On the Automorphism Group of Solomon's Descent Algebra

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By Δ_n we denote the subalgebra of $\mathbb{Q}S_n$ which is known as Solomon's Descent Algebra [7, 6, 1]. This article gives a sketch of a proof for the following theorem about the automorphisms of Δ_n. (Details and related developments will be given in a forthcoming publication.)

Main Theorem *For the automorphism group of Solomon's Descent Algebra Δ_n the following holds:*

$$\mathrm{Aut}(\Delta_n) = \begin{cases} \mathrm{Inn}(\Delta_n) & \text{if } n \text{ odd,} \\ \mathrm{Inn}(\Delta_n) \times C_2 & \text{if } n \text{ even.} \end{cases}$$

The proof of this theorem splits into four main steps:

- the reduction of the problem to show that a stabilizer of a certain set is generated by inner automorphisms,
- the definition of a graph Γ_n,
- the mentioned stabilizer is determined by its action on Γ_n,
- the action of the stabilizer on Γ_n is induced by conjugation by invertible elements of Δ_n, i.e. by inner automorphisms, and possibly a central involutionary outer automorphism.

1 Notation

We write $\mathbb{N}$ for the set of positive integers $\{1, 2, \dots\}$, $\mathbb{N}^*$ for the free monoid generated by $\mathbb{N}$ and $\mathbb{Q}\mathbb{N}^*$ for the free algebra generated by $\mathbb{N}$ over the field of the rational numbers $\mathbb{Q}$. The elements of $\mathbb{N}^*$ are written as words in the alphabet $\mathbb{N}$, e.g. $1\,3\,2 \in \mathbb{N}^*$ is the product of 1, 3 and 2. If $w = w_1 \dots w_k$ is a word, we set $|w| := k$, the *length* of w.

Let $n \in \mathbb{N}$ and $p = p_1 \dots p_k \in \mathbb{N}^*$ such that $p_1, \dots, p_k \in \mathbb{N}$. The word p is called a *partition* of n ($p \vdash n$) if $p_1 + \dots + p_k = n$ and $p_1 \geq p_2 \geq \dots \geq p_k$. We write $p(n)$ for the number of all partitions of n. For each letter c we set

$$a_c(p) := |\{i \mid p_i = c\}|,$$

the number of occurences of the letter c in p. Then we may write

$$p = n^{a_n(p)}(n-1)^{a_{n-1}(p)} \dots 1^{a_1(p)}.$$

2 Reduction of the problem

By a theorem of Malcev [4], [5, 11.6] we know that the set $1 + \mathrm{Rad}(\Delta_n)$ of invertible elements acts transitively by conjugation on the set of complements of $\mathrm{Rad}(\Delta_n)$.

By Frattini's lemma [3, 3.3][1], for each complement H of $\mathrm{Rad}(\Delta_n)$ we have:

$$\mathrm{Aut}(\Delta_n) \;=\; \mathrm{Stab}_{\mathrm{Aut}(\Delta_n)}(H)\;\overline{(1 + \mathrm{Rad}(\Delta_n))}, \tag{1}$$

where $\overline{(1 + \mathrm{Rad}(\Delta_n))}$ denotes the group of automorphisms induced by conjugation by the elements of $1 + \mathrm{Rad}(\Delta_n)$.

Now we construct a complement H of $\mathrm{Rad}(\Delta_n)$ to which the above mentioned theorem and lemma will be applied.

In [2], D. Blessenohl and H. Laue define elements[2] ν^p, $p \vdash n$, with the following properties [2, 1.2 Proposition]:

(a) ν^p is an idempotent for each $p \vdash n$.

(b) $\nu^p \nu^r \;=\; 0$ for all partitions p, r such that $p \neq r$.

(c) $\sum_{p \vdash n} \nu^p \;=\; 1$.

[1] Though the lemma is only stated for finite groups, it holds for infinte groups, too.

[2] In [2] these elements are not indexed by the partitions themselves but by listing them in the lexicographically decreasing order: $\nu^{(j)}$ instead of ν^p if p is the j-th partition.

(d) $\Delta_n = \langle\, \nu^p \mid p \vdash n \,\rangle_{\mathbb{Q}} \oplus \operatorname{Rad}(\Delta_n)$.

The set $H := \langle\, \nu^p \mid p \vdash n \,\rangle_{\mathbb{Q}}$ is a subalgebra-complement of $\operatorname{Rad}(\Delta_n)$ in Δ_n. We observe that

$$H \cong \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{p(n)},$$

and that $\{\nu^p \mid p \vdash n\}$ is the unique set of $p(n)$ mutually orthogonal idempotents $\neq 0$ of H.

By Equation (1) it suffices to show that $\operatorname{Stab}_{\operatorname{Aut}(\Delta_n)}(H)$ is generated by inner automorphisms in the case that n is even and by inner automorphisms and an involutionary outer automorphism in the case that n is odd.

3 Directed graph of partitions

Now we define a directed graph Γ_n, the nodes of which are the partitions of n. The node r is called connected with p ($r \mathrel{|\!\!\sim\!\cdot_{\mathrm{pf}}} p$) if $|r| = |p| + 1$ and there exist letters $c, d \in \mathbb{N}$ such that $c \neq d$, $a_c(r) = a_c(p) + 1$, $a_d(r) = a_d(p) + 1$ and $a_{c+d}(r) + 1 = a_{c+d}(p)$, i.e. p can be obtained from r by coalescing two different letters of r and reordering the letters to obtain a partition.

The shape of the graph Γ_7 is shown in Figure 1.

Obviously Γ_n has two (three resp.) connected subgraphs in the case that n is odd (even resp.). More precisely, the partition $1\ldots1$ if n is odd and $1\ldots1$ and $2\ldots2$ if n is even are not connected with any other node. We observe further that for each $k < n$ the subgraph of Γ_n induced by all partitions which include the letter k is isomorphic to Γ_{n-k}. It is not at all trivial to see that, if φ is an automorphism of Γ_n, for each $k \in \mathbb{N}$ this subgraph is invariant under φ. We get by an inductive argument the following lemma about Γ_n:

Lemma 1 *For the automorphism group of Γ_n holds:*

$$\operatorname{Aut}(\Gamma_n) = \begin{cases} \{\mathrm{id}\} & \text{if } n \text{ odd,} \\ \{\mathrm{id}, \tau\} & \text{if } n \text{ even,} \end{cases}$$

where τ is the automorphism of Γ_n that exchanges the nodes $1\ldots1$ and $2\ldots2$ and fixes the other nodes.

4 The action of $\operatorname{Stab}_{\operatorname{Aut}(\Delta_n)}(H)$ on Γ_n

Now let φ be an automorphism of Δ_n such that $H^\varphi \subseteq H$.

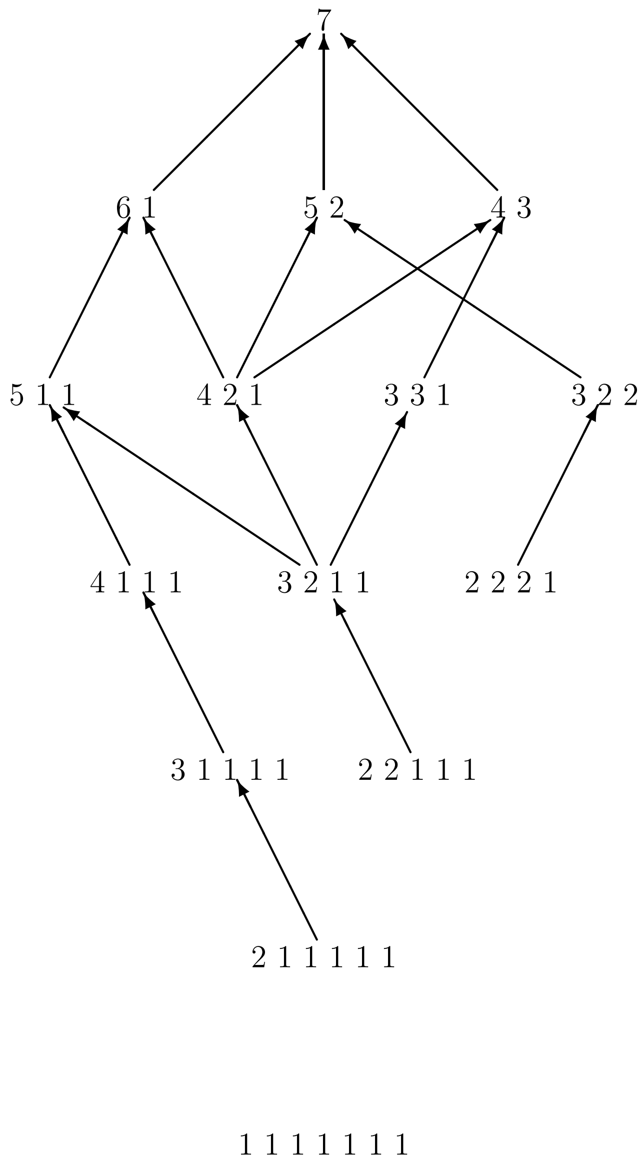

Figure 1: The graph Γ_7

Since the elements ν^p, $p \vdash n$, are mutually orthogonal idempotents, φ acts on the set $\{\nu^p \mid p \vdash n\}$. Therefore φ induces an action on the set of partitions of n.

It turns out that the action of φ on the set of partitions of n is compatible with the relation $\mathrel{|\!\!\sim\!\cdot}_{\mathrm{pf}}$, i.e. for all partitions r, p of n we have:

$$r \mathrel{|\!\!\sim\!\cdot}_{\mathrm{pf}} p \iff r^{\varphi} \mathrel{|\!\!\sim\!\cdot}_{\mathrm{pf}} p^{\varphi}.$$

Therefore φ induces an automorphism of Γ_n.

By Lemma 1 we see that the spaces $\nu^p \Delta_n \nu^r$ are φ-invariant for all partitions p, r such that $r \mathrel{|\!\!\sim\!\cdot}_{\mathrm{pf}} p$.

By [1, 2.2 Corollary] [3] we get then

Lemma 2 *Let p, r partitions of n such that $r \mathrel{|\!\!\sim\!\cdot}_{pf} p$. Then $\nu^p \Delta_n \nu^r$ is a one-dimensional φ-invariant space, and $\varphi|_{\nu^p \Delta_n \nu^r}$ has an eigenvalue $\neq 0$.*

By [1, 2.4 Corollary] we can deduce

$$\mathrm{Rad}(\Delta_n) \;=\; \langle\; \nu^p \Delta_n \nu^r \mid r \mathrel{|\!\!\sim\!\cdot}_{\mathrm{pf}} p \;\rangle_{(+,\cdot)}$$

from which we see that the action of φ on $\mathrm{Rad}(\Delta_n)$ is determined by the action on the subspaces $\nu^p \Delta_n \nu^r$, $r \mathrel{|\!\!\sim\!\cdot}_{\mathrm{pf}} p$.

Now we define a certain subgraph Γ_n^* of Γ_n, the nodes of which are the partitions of n, too. At first, we set

$$P_n^* := \{p \in P_n \mid p = p_1 \dots p_k, p_1 \neq p_k\},$$

i.e. the set of all partitions of n that have at least two different letters. For each partition $p = p_1 \dots p_k \in P_n^*$ let

$$\iota(p) := \min\{i \mid i \in \{1, \dots, k\}, p_i \neq p_1\}$$

and

$$\zeta(p) := (p_1 + p_{\iota(p)}) p_2 \cdots p_{\iota(p)-1} p_{\iota(p)+1} \cdots p_k,$$

i.e. we form the sum of the two largest different letters that occur in p, delete these two letters from p and add the sum as a new letter. If $p \in P_n^*$ then $\zeta(p)$ is a partition of n and it holds $p \mathrel{|\!\!\sim\!\cdot}_{\mathrm{pf}} \zeta(p)$.

The partitions p, r of n are called connected in Γ_n^* ($p \succ r$), if $p \in P_n^*$ and $r = \zeta(p)$ or if $r = d^k$ for some $d, k \in \mathbb{N}$ and $p = d^{k-1}(d-1)1$.

The relation $\succ$ is coarser than $\mathrel{|\!\!\sim\!\cdot}_{\mathrm{pf}}$ and defines a spanning tree for the "big" connected component of Γ_n, seen as an undirected graph, that contains the node n. Figure 2 shows Γ_7^*.

We obtain

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[3] In [1], the spaces $\omega_p \Lambda^r$ are treated. But these are isomorphic to $\nu^p \Delta_n \nu^r$.

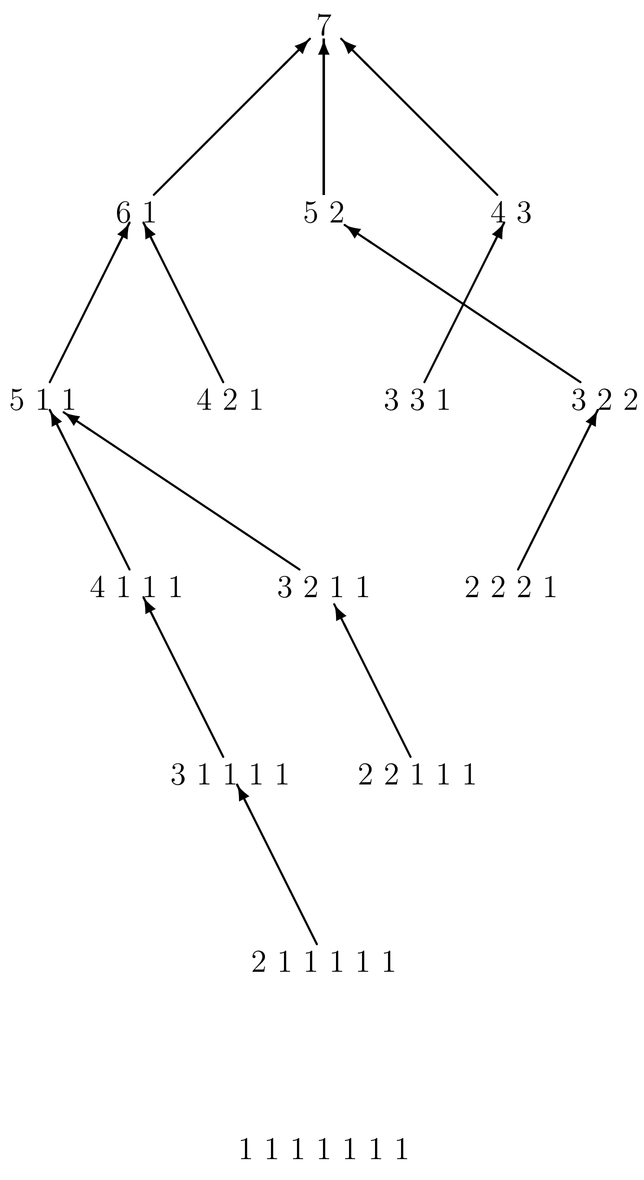

Figure 2: The edges defined by $\succ$

Main Lemma *Let* $\varphi \in \mathrm{Stab}_{\mathrm{Aut}(\Delta_n)}(H)$ *such that* $\varphi|_{\nu^p \Delta_n \nu^r} = \mathrm{id}$ *for all partitions* p, r *such that* $r \succ p$. *Then* $\varphi|_{\mathrm{Rad}(\Delta_n)} = \mathrm{id}$.

The proof of the Main Lemma needs hard conclusions. For this reason we give only some instructive examples in Section 5 that demonstrate the important ideas.

But now we may easily construct for each $\varphi \in \mathrm{Stab}_{\mathrm{Aut}(\Delta_n)}(H)$ an invertible element $h \in H$ such that the conjugation by h coincides with the action of φ on the subspaces $\nu^p \Delta_n \nu^r$, $r \succ p$. By the Main Lemma, we see that the action of h and φ coincide on $\mathrm{Rad}(\Delta_n)$.

Therefore the automorphism $\psi : \Delta_n \to \Delta_n, x \mapsto (x^\varphi)^{h^{-1}}$ centralizes $\mathrm{Rad}(\Delta_n)$ and stabilizes the set $\{\nu^p \mid p \vdash n\}$. By what we have observed above, ψ induces an automorphism of Γ_n. Now Lemma 1 implies, that ψ is the identity or an involution. Hence, it follows

Lemma 3

$$\mathrm{Stab}_{\mathrm{Aut}(\Delta_n)}(H) = \begin{cases} \{\textit{inner automorphisms induced by } H\} & \textit{if } n \textit{ odd,} \\ \{\textit{inner automorphisms induced by } H\} \times C_2 & \textit{if } n \textit{ even.} \end{cases}$$

5 Example for $n = 7$

We illustrate the proof of the Main Lemma by discussing the example of $n = 7$ which provides a spectrum of all three typical cases which may occur in general. This discussion therefore does not only give a flavour of the general proof but presents all its basic elements in a concrete form.

In [1, p. 718] a basis of Δ_n consisting of idempotents ν_q, $q \models n$, is given. In the following, we consider the linear extension $\nu : \langle q \mid q \models n \rangle_{\mathbb{Q}} \to \Delta_n$ of the mapping $\{q \mid q \models n\} \to \Delta_n$, $q \mapsto \nu_q$.

We use the Lie product $\circ$ on $\mathbb{Q}\mathbb{N}^*$ defined by $a \circ b := ab - ba$ for all $a, b \in \mathbb{Q}\mathbb{N}^*$.

E.g. we write

$$\nu_{1\circ 2} = \nu_{12-21} = \nu_{12} - \nu_{21}.$$

In order to illustrate the proof of the Main Lemma we may assume φ fixes elementwise the subspaces $\nu^7\Delta_n\nu^{61}$, $\nu^7\Delta_n\nu^{52}$, $\nu^7\Delta_n\nu^{43}$, $\nu^{61}\Delta_n\nu^{511}$, $\nu^{61}\Delta_n\nu^{421}$, $\nu^{52}\Delta_n\nu^{322}$, $\nu^{43}\Delta_n\nu^{331}$, $\nu^{511}\Delta_n\nu^{4111}$, $\nu^{511}\Delta_n\nu^{3211}$, $\nu^{322}\Delta_n\nu^{2221}$, $\nu^{4111}\Delta_n\nu^{31111}$, $\nu^{3211}\Delta_n\nu^{22111}$, $\nu^{31111}\Delta_n\nu^{211111}$.

We have to show that the subspaces $\nu^{52}\Delta_n\nu^{421}$, $\nu^{43}\Delta_n\nu^{421}$, $\nu^{421}\Delta_n\nu^{3211}$, $\nu^{331}\Delta_n\nu^{3211}$ are fixed elementwise by φ, too.

Figure 3 shows this situation. The thick edges represent the subspaces on which the action of φ is assumed as identity. The thin edges represent the subspaces on which the action of φ is not known. The eigenvalues of φ on these eigenspaces are denoted by a, b, c and d. We have to show that $a = b = c = d = 1$.

In the following considerations we use rules described in [1, 1.5 Theorem, 2.1 Proposition].

Case 1: The calculation of a and b. There are the one-dimensional spaces

$$\begin{aligned} & (\nu^7 \Delta_n \nu^{61})(\nu^{61} \Delta_n \nu^{421}) \\ = & \langle\ \nu^7 \nu_7 \nu_{61} \nu_{421}\ \rangle_{\mathbb{Q}} \\ = & \langle\ \nu^7 \nu_{6\circ 1} \nu_{421}\ \rangle_{\mathbb{Q}} \\ = & \langle\ \nu^7 \nu_{(4\circ 2)\circ 1}\ \rangle_{\mathbb{Q}} \end{aligned}$$

with eigenvalue 1,

$$\begin{aligned} & (\nu^7 \Delta_n \nu^{52})(\nu^{52} \Delta_n \nu^{421}) \\ = & \langle\ \nu^7 \nu_7 \nu_{52} \nu_{421}\ \rangle_{\mathbb{Q}} \\ = & \langle\ \nu^7 \nu_{5\circ 2} \nu_{421}\ \rangle_{\mathbb{Q}} \\ = & \langle\ \nu^7 \nu_{(4\circ 1)\circ 2}\ \rangle_{\mathbb{Q}} \\ = & \langle\ \nu^7 \nu_{(1\circ 4)\circ 2}\ \rangle_{\mathbb{Q}} \end{aligned}$$

with eigenvalue a,

$$\begin{aligned} & (\nu^7 \Delta_n \nu^{71})(\nu^{43} \Delta_n \nu^{421}) \\ = & \langle\ \nu^7 \nu_7 \nu_{43} \nu_{421}\ \rangle_{\mathbb{Q}} \\ = & \langle\ \nu^7 \nu_{4\circ 3} \nu_{421}\ \rangle_{\mathbb{Q}} \\ = & \langle\ \nu^7 \nu_{4\circ(2\circ 1)}\ \rangle_{\mathbb{Q}} \\ = & \langle\ \nu^7 \nu_{(2\circ 1)\circ 4}\ \rangle_{\mathbb{Q}} \end{aligned}$$

with eigenvalue b.

Applying the Jacobi identity $((x \circ y) \circ z + (y \circ z) \circ x + (z \circ x) \circ y = 0)$ we get

$$\begin{aligned} 0 &= 0^\varphi \\ &= (\nu^7)^\varphi (\overbrace{\nu_{(4\circ 2)\circ 1} + \nu_{(1\circ 4)\circ 2} + \nu_{(2\circ 1)\circ 4}}^{=0})^\varphi \\ &= (\nu^7)^\varphi (\nu_{(4\circ 2)\circ 1} + a\nu_{(1\circ 4)\circ 2} + b\nu_{(2\circ 1)\circ 4}) \\ &= (\nu^7)^\varphi ((-\nu_{(1\circ 4)\circ 2} - \nu_{(2\circ 1)\circ 4}) + a\nu_{(1\circ 4)\circ 2} + b\nu_{(2\circ 1)\circ 4}) \\ &= (\nu^7)^\varphi ((a-1)\nu_{(1\circ 4)\circ 2} + (b-1)\nu_{(2\circ 1)\circ 4}). \end{aligned}$$

The summands in the last equation are linearly independent. It follows that $a = 1$ and $b = 1$.

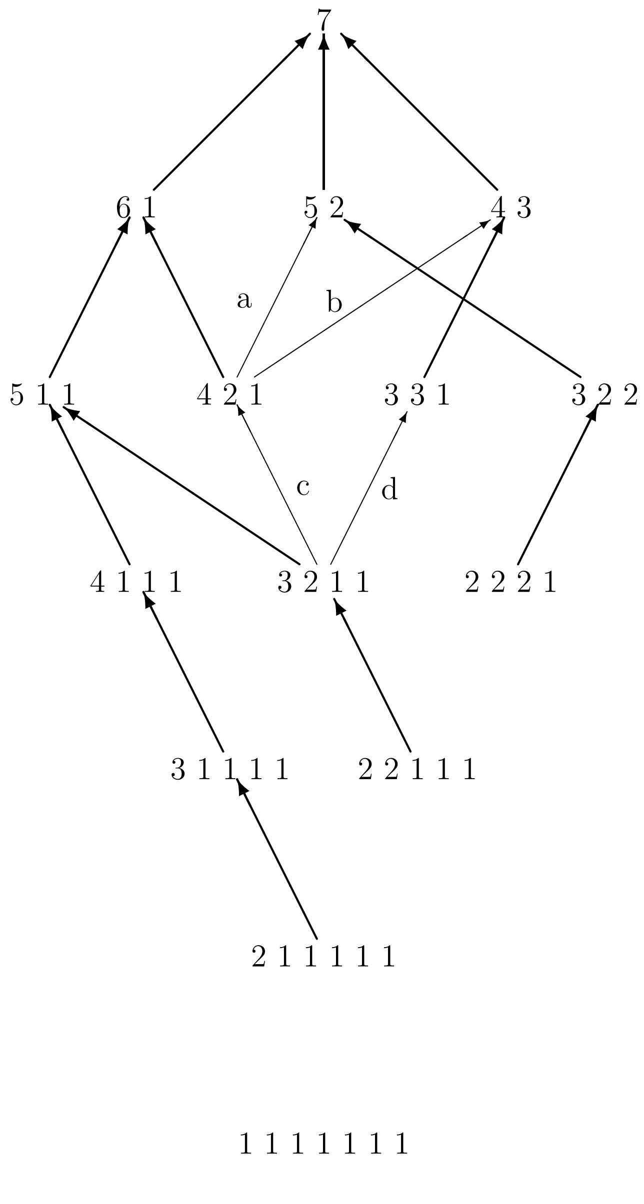

Figure 3: Γ_7 with the eigenvalues as weights

In a similar manner we treat the other two cases.

Case 2: The calculation of c. There are the one-dimensional spaces

$$(\nu^{61}\Delta_n\nu^{511})(\nu^{511}\Delta_n\nu^{3211})(\nu^{3211}\Delta_n\nu^{22111}) \;=\; \langle\; \nu^{61}\nu_{(((2\circ 1)\circ 2)\circ 1)1)} \;\rangle_{\mathbb{Q}}$$

with eigenvalue 1,

$$(\nu^{61}\Delta_n\nu^{421})(\nu^{421}\Delta_n\nu^{3211})(\nu^{3211}\Delta_n\nu^{22111}) \;=\; \langle\; \nu^{61}\nu_{(((2\circ 1)\circ 1)\circ 2)1)} \;\rangle_{\mathbb{Q}}$$

with eigenvalue c.

The anticommutative law and the Jacobi identity imply that both spaces are generated by the same element. It follows that $c = 1$.

Case 3: The calculation of d. There are the one-dimensional spaces

$$(\nu^{43}\Delta_n\nu^{421})(\nu^{421}\Delta_n\nu^{3211}) \;=\; \langle\; \nu^{43}\nu_{(3\circ 1)(2\circ 1)} \;\rangle_{\mathbb{Q}}$$

with eigenvalue $b = 1$,

$$(\nu^{43}\Delta_n\nu^{331})(\nu^{331}\Delta_n\nu^{3211}) \;=\; \langle\; \nu^{43}\nu_{(3\circ 1)(2\circ 1)} \;\rangle_{\mathbb{Q}}$$

with eigenvalue d. Both spaces are generated by the same element. It follows that $d = b = 1$.

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