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Group Actions on Magic Squares

Wolfgang Müller

Abstract: Two types of magic $n \times n$-squares are studied – the pandiagonal squares and the W-squares, in which the coefficients of all 2×2-submatrices have the same sum. For $4 \leq n \leq 8$ big groups – in general wreath products – are described which act on sets of these magic squares.

In the paper [2] we have studied two types of magic 4×4-squares: P- and W-squares. These types admit cyclic permutations of rows and columns. Here we give some results on such $n \times n$-squares, especially for $n \leq 8$. First we recall some definitions.

A magic $n \times n$-square is a $n \times n$-matrix (a_{ij}) with coefficients

$$\{a_{ij} \mid i, j = 1, \ldots, n\} = \{1, 2, \ldots, n^2\} \subset \mathbb{N},$$

such that the coefficients of every row and every column yield the same sum

$$s := \frac{n^2(n^2+1)}{2} \cdot \frac{1}{n} = \frac{n}{2}(n^2+1).$$

a) A magic square is called perfect or pandiagonal or P-square, if the coefficients of every diagonal (and its parallels) have also sum s.

b) A magic square is called W-square, if the coefficients of every 2×2-submatrix
$$\begin{pmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{pmatrix}$$
with $i, j = 1, \ldots, n$ and cyclic notation of the indices have the same sum t. (The only possibility is $t := 2(n^2+1)$.)

Now we list simple properties of such magic squares.

Lemma 1.

a) *The set of all P-squares resp. W-squares is closed under cyclic permutations of rows and columns.*

b) *Every W-square is completely determined by its first row and its first column.*

c) *If n is odd then there exists no W-square.*

d) *If $n \equiv 2 \bmod 4$ then there exists no P-square.*

Proof. a,b) Trivial

c) Considering pairs of neighbouring 2×2-submatrices we see that

$$a_{11} + a_{12} = a_{31} + a_{32} = a_{51} + a_{52} = \ldots,$$

$$a_{11} + a_{21} = a_{13} + a_{23} = a_{15} + a_{25} = \ldots$$

If n is odd then we get $a_{11} + a_{12} = a_{21} + a_{22}$ and $a_{11} + a_{21} = a_{12} + a_{22}$. By adding these equations it follows $a_{11} = a_{22}$. A contradiction to the condition that the coefficients of a magic square are pairwise different.

d) Let be $n = 4m + 2$ with $m \in \mathbb{N}$. If we denote the sum of all coefficients in the k-th row resp. column resp. diagonal by

$$\begin{aligned} R_k &:= \sum_{j=1}^{n} a_{kj}, & C_k &:= \sum_{i=1}^{n} a_{ik}, \\ D_k &:= \sum_{i=1}^{n} a_{i,k+i-1}, & D'_k &:= \sum_{i=1}^{n} a_{i,k-i+1}, \end{aligned} \tag{1}$$

we get

$$\begin{aligned} \sum_{k=2}^{n} (k-1) R_k + \sum_{k=1}^{n-1} k(D_k - C_k) &= \\ &= n\Big(\sum_{i=3}^{n} \sum_{j=1}^{i-2} a_{ij} + \sum_{i=2}^{n} a_{i,n}\Big) \equiv 0 \bmod 2. \end{aligned} \tag{2}$$

Since in a P-square the sums R_k, C_k, D_k are all equal to s the expression (2) equals

$$\sum_{k=2}^{n} (k-1)s = \frac{n(n-1)}{2} \cdot \frac{n}{2}(n^2+1) = (2m+1)^2(n-1)(n^2+1) \equiv 1 \bmod 2.$$

A contradiction. $\square$

Lemma 2. *In every $n \times n$-W-square (a_{ij}) the following diagonal sums (notation as in (1)) are equal:*

$$D_1 = D_3 = \ldots = D_{n-1} = D'_1 = D'_3 = \ldots = D'_{n-1}$$

and

$$D_2 = D_4 = \ldots = D_n = D'_2 = D'_4 = \ldots = D'_n.$$

Therefore every $n \times n$-W-square (a_{ij}) with $D_1 = s$ is a P-square.

Proof. Let $n = 2m$ be an even integer. In an $n \times n$-W-square the coefficients depend on the first row and first column as follows:

$$a_{i,j} = \begin{cases} -a_{1,1} + a_{1,j} + a_{i,1} & \text{if } i \text{ and } j \text{ odd} \\ a_{1,1} + a_{1,j} - a_{i,1} & \text{if } i \text{ odd and } j \text{ even} \\ a_{1,1} - a_{1,j} + a_{i,1} & \text{if } i \text{ even and } j \text{ odd} \\ t - a_{1,1} - a_{1,j} - a_{i,1} & \text{if } i \text{ and } j \text{ even} \end{cases}$$

By inserting these expressions in the "half diagonals"

$$d_{ij} := \sum_{k=0}^{m-1} a_{i+2k,j+2k} \text{ and } d'_{ij} := \sum_{k=0}^{m-1} a_{i+2k,j-2k}$$

we obtain using the abbreviations

$$\begin{aligned} U &:= \sum_{k=1}^{m} a_{1,2k-1}, \quad V := \sum_{k=1}^{m} a_{1,2k}, \\ U^t &:= \sum_{k=1}^{m} a_{2k-1,1}, \quad V^t := \sum_{k=1}^{m} a_{2k,1} \end{aligned} \tag{3}$$

only four different values

$$d_{ij} = d'_{ij} = \begin{cases} -4a_{11} + U + U^t & \text{if } i \text{ and } j \text{ odd} \\ 4a_{11} + V - U^t & \text{if } i \text{ odd and } j \text{ even} \\ 4a_{11} - U + V^t & \text{if } i \text{ even and } j \text{ odd} \\ 4(t - a_{11}) - V - V^t & \text{if } i \text{ and } j \text{ even} \end{cases}$$

From these relations the assertion follows. □

For later use we denote by $\mathcal{S}_n$ the symmetric group over n elements and by M_n the hyper-octahedral group, i.e. the group of all rigid motions τ of the

n-dimensional hypercube $[-1,1]^n \subset \mathbb{R}^n$ with $\tau([-1,1]^n) \subset [-1,1]^n$. M_n is a subgroup of the orthogonal group O_n and isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathcal{S}_n$. Its center $C(M_n)$ is generated by the antipodal map, so $C(M_n)$ has order 2.

Now let $(a_{ij})_{i,j=1,\ldots,n}$ be a W-square. Starting from the first row and first column we get by means of the sum condition in the 2×2-submatrices

$$a_{i,j} + a_{i,j+1} = \begin{cases} a_{1,j} + a_{1,j+1} & \text{if } i \text{ odd} \\ t - a_{1,j} - a_{1,j+1} & \text{if } i \text{ even} \end{cases}$$

and

$$a_{i,j} + a_{i+1,j} = \begin{cases} a_{i,1} + a_{i+1,1} & \text{if } j \text{ odd} \\ t - a_{i,1} - a_{i+1,1} & \text{if } j \text{ even} \end{cases}$$

It follows that the set of all W-squares is closed under the following operations:

Permutations of the odd rows $1, 3, \ldots, n-1$
Permutations of the even rows $2, 4, \ldots, n$
Permutations of the odd columns $1, 3, \ldots, n-1$
Permutations of the even columns $2, 4, \ldots, n$
Interchanging of odd and even rows $\{1, 3, \ldots, n-1\} \leftrightarrow \{2, 4, \ldots, n\}$
Interchanging of odd and even columns $\{1, 3, \ldots, n-1\} \leftrightarrow \{2, 4, \ldots, n\}$
Transposing the square

So the following theorem is clear.

Theorem 3. *Let m be an integer and $n = 2m$. The wreath product*

$$(\mathcal{S}_m \wr \mathcal{S}_2) \wr \mathcal{S}_2 \cong (((\mathcal{S}_m \times \mathcal{S}_m) \rtimes \mathcal{S}_2) \times ((\mathcal{S}_m \times \mathcal{S}_m) \rtimes \mathcal{S}_2)) \rtimes \mathcal{S}_2$$

acts faithfully on the set of all $n \times n$-W-squares. By this action the set $\{D_1, D'_n\}$ of diagonal sums is preserved.

4×4-Squares

In [2] we have proved that every P-square is a W-square and that there is a close connection to the cube problem:

How to relate the integers $1, 2, \ldots, 16$ to the 16 vertices of the 4-dimensional cube $[-1, 1]^4$ such that all squares on its surface have equal sums $s = t = 34$?

If this problem is solved then there is the question how to put the integer x_{abcd} related to the vertex (a, b, c, d) where $a, b, c, d \in \{-1, 1\}$, into the 4×4-matrix such that every 2×2-submatrix corresponds to a square on the cube? This can be done in the following way (unique up to motions in M_4).

$$\begin{matrix} x_{----} & x_{---+} & x_{--++} & x_{--+-} \\ x_{-+--} & x_{-+-+} & x_{-+++} & x_{-++-} \\ x_{++--} & x_{++-+} & x_{++++} & x_{+++-} \\ x_{+---} & x_{+--+} & x_{+-++} & x_{+-+-} \end{matrix}$$

In this matrix only "$-$" is written for "-1" and "$+$" for "$+1$". From [2] we can deduce that every P-square arises from one rigid motion of the marked hypercube $[-1, 1]^4$ where the integers $8, 12, 14, 15$ are neighbours of the integer 1, and that every W-square arises from one rigid motion of the 5-dimensional marked hypercube $[-1, 1]^5$ with equal square sums $s = 34$ where the integers $8, 12, 14, 15, 16$ are neighbours of the integer 1. In this latter cube antipodal vertices have got the same integer. Thus the action of the antipodal map is trivial on the latter cube. Summing up we have

Theorem 4.

a) *The hyper-octahedral group M_4 acts faithfully and transitively on the set of all 4×4-P-squares, i.e. there exist precisely $|M_4| = 2^4 \cdot 4! = 384$ 4×4-P-squares.*

b) *The factor group $M_5/C(M_5)$ acts faithfully and transitively on the set of all 4×4-squares, i.e. there exist precisely $|M_5|/2 = 2^4 \cdot 5! = 1920$ 4×4-W-squares.*

5×5-Squares

By Lemma 1c) only P-squares can exist. By the condition that the coefficients of every row, column and diagonal yield the same sum $s = 65$ we have 20 equations with 25 unknowns. It is easy to see that the rank of the coefficient matrix of this linear equation system is 17. So the solution space is an affine space (over $\mathbb{Q}$) spanned by every 9 of the following 10 permutation matrices

$$\begin{matrix} 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{matrix} \qquad \begin{matrix} 0&1&0&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{matrix} \qquad \begin{matrix} 0&0&1&0&0\\ 0&0&0&0&1\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0 \end{matrix} \qquad \begin{matrix} 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&1&0&0&0 \end{matrix} \qquad \begin{matrix} 0&0&0&0&1\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&1&0&0 \end{matrix}$$

$$\begin{matrix} 1&0&0&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&1&0&0 \end{matrix} \qquad \begin{matrix} 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0 \end{matrix} \qquad \begin{matrix} 0&0&1&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 0&0&0&0&1 \end{matrix} \qquad \begin{matrix} 0&0&0&1&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 1&0&0&0&0 \end{matrix} \qquad \begin{matrix} 0&0&0&0&1\\ 0&0&1&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0 \end{matrix}$$

Note that in these matrices there is exactly one integer "1" in every row resp. column resp. diagonal. Therefore every P-square is a linear combination of these matrices, i.e. it has the form

$$(4) \qquad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5\\ a_4 & a_5 & a_1 & a_2 & a_3\\ a_2 & a_3 & a_4 & a_5 & a_1\\ a_5 & a_1 & a_2 & a_3 & a_4\\ a_3 & a_4 & a_5 & a_1 & a_2 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5\\ b_3 & b_4 & b_5 & b_1 & b_2\\ b_5 & b_1 & b_2 & b_3 & b_4\\ b_2 & b_3 & b_4 & b_5 & b_1\\ b_4 & b_5 & b_1 & b_2 & b_3 \end{pmatrix}$$

where $a_i, a_j \in \mathbb{Q}$ and

$$(5) \qquad \{a_i + b_j \mid i,j = 1,\ldots,n\} = \{1,2,\ldots,25\}.$$

From $a_i + b_j \in \mathbb{Z}$ for $i,j = 1,\ldots,5$ it follows that

$$a_1 \equiv a_2 \equiv a_3 \equiv a_4 \equiv a_5 \bmod 1$$

and

$$b_1 \equiv b_2 \equiv b_3 \equiv b_4 \equiv b_5 \mod 1.$$

So without loss of generality it suffices to look for integers a_i, b_j. It is an elementary task to prove the following

Lemma 5. *The unique solution of* (5) *with* $a_i, b_j \in \mathbb{Z}$ *and*

$$1 = a_1 < a_2 < a_3 < a_4 < a_5 \text{ and } 0 = b_1 < b_2 < b_3 < b_4 < b_5$$

is

$$(a_1, a_2, a_3, a_4, a_5) = (1, 2, 3, 4, 5),\ (b_1, b_2, b_3, b_4, b_5) = (0, 5, 10, 15, 20).$$

Since every permutation of $a_1, \dots, a_5$ and of $b_1, \dots, b_5$ and also interchanging of $\{a_1, \dots, a_5\}$ and $\{b_1, \dots, b_5\}$ gives a solution of (5) we have

Theorem 6. [3], [4, §22] *Every* 5×5-P*-square is of the form* (4) *with*

$$\Big\{\{a_1, \dots, a_5\}, \{b_1, \dots, b_5\}\Big\} = \Big\{\{1, 2, 3, 4, 5\}, \{0, 5, 10, 15, 20\}\Big\}.$$

This fact has as group theoretical consequence

Theorem 7. *The wreath product* $\mathcal{S}_5 \wr \mathcal{S}_2 \cong (\mathcal{S}_5 \times \mathcal{S}_5) \rtimes \mathcal{S}_2$ *acts faithfully and transitively on the set of all* 5×5-P*-squares.*

6×6-Squares

By Lemma 1d) we have to consider only W-squares. By the aid of a computer we have found

Theorem 8. *There are* 41 $(\mathcal{S}_3 \wr \mathcal{S}_2) \wr \mathcal{S}_2$*-orbits on the set of all* 6×6-W*-squares. Representatives of these orbits are given by* $a_{11} = 1$ *and*

a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
34	2	35	3	36	9	19	24	25	33
34	2	35	3	36	9	19	24	31	27
30	4	33	7	36	9	19	26	21	35
30	4	33	7	36	9	19	26	29	27
30	4	33	7	36	9	19	27	20	35
34	2	35	3	36	9	19	27	22	33
30	4	33	7	36	9	20	25	21	35
30	4	33	7	36	9	20	25	29	27
30	4	33	7	36	9	21	25	29	26
34	2	35	3	36	9	22	21	25	33
34	2	35	3	36	9	22	21	31	27
34	2	35	3	36	9	25	21	31	24

a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
28	5	32	9	36	11	15	24	25	35
28	5	32	9	36	11	15	24	27	33
28	5	32	9	36	11	15	33	16	35
28	5	32	9	36	11	16	23	25	35
28	5	32	9	36	11	16	23	27	33
28	5	32	9	36	11	25	23	27	24
27	7	32	9	35	12	13	24	25	36
27	7	32	11	33	12	13	24	25	36
27	7	33	8	35	12	13	24	25	36
27	8	31	9	35	12	13	24	25	36
27	8	31	11	33	12	13	24	25	36
27	9	31	11	32	12	13	24	25	36
31	3	32	9	35	12	13	24	25	36
31	3	32	11	33	12	13	24	25	36
31	3	33	8	35	12	13	24	25	36
32	3	33	7	35	12	13	24	25	36
34	2	35	3	36	21	7	24	25	33
34	2	35	3	36	21	7	24	31	27
34	2	35	3	36	21	7	27	22	33
28	5	32	9	36	23	3	24	25	35
28	5	32	9	36	23	3	24	27	33
28	5	32	9	36	23	3	33	16	35
29	4	33	8	36	24	3	25	23	35
28	5	32	9	36	24	3	33	15	35
34	2	35	3	36	24	7	27	19	33
30	4	33	7	36	25	3	26	21	35
30	4	33	7	36	25	3	26	29	27
30	4	33	7	36	25	3	27	20	35
30	4	33	7	36	26	3	27	19	35 .

It is eye-catching that ten orbits belong to each of the regular vectors

$$(1, 34, 2, 35, 3, 36)$$
$$(1, 30, 4, 33, 7, 36)$$
$$(1, 28, 5, 32, 9, 36)$$
$$(1, 12, 13, 24, 25, 36)$$

If these vectors are in a row resp. column of a W-square then even the symmetric group $\mathcal{S}_6$ is acting on the rows resp. columns.
For example, two neighbouring rows can be interchanged as follows:

$$\begin{matrix} a & b & c & d & e & f \\ a' & b' & c' & d' & e' & f' \end{matrix} \qquad \longrightarrow \qquad \begin{matrix} e' & f' & c' & d' & a' & b' \\ e & f & c & d & a & b \end{matrix}$$

while the other rows remain fixed. (Compare the values $a+b$, $b+c$, $c+d$, $d+e$, $e+f$ with $a'+b'$, $b'+c'$, $c'+d'$, $d'+e'$, $e'+f'$. They form an arithmetic sequence.)
As a consequence we can consider the corresponding 40 orbits as 8 orbits of the group $(\mathcal{S}_3 \wr \mathcal{S}_2) \times \mathcal{S}_6$.

7×7-Squares

By Lemma 1c) only P-squares can exist. Similar to the case $n = 5$ one can show that every 7×7-P-square is the sum of the four matrices

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_6 & a_7 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_4 & a_5 & a_6 & a_7 & a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_1 \\ a_7 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 & a_1 & a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_1 & a_2 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ b_5 & b_6 & b_7 & b_1 & b_2 & b_3 & b_4 \\ b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_1 \\ b_6 & b_7 & b_1 & b_2 & b_3 & b_4 & b_5 \\ b_3 & b_4 & b_5 & b_6 & b_7 & b_1 & b_2 \\ b_7 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_4 & b_5 & b_6 & b_7 & b_1 & b_2 & b_3 \end{pmatrix}$$

$$+ \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ c_4 & c_5 & c_6 & c_7 & c_1 & c_2 & c_3 \\ c_7 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_3 & c_4 & c_5 & c_6 & c_7 & c_1 & c_2 \\ c_6 & c_7 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_1 \\ c_5 & c_6 & c_7 & c_1 & c_2 & c_3 & c_4 \end{pmatrix} + \begin{pmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 \\ d_3 & d_4 & d_5 & d_6 & d_7 & d_1 & d_2 \\ d_5 & d_6 & d_7 & d_1 & d_2 & d_3 & d_4 \\ d_7 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_1 \\ d_4 & d_5 & d_6 & d_7 & d_1 & d_2 & d_3 \\ d_6 & d_7 & d_1 & d_2 & d_3 & d_4 & d_5 \end{pmatrix}$$

where the four vectors

$$(a_1, \ldots, a_7),\ (b_1, \ldots, b_7),\ (c_1, \ldots, c_7),\ (d_1, \ldots, d_7)$$

are in $\mathbb{Z}^7$ such that

$$(6) \qquad \{a_i + b_{i+k} + c_{i+2k} + d_{i+3k} \mid i,k = 1,\dots,7\} = \{1,2,\dots,49\}$$

This condition is fulfilled if two of these vectors are zero vectors and the other two are equal to $(1,2,\dots,7)$ and $(0,7,\dots,42)$. Starting with these special vectors we get by permuting the components of each non-constant vector altogether $(7!)^2 \cdot 12 \quad 7\times 7$-$P$-squares. So we have an action of the group $\mathcal{S}_7 \times \mathcal{S}_7$ and 12 orbits.

The following question is open: Are there more P-squares, i.e. exists a solution of (6) with three or four non-constant vectors in $\mathbb{Z}^7$?

8×8-Squares

From now on a magic square which is simultaneously a P- and a W-square is called a PW-square.

We proceed analogous to the case $n = 4$: We relate the integers $1,2,\dots,64$ to the vertices of a 6-dimensional cube, i.e.

$$\Big\{x_{abcdef} \mid a,b,c,d,e,f \in \{-1,1\}\Big\} = \{1,2,\dots,64\},$$

by choosing the integers $32, 48, 56, 60, 62, 63$ as neighbours of the integer 1 and by using the sum condition that every square on the surface of this hypercube has sum $t = 130$. So we get a marked hypercube.

There are four essentially different ways (with respect to the motion group M_6) how to put these integers into a 8×8-matrix such that every 2×2-submatrix corresponds to a square on the surface of this hypercube. (One sees easily that in every 2×2-submatrix

$$\begin{pmatrix} x_\alpha & x_\beta \\ x_\gamma & x_\delta \end{pmatrix} \text{ with } \alpha,\beta,\gamma,\delta \in \{-1,1\}^6$$

the vectors in every pair

$$(\alpha,\beta),\ (\alpha,\gamma),\ (\beta,\delta),\ (\gamma,\delta)$$

only differ in one component and this component is the same in (α, β) and (β, δ) and also in (α, γ) and (β, δ). Going from left to right in a row or from top to bottom in a column three components can be changed as follows:

$$---\quad --+\quad -++\quad +++\quad +-+\quad +--\quad ++-\quad -+-$$

or

$$---\quad --+\quad -++\quad -+-\quad ++-\quad +++\quad +-+\quad +--\quad .)$$

One of these embeddings is given by

−−−−−−	−−−−−+	−−−−++	−−−+++	−−−+−+	−−−+−−	−−−++−	−−−−+−
−−+−−−	−−+−−+	−−+−++	−−++++	−−++−+	−−++−−	−−+++−	−−+−+−
−++−−−	−++−−+	−++−++	−+++++	−+++−+	−+++−−	−++++−	−++−+−
+++−−−	+++−−+	+++−++	++++++	++++−+	++++−−	+++++−	+++−+−
+−+−−−	+−+−−+	+−+−++	+−++++	+−++−+	+−++−−	+−+++−	+−+−+−
+−−−−−	+−−−−+	+−−−++	+−−+++	+−−+−+	+−−+−−	+−−++−	+−−−+−
++−−−−	++−−−+	++−−++	++−+++	++−+−+	++−+−−	++−++−	++−−+−
−+−−−−	−+−−−+	−+−−++	−+−+++	−+−+−+	−+−+−−	−+−++−	−+−−+−

For example we get:

1	63	4	57	6	60	7	62
56	10	53	16	51	13	50	11
25	39	28	33	30	36	31	38
8	58	5	64	3	61	2	59
41	23	44	17	46	20	47	22
32	34	29	40	27	37	26	35
49	15	52	9	54	12	55	14
48	18	45	24	43	21	42	19

Theorem 9. *In every square arising from the above marked* 6*-dimensional cube the following relations hold:*

$$\begin{aligned} d_{ij} &:= a_{i,j} + a_{i+2,j+2} + a_{i+4,j+4} + a_{i+6,j+6} = t = 130 \\ d'_{ij} &:= a_{i,j} + a_{i+2,j-2} + a_{i+4,j-4} + a_{i+6,j-6} = t = 130 \end{aligned} \tag{7}$$

with $i, j = 1, \ldots, n$ *and cyclic notation of the indices. These squares are lying in* $\binom{6}{3} = 20$ *orbits, consisting only of PW-squares, of the group* $(\mathcal{S}_4 \wr \mathcal{S}_2) \wr \mathcal{S}_2$.

Proof. With $m = 4$ we proceed as in the proof of Lemma 2. So we get for d_{ij}, d'_{ij} only four different values

$$\text{(8)} \qquad \begin{array}{ll} -4a_{11} + U + U^t, & 4(t - a_{11}) - V - V^t, \\ 4a_{11} + V - U^t, & \quad 4a_{11} - U + V^t. \end{array}$$

Now we take the embedding above, i.e. $a_{11} = 1$ and

$$\text{(9)} \qquad \{a_{12}, a_{16}, a_{18}, a_{21}, a_{61}, a_{81}\} = \{32, 48, 56, 60, 62, 63\}.$$

Using the square sums we get:

$$\begin{array}{ll} a_{17} = t - a_{11} - a_{16} - a_{18}, & a_{13} = t - a_{11} - a_{12} - a_{18}, \\ a_{15} = t - a_{11} - a_{12} - a_{16}, & a_{14} = 2a_{11} + a_{12} + a_{16} + a_{18} - t \end{array}$$

and further

$$\text{(10)} \qquad U = 3t - 2(a_{11} + a_{12} + a_{16} + a_{18}),\ V = 2(a_{11} + a_{12} + a_{16} + a_{18}) - t.$$

Similarly we have

$$\text{(11)} \qquad U^t = 3t - 2(a_{11} + a_{21} + a_{61} + a_{81}),\ V^t = 2(a_{11} + a_{21} + a_{61} + a_{81}) - t.$$

By inserting (10) and (11) into (8) and then using (9) the assertion follows. In case of the other embeddings one can proceed in an analogous manner. The assertion on the orbits follows from Theorem 3. □

Now it should be mentioned that one can get $4 \cdot 2^6 \cdot 7!$ 8×8-W-squares by means of the marked 7-dimensional cube in which the neighbours of 1 are $32, 48, 56, 60, 62, 63, 64$ and all square sums are equal to $t = 130$. These squares are lying in $\frac{1}{2}\binom{7}{3}\binom{4}{3} = 70$ orbits of the group $(\mathcal{S}_4 \wr \mathcal{S}_2) \wr \mathcal{S}_2$.

Much more 8×8-W-squares, namely 40 orbits of the group $(\mathcal{S}_4 \wr \mathcal{S}_2) \times \mathcal{S}_8$ can be obtained from the following 20 regular W-squares and their corresponding transposed W-squares using the special interchange (described in the case $n = 6$) of neighbouring columns resp. rows:

a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	a_{71}	a_{81}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	
61	2	62	3	63	4	64	60	53	16	17	44	37	32	
							8	53	52	45	44	25	32	*
							60	9	52	45	24	37	32	
							60	53	16	45	24	25	36	
58	3	60	5	62	7	64	63	50	16	17	47	34	32	
							8	50	55	42	47	25	32	*
							63	9	55	42	24	34	32	
							63	50	16	42	24	25	39	
52	5	56	9	60	13	64	63	50	16	17	47	34	32	
							14	50	61	36	47	19	32	*
							63	3	61	36	30	34	32	
							63	50	16	36	30	19	45	
40	9	48	17	56	25	64	63	38	28	5	59	34	32	
							26	38	61	36	59	7	32	*
							63	3	61	36	30	34	32	
							63	38	28	36	30	7	57	
16	17	32	33	48	49	64	63	14	52	5	59	10	56	
							50	14	61	12	59	7	56	*
							63	3	61	12	54	10	56	
							63	14	52	12	54	7	57	

In these $(\mathcal{S}_4 \wr \mathcal{S}_2) \times \mathcal{S}_8$-orbits of W-squares one can find 60 $(\mathcal{S}_4 \wr \mathcal{S}_2) \wr \mathcal{S}_2$-orbits of PW-squares. Representatives of these PW-squares arise from the preceding lines indicated with a star "$*$":
In the corresponding PW-squares $(c_1, \ldots, c_8)$ where c_i denotes the column i, the columns are changed by using three maps (later written above the columns)

$$\begin{aligned}
{}^{-} : \mathbb{Z}^8 \to \mathbb{Z}^8, (a,b,c,d,e,f,g,h) &\longmapsto (g,h,e,f,c,d,a,b), \\
{}^{\bullet} : \mathbb{Z}^8 \to \mathbb{Z}^8, (a,b,c,d,e,f,g,h) &\longmapsto (h,g,f,e,d,c,b,a), \\
{}^{\sim} : \mathbb{Z}^8 \to \mathbb{Z}^8, (a,b,c,d,e,f,g,h) &\longmapsto (b,a,d,c,f,e,h,g).
\end{aligned}$$

From the PW-square $(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$ five PW-squares are deduced:

$$
\begin{aligned}
&(c_1, c_2, c_3, c_4, \overline{c}_6, \overline{c}_5, \overline{c}_8, \overline{c}_7),\\
&(c_1, c_2, \overline{c}_4, \overline{c}_3, c_5, c_6, \overline{c}_8, \overline{c}_7),\\
&(c_1, \dot{c}_3, \dot{c}_2, c_4, c_5, \dot{c}_7, \dot{c}_6, c_8),\\
&(c_1, \dot{c}_3, \dot{c}_2, c_4, \overline{c}_8, \widetilde{c}_6, \widetilde{c}_7, \overline{c}_5),\\
&(c_1, \dot{c}_7, c_3, \dot{c}_5, \dot{c}_4, c_6, \dot{c}_2, c_8).
\end{aligned}
$$

To each of these six PW-squares the map

$$(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) \longmapsto (c_1, \widetilde{c}_2, c_3, \widetilde{c}_4, c_5, \widetilde{c}_6, c_7, \widetilde{c}_8)$$

supplies another PW-square. Is there an extension of the group $\mathcal{S}_4 \wr \mathcal{S}_2$ behind these 12 column transformations?

At the end of his paper [1] Fitting has given two construction methods for 8×8-P-squares which are in general no W-squares. Each method supplies more than 10^{14} P-squares. How to describe the groups acting on these sets?

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Mathematisches Institut
Universität Bayreuth

95440 Bayreuth