma(i+1))} + 1$.

(g) If $I = k_i + 1$ we have to prove $c_J \leq E_1^{(i)} - 1$ and $d_{J-1} \leq E_2^{(i)}$. In this case the assignment $a_{k_i+1} := E_1^{(i)}$ applies. By (26a) we have $c_J < a_{k_i+1} = E_1^{(i)}$. By (25) and (14), we have $d_{J-1} < d_J \leq E_2^{(i+1)} \leq E_2^{(i)} + 1$.

(h) If $J = k_{i+1}$ we have to prove $a_{I-1} \leq E_1^{(i+1)} - 1$. By (24) and (14) we have $a_{I-1} \leq E_1^{(i)} - 1 \leq E_1^{(i+1)} - 1$.

Obviously the map (27) is weight-preserving with respect to $w_{\mathbf{x},\mathbf{y}}$ and reverses the signs of the associated permutations.

Next we claim that the map (27) is an involution. To establish this claim it suffices to show that S is also a crossing point of $\bar{P}_i$ and $\bar{P}_{i+1}$, and also maximal in the sense explained above. Once this is shown it easily seen that an application of our mapping (27) to $(\bar{\mathcal{P}}, \sigma\circ(i, i+1))$ gives $(\mathcal{P}, \sigma)$ again. Suppose that $\bar{P}_i = (\bar{a} \mid \bar{b}) = (\dots \bar{a}_{\bar{k}_i} \mid \dots \bar{b}_{\bar{k}_i})$ and $\bar{P}_{i+1} = (\bar{c} \mid \bar{d}) = (\dots \bar{c}_{\bar{k}_{i+1}} \mid \dots \bar{d}_{\bar{k}_{i+1}})$. Let $\bar{I}$ and $\bar{J}$ be chosen such that $\bar{a}_{\bar{I}} = a_I$ and $\bar{d}_{\bar{J}} = d_J$ (compare with (27)). Then we have $\bar{b}_{\bar{I}-1} = d_{J-1}$ and $\bar{c}_{\bar{J}} = a_{I-1}$. Because of $a_{I-1} < a_I$ and $d_{J-1} < d_J$ we conclude $\bar{c}_{\bar{J}} < \bar{a}_{\bar{I}}$ and $\bar{b}_{\bar{I}-1} < \bar{d}_{\bar{J}}$. By (23) this is equivalent to saying that $(\bar{a}_{\bar{I}}, \bar{d}_{\bar{J}}) = (a_I, d_J) = S$ is a crossing point of $\bar{P}_i$ and $\bar{P}_{i+1}$. (These arguments also hold in the "degenerate" cases $I = 1$, $J = 0, 1$, etc.) That S is also maximal for $\bar{\mathcal{P}}$ follows from the fact that when applying the map (27) to P_i and P_{i+1} nothing was changed to the right of a_I, b_I, d_J, and c_{J+1}.

Finally we have to show that, given a pair $(\mathcal{P}, \sigma)$, $\mathcal{P} = (P_1, \dots, P_r)$, $\sigma \neq \mathrm{id}$, there exist neighbouring two-rowed arrays P_i and P_{i+1} having a crossing point. If $\sigma \neq \mathrm{id}$ then there must exist an i with $\sigma(i+1) < i+1$. Without loss of generality we may assume that i is minimal with this property. Then we have $\sigma(i) \geq i$. Besides, from $\sigma(i) \geq i \geq \sigma(i+1)$ it follows that $\sigma(i) > \sigma(i+1)$. Let $P_i = (a \mid b) = (\dots a_{k_i} \mid \dots b_{k_i})$ and $P_{i+1} = (c \mid d) = (\dots c_{k_{i+1}} \mid \dots d_{k_{i+1}})$. Because of $\sigma(i) - i \geq 0$ the sequence $\dots b_0$ is empty, hence the assignment $b_0 := A_2^{(\sigma(i))}$ applies. Because of $\sigma(i+1) - i - 1 < 0$ the sequence $\dots c_0$ is empty, hence the assignment $c_0 := A_1^{(\sigma(i+1))} - 1$ applies. Because of $\sigma(i) > \sigma(i+1)$ and (13) the inequalities $c_0 = A_1^{(\sigma(i+1))} - 1 \le A_1^{(\sigma(i))} < a_1$ and $b_0 = A_2^{(\sigma(i))} \le A_2^{(\sigma(i+1))} < d_0$ hold. By (23) this means that (a_1, d_0) is a crossing point of P_i and P_{i+1}. $\square$

Remark. The operation (24)/(25) → (28)/(29) that is the backbone of the preceding proof and is an analogue of the Gessel–Viennot involution in the context of two-rowed arrays is inspired by [10, operation (5.3.6)/(5.3.10)].

Proof of Theorem 5. Again we work with families of two-rowed arrays. This time we consider triples $(\mathcal{P}, \sigma, \eta)$, where σ is a permutation in $\mathfrak{S}_r$, $\eta \in \{-1, 1\}^r$, and $\mathcal{P} = (P_1, \dots, P_r)$ is a family of two-rowed arrays, with P_i being of type $\eta_i \sigma(i) - i$ and the bounds of P_i being given by

$$\begin{matrix} A_1^{(\sigma(i))} \le & \dots & \le E_1^{(i)} - 1 \\ A_2^{(\sigma(i))} + 1 \le & \dots & \le E_2^{(i)} \end{matrix}, \quad \text{for } \eta = 1, \tag{30a}$$

respectively

$$\begin{matrix} A_2^{(\sigma(i))} + 1 \le & \dots & \le E_1^{(i)} - 1 \\ A_1^{(\sigma(i))} \le & \dots & \le E_2^{(i)} \end{matrix}, \quad \text{for } \eta = -1. \tag{30b}$$

Define $\operatorname{sgn} \eta := \prod_{i=1}^r \eta_i$. It is easy to see that (17) is the generating function

$$\sum_{(\mathcal{P}, \sigma, \eta)} \operatorname{sgn} \eta \, \operatorname{sgn} \sigma \, w_{\mathbf{x},\mathbf{x}}(\mathcal{P}) \tag{31}$$

where the sum is over all triples which were described above.

We adopt the notion of "crossing of neighbouring arrays" from the previous proof. In addition, we have to consider crossings of P_1 with the line $x = y$. Let $P_1 = (a \mid b)$, or with bounds included

$$\begin{matrix} L_1 \le & \dots\dots\dots \; a_I \; a_{I+1} \; \dots \; a_{k_1} & \le E_1^{(1)} - 1 \\ L_2 \le & \dots \; b_{I-1} \; b_I \; \dots\dots\dots \; b_{k_1} & \le E_2^{(1)} \end{matrix}. \tag{32}$$

Recall that L_1 and L_2 are $A_1^{(\sigma(1))} + 1$ and $A_2^{(\sigma(1))}$ or the other way round, depending on whether $\eta_1 = 1$ or $\eta = -1$. Let us, by convention, set $a_0 := L_1 - 1$. We say that (b_I, b_I) is a *crossing point of* P_1 *and* $x = y$ if $I \ge 0$ and $a_I < b_I$. Again it is understood that this inequality only holds if both a_I and b_I are defined. In particular, this means that in case $I = 0$ the inequality only makes sense if the sequence $\dots a_0$ is empty (in which case the assignment $a_0 := L_1 - 1$ applies). We say that a family $\mathcal{P} = (P_1, \dots, P_r)$ is a *crossing family* if either their is a crossing of neighbouring arrays or their is a crossing of P_1 and $x = y$.

Similarly to the proof of Theorem 4 we shall show that in the sum (31) all contributions corresponding to triples $(\mathcal{P}, \sigma, \eta)$ where $\mathcal{P}$ is a crossing family of two-rowed arrays cancel by constructing a weight-preserving (with respect to $w_{\mathbf{x},\mathbf{x}}$), sign-reversing (with respect to $\operatorname{sgn} \eta \operatorname{sgn} \sigma$) involution on those triples. Finally we show that in a triple $(\mathcal{P}, \sigma, \eta)$ with $(\sigma, \eta) \ne (\mathrm{id}, (1, 1, \dots, 1))$ the family $\mathcal{P}$ must be crossing. This establishes that only triples $(\mathcal{P}, \mathrm{id}, (1,1,\dots,1))$ where $\mathcal{P}$ is a non-crossing family of two-rowed arrays contribute to the sum (31). But these triples exactly correspond to the families of non-crossing paths under consideration, hence Theorem 5 would be proved.

Let $(\mathcal{P}, \sigma, \eta)$ be a triple where $\mathcal{P} = (P_1, \dots, P_r)$ is a crossing family of two-rowed arrays. Consider all crossing points of neighbouring arrays and all crossing points of P_1 and $x = y$. Among these points choose those with maximal x-coordinate, and among all those choose the crossing point with maximal y-coordinate. Denote this crossing point by S. If S is a crossing point of neighbouring arrays let i be minimal such that S is a crossing point of P_i and P_{i+1}. Map $(\mathcal{P}, \sigma, \eta)$ to $(\bar{\mathcal{P}}, \sigma \circ (i, i+1), \eta^{(i,i+1)})$, where $\eta^{(i,i+1)} = (\eta_1, \dots, \eta_{i+1}, \eta_i, \dots, \eta_r)$ and $\bar{\mathcal{P}} = (\dots, \bar{P}_i, \bar{P}_{i+1}, \dots)$ with $\bar{P}_i$ and $\bar{P}_{i+1}$ being constructed by (28) and (29), respectively, as in the proof of Theorem 4. Note that $\operatorname{sgn} \eta^{(i,i+1)} \operatorname{sgn}\big(\sigma \circ (i, i+1)\big) = -\operatorname{sgn}\eta \operatorname{sgn}\sigma$ so that the map indeed changes the sign of the corresponding terms in the generating function (31).

If S is no crossing point of neighbouring arrays, then it has to be a crossing point of P_1 and $x = y$. In this case we map $(\mathcal{P}, \sigma, \eta)$ to the triple $(\bar{\mathcal{P}}, \sigma, \eta^{(1)})$ where $\eta^{(1)} = (-\eta_1, \eta_2, \dots, \eta_r)$ and $\bar{\mathcal{P}} = (\bar{P}_1, P_2, \dots, P_r)$ with $\bar{P}_1$ being constructed as follows. Let P_1 be given by (32) and let I be the index such that $S = (b_I, b_I)$. Then $\bar{P}_1$ is defined by

$$\begin{array}{llll} L_2 \le & \dots\dots\ b_{I-1}\ a_{I+1}\ \dots\ a_{k_1} & \le E_1^{(1)} - 1 \\ L_1 \le & \dots\ a_I\ \ b_I\ \ \dots\dots\dots\ b_{k_1} & \le E_2^{(1)} \end{array}. \tag{33}$$

To check that this is well-defined we have to verify $b_{I-1} < a_{I+1}$. In fact, if we assume $b_{I-1} \ge a_{I+1}$, we have $a_{I+1} < b_{I+1}$ (because otherwise $a_{I+1} \ge b_{I+1} > b_I > b_{I-1}$, which is a contradiction to our assumption). But this means that (b_{I+1}, b_{I+1}) is a crossing point of P_1 and $x = y$ with larger coordinates than $\mathcal{S} = (b_I, b_I)$, which contradicts the “maximality” of $\mathcal{S}$. Also for being well-defined, we have to check $L_1 \le b_0$ in case that $I = 0$ and $\dots a_0$ is empty. But in this case the assignment $a_0 := L_1 - 1$ applies. And, since $I = 0$ we have $a_0 < b_0$. But this is exactly equivalent to $L_1 \le b_0$.

Next we note that the type of $\bar{P}_1$ is $-\eta_1\sigma(1) + 1 - 2 = -\eta_1\sigma(1) - 1 = \eta_1^{(1)}\sigma(1) - 1$. Trivially, $\operatorname{sgn}\eta^{(1)} = -\operatorname{sgn}\eta$ so also this mapping changes the sign of the corresponding terms in the generating function (31). Therefore $(\bar{\mathcal{P}}, \sigma, \eta^{(1)})$ is indeed again a triple under consideration for the generating function (31).

That (b_I, b_I) is also a crossing point of $\bar{P}_1$ is obvious. It is also maximal in the sense explained above since the entries to the right of a_{I+1} and b_I were not changed. Therefore, applying the map again to the triple $(\bar{\mathcal{P}}, \sigma, \eta^{(1)})$ gives $(\mathcal{P}, \sigma, \eta)$.

Finally we have to show that if $(\sigma, \eta) \ne (\mathrm{id}, (1,1,\dots,1))$ in the triple $(\mathcal{P}, \sigma, \eta)$ the family $\mathcal{P}$ must be a crossing family. Let $\eta \ne (1,1,\dots,1)$. Let i be minimal with $\eta_i = -1$. If $i = 1$ then either there is an $I \ge 1$ with $a_I < b_I$, i.e. a crossing of P_1 and $x = y$. Or for all $I \ge 1$ there holds $a_I \ge b_I$. By (13) and (16) we have $b_0 \ge L_2 + \sigma(1) = A_1^{(\sigma(1))} + \sigma(1) \ge A_1^{(1)} + 1 > A_2^{(1)} \ge A_2^{(\sigma(1))} = L_1 - 1 = a_0$. Hence (b_0, b_0) is a crossing of P_1 and $x = y$. (Note that b_{-1} and b_0 exist since for $\eta_1 = -1$ the type of P_1 is $-\sigma(1) - 1 \le -2$.) Now suppose that $i \ge 2$. In addition assume that there is no crossing point of neighbouring arrays. We claim that again there must be a crossing point of P_1 and $x = y$. Because of $\eta_i = -1$, P_i is of the type $-\sigma(i) - i$. Let

$P_{i-1}=(a \mid b)$ and P_i be given by

$$\begin{array}{rcl} A_2^{(\sigma(i))}+1\le & c_1\,\dots\,c_{k_i} & \le E_1^{(i)}-1 \\ A_1^{(\sigma(i))}\le & \dots\,d_0\;d_1\,\dots\,d_{k_i} & \le E_2^{(i)} \end{array}.$$

By assumption P_i and P_{i-1} are non-crossing. Because $\dots c_0$ is empty the assignment $c_0:=A_2^{(\sigma(i))}$ applies. By (13) there hold the inequalities $d_0\ge A_1^{(\sigma(i))}+\sigma(i)+i-1\ge A_1^{(1)}+1+i-1>A_2^{(1)}\ge A_2^{(\sigma(i))}=c_0$. Now let j be maximal with $a_j\le c_0$. Then there holds $c_0<a_{j+1}$. If $b_j<d_0$ then (a_{j+1},d_0) would be a crossing point of P_{i-1} and P_i which contradicts our assumptions. Hence we have $a_j\le c_0<d_0\le b_j$. The same argument is repeated with P_{i-2} and P_{i-1}, etc., until we arrive at P_1. Obviously, this gives us a crossing point of P_1 and $x=y$, as was claimed.

If $\sigma\ne \mathrm{id}$ and $\eta=(1,1,\dots,1)$, the same arguments as in the proof of Theorem 4 apply to establish that there must be a crossing of neighbouring arrays. □

Remark. The operation (32) → (33) that is an analogue of the reflection principle (see e.g. [3, p. 22]) for two-rowed arrays is inspired by [13, (2.12); 10, (5.2.4)].

Sketch of Proof of Theorem 6. Here we encode paths by their EN-turns. For example the two-rowed array representation of the path in Figure 1 would be

$$\begin{array}{c} 2\;5\;6 \\ 1\;3\;4 \end{array},$$

or with bounds included,

$$\begin{array}{rcl} 2\le & 2\;5\;6 & \le 6 \\ -1\le & 1\;3\;4 & \le 5 \end{array}.$$

For the combinatorial interpretation of the determinant (20) we consider triples $(\mathcal{P},\sigma,\eta)$, where σ is a permutation in $\mathfrak{S}_r$, $\eta\in\{-1,1\}^r$, and $\mathcal{P}=(P_1,\dots,P_r)$ is a family of two-rowed arrays, with P_i being of type $\eta_i\sigma(i)-i+1-\eta_i$ and the bounds of P_i being given by

$$\begin{array}{rcl} A_1^{(\sigma(i))}+1\le & \dots & \le E_1^{(i)} \\ A_2^{(\sigma(i))}\le & \dots & \le E_2^{(i)}-1 \end{array},\quad \text{for } \eta=1, \tag{34a}$$

respectively

$$\begin{array}{rcl} A_2^{(\sigma(i))}\le & \dots & \le E_1^{(i)} \\ A_1^{(\sigma(i))}+1\le & \dots & \le E_2^{(i)}-1 \end{array},\quad \text{for } \eta=-1. \tag{34b}$$

It is easy to see that (20) is the generating function

$$\sum_{(\mathcal{P},\sigma,\eta)} \operatorname{sgn}\eta\,\operatorname{sgn}\sigma\, w_{\mathbf{x},\mathbf{x}}(\mathcal{P}) \tag{35}$$

where the sum is over all triples which were described above.

Since now we are considering EN-turns, we have to redefine what we mean by a crossing of two-rowed arrays and a crossing of a two-rowed array and $x = y$. Let M_1 and M_2 be two-rowed arrays, given by

$$M_1 : \quad \begin{matrix} A_1+1 \le & \dots\ a_k & \le E_1 \\ A_2 \le & \dots\ b_k & \le E_2 - 1 \end{matrix}$$

and

$$M_2 : \quad \begin{matrix} B_1+1 \le & \dots\ c_l & \le F_1 \\ B_2 \le & \dots\ d_l & \le F_2 - 1 \end{matrix},$$

respectively. By definition we set $d_{l+1} := F_2$, we set $c_0 := B_1$ in case that the sequence $\dots\ c_0$ is empty, and we set $b_0 := A_2 - 1$ in case that the sequence $\dots\ b_0$ is empty. We say that (a_I, d_J) is a *crossing point* of M_1 and M_2 if

$$c_{J-1} < a_I \tag{36a}$$

$$b_I < d_J \tag{36b}$$

and

$$0 \le I \le k, \quad 1 \le J \le l+1. \tag{36c}$$

To define crossings of a two-rowed array M with $x = y$, let M be given by

$$\begin{matrix} L_1 \le & \dots\ a_{k_1} & \le U_1 \\ L_2 \le & \dots\ b_{k_1} & \le U_2 \end{matrix}.$$

By convention we set $b_{k_1+1} := U_2 + 1$. The point (b_{I+1}, b_{I+1}) is called a crossing point of M and $x = y$ if $1 \le I \le k_1 + 1$ and $a_I < b_{I+1}$. Note that if M_1, M_2, M correspond to lattice paths (by the EN-turn encoding) P_1, P_2, P, respectively, the first definition exactly is equivalent to P_1 and P_2 having a crossing, while the second is equivalent to P and $x = y$ having a crossing.

The arguments are now completely analogous to those in the proof of Theorem 5. So we shall not give all the details. Again we construct a weight-preserving (with respect to $w_{\mathbf{x},\mathbf{x}}$), sign-reversing (with respect to $\operatorname{sgn}\eta \operatorname{sgn}\sigma$) involution on triples $(\mathcal{P}, \sigma, \eta)$ where $\mathcal{P}$ is a crossing (in the new sense) family. Also here we have to distinguish between two cases. Either there is a crossing of neighbouring paths, P_i and P_{i+1} say. Or else their is a crossing of P_1 and $x = y$. In the first case assume that P_i is given by

$$\begin{matrix} \dots\ a_{I-1}\ a_I\ \dots\dots \\ \dots\dots\ b_I\ b_{I+1}\ \dots \end{matrix}, \tag{37a}$$

and P_{i+1} be given by

$$\begin{matrix} \dots\ c_{J-1}\ c_J\ \dots \\ \dots\ d_{J-1}\ d_J\ \dots \end{matrix}. \tag{37b}$$

As in the proof of Theorem 5 it is assumed that $S = (a_I, d_J)$ is a "maximal" crossing point, and in particular a crossing point of P_i and P_{i+1}. Then we map $(\mathcal{P}, \sigma, \eta)$ to $(\bar{\mathcal{P}}, \sigma \circ (i, i+1), \eta^{(i,i+1)})$ where $\mathcal{P} = (P_1, \dots, \bar{P}_i, \bar{P}_{i+1}, \dots, P_r)$ with $\bar{P}_i$ being given by

$$\begin{matrix} \dots\ c_{J-1} & a_I & \dots\dots \\ \dots\dots\ d_{J-1} & b_{I+1} & \dots \end{matrix}, \tag{38a}$$

and $\bar{P}_{i+1}$ being given by

$$\begin{matrix} \dots & a_{I-1} & c_J & \dots \\ \dots & b_I & d_J & \dots \end{matrix}. \tag{38b}$$

In the second case let P_1 be given by

$$\begin{matrix} \dots & a_I & a_{I+1} & \dots \\ \dots & b_I & b_{I+1} & \dots \end{matrix}. \tag{39}$$

Here we assume that $S = (b_{I+1}, b_{I+1})$ is a "maximal" crossing point of P_1 and $x = y$. Then we map $(\mathcal{P}, \sigma, \eta)$ to $(\bar{\mathcal{P}}, \sigma, \eta^{(1)})$ where $\mathcal{P} = (\bar{P}_1, P_2, \dots, P_r)$ with $\bar{P}_1$ being given by

$$\begin{matrix} \dots & b_I & a_{I+1} & \dots \\ \dots & a_I & b_{I+1} & \dots \end{matrix}. \tag{40}$$

It can be shown that this mapping is a weight-preserving, sign-reversing involution and that "non-crossing triples" can only occur for $(\sigma, \eta) = (\mathrm{id}, (1, 1, \dots, 1))$. These exactly correspond to the families of non-crossing paths under consideration. □

Remark. The operation (37a)/(37b) → (38a)/(38b) that is another analogue of the Gessel–Viennot involution in the context of two-rowed arrays is again inspired by [10, operation (5.3.6)/(5.3.10)]. (In fact, it is a simple translation of the operation (24)/(25) → (28)/(29)). The operation (39) → (40) that is another analogue of the reflection principle (see e.g. [3, p. 22]) for two-rowed arrays is inspired by [13, (2.11); 10, (5.2.24)].

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