Séminaire Lotharingien de Combinatoire **46** (2001), Article B46a

ON A FUNCTIONAL–DIFFERENCE EQUATION OF RUNYON, MORRISON, CARLITZ, AND RIORDAN

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ABSTRACT. A certain functional–difference equation that Runyon encountered when analyzing a queuing system was solved in a combined effort of Morrison, Carlitz, and Riordan. We simplify that analysis by exclusively using generating functions, in particular the *kernel method,* and the *Lagrange inversion formula.*

1. The equation

The functional–difference equation in the title is

$$(x-\alpha)(\alpha-\beta)^{n-1}g_n(x) = \alpha(x-\beta)^n g_{n-1}(\alpha) - x(\alpha-\beta)^n g_{n-1}(x), \quad n \ge 1,\ g_0(x) = 1. \tag{1}$$

J. P. Runyon encountered it in a study of a queuing system in which a group of servers handles traffic from two sources, one of which is preferred over the other[1].

The aim of this note is to present a (possibly) simpler solution than the (combined) solution by Morrison, Carlitz, and Riordan [6, 2, 7]. Note that the $g_n(x)$ are polynomials in x with rational coefficients in α, β. (Our arguments will re-establish that fact.)

We introduce the generating function

$$G(t, x) := \sum_{n\ge 0} (\alpha - \beta)^{n-1} g_n(x) t^n.$$

Multiplying (1) by t^n and summing we get

$$(x - \alpha)G(t, x) - \frac{x - \alpha}{\alpha - \beta} = \alpha(\alpha - \beta)G(\alpha, \tfrac{x-\beta}{\alpha-\beta} t) - xt(\alpha - \beta)^2 G(t, x),$$

or

$$G(t, x) = \frac{\alpha \sum_{n\ge 1} (x - \beta)^n t^n g_{n-1}(\alpha) + \frac{x-\alpha}{\alpha-\beta}}{x - \alpha + xt(\alpha - \beta)^2}. \tag{2}$$

Now for

$$x = \bar{x} := \frac{\alpha}{1 + t(\alpha - \beta)^2}$$

the denominator of (2) vanishes. Consequently, the numerator must also vanish. (A more elaborate argument would be that the power series expansion must exist for that

1991 *Mathematics Subject Classification.* Primary: 05A15.

Key words and phrases. functional-difference equation, generating function, Lagrange inversion formula, kernel method.

[1]This is the only information found in Morrison's paper; apparently, Runyon was his colleague at Bell Telephone Laboratories and asked him this question. Zentralblatt and Mathematical Reviews don't give a hint to any publications of Runyon in the open literature.

combination of values.) This is reminiscent of Knuth's trick [5, page 537], which is called *kernel method* by some french authors; see e. g. [1].

It leads to

$$\sum_{n\ge1}(\bar{x}-\beta)^n t^n g_{n-1}(\alpha)=\frac{\bar{x}-\alpha}{(\beta-\alpha)\alpha}.$$

Now we set $T=(\bar{x}-\beta)t$, i. e.

$$t=\frac{1-T(\alpha-\beta)-\sqrt{1-2T(\alpha+\beta)+T^2(\alpha-\beta)^2}}{2\beta(\alpha-\beta)}.$$

So

$$\sum_{n\ge0}T^n g_n(\alpha)=\frac{1+T(\alpha-\beta)-\sqrt{1-2T(\alpha+\beta)+T^2(\alpha-\beta)^2}}{2T\alpha}.$$

The expansion of this generating function is well known, from the context of the Narayana (Runyon!) numbers [8] or elsewhere. In any instance, the coefficients could be easily detected by the Lagrange inversion formula, with the result

$$g_n(\alpha)=\frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}\beta^{n-k}\alpha^k,\quad n\ge1,\ g_0(\alpha)=1.$$

In the next section, we will see a more impressive occurrence of the Lagrange inversion formula.

2. The general case

In this section we move from the particular case of $g_n(\alpha)$ to the general case of $g_n(x)$.

Now that the series in the numerator of (2) is established, the generating function $G(t,x)$ is fully explicit:

$$G(t,x)=\frac{1+t(x-\beta)(\alpha-\beta)-\sqrt{1-2t(x-\beta)(\alpha+\beta)+t^2(x-\beta)^2(\alpha-\beta)^2}+\frac{2(x-\alpha)}{\alpha-\beta}}{2\big(x-\alpha+xt(\alpha-\beta)^2\big)},\tag{3}$$

and one could work out some clumsy expressions for the coefficients, e. g. (for $x\ne\alpha$)

$$g_n(x)=\frac{(\alpha-\beta)^n x^n}{(\alpha-x)^n}-\alpha\sum_{k=1}^{n}x^{n-k}(\alpha-x)^{k-1-n}(\alpha-\beta)^{n+1-2k}(x-\beta)^k g_{k-1}(\alpha).$$

This was obtained by Morrison without using the generating function. Carlitz [2] set

$$g_n(x)=\sum_{k=0}^{n-1}A_k^{(n)}(\alpha-\beta)^{-k}(x-\beta)^k\tag{4}$$

and managed to express the coefficients as follows:

$$A_r^{(n)}=\beta\phi_{r,n-1}-\alpha\sum_{s=1}^{r-1}g_{r-s}(\alpha)\,\phi_{s-1,n-r+s-1}-\beta\phi_{r-1,n-1},$$

with

$$\phi_{r,k} = \sum_{j=0}^{\min\{r,k\}} \binom{r}{j}\binom{k}{j}\alpha^j\beta^{k-j}, \quad k \ge 0, \qquad \phi_{r,k} = 0, \quad k < 0.$$

He asked whether the expressions

$$\mathcal{C}_{r,n} := \sum_{s=1}^{r-1} g_{r-s}(\alpha)\,\phi_{s-1,n-r+s-1}$$

can be simplified. Now Riordan [7] proved that

$$A_k^{(n)} = (n-k)\sum_{j=1}^{k}\frac{1}{j}\binom{n-1}{j-1}\binom{k-1}{j-1}\alpha^j\beta^{n-j}, \qquad 1 \le k < n,$$

and $A_0^{(n)} = \beta^n$. (This was then generalized by Carlitz [3] who produced a q–version of that.) Riordan's answer translates as

$$\mathcal{C}_{r,n} = \sum_{j=1}^{\min\{r,n\}} \binom{\min\{r,n\}}{j}\binom{\max\{r,n\}-1}{j-1}\alpha^{j-1}\beta^{n-j}.$$

We are going to prove Riordan's result, purely by the use of generating functions and Lagrange's inversion formula, avoiding any recursions and any guesswork (as in [7]).

We start with the expression for $G(t,x)$ in (3) and write it as

$$\sum_{t\ge 0} t^n g_n(x) = \frac{1}{1-y}$$

with

$$y = \frac{\alpha-\beta+t(\beta-\alpha)(\beta-x)-\sqrt{(\beta-\alpha)^2-2(\beta-\alpha)(\alpha+\beta)(\beta-x)t+(\beta-\alpha)^2(\beta-x)^2t^2}}{2(x-\beta)}.$$

Consequently (see e. g. [9, 10] for the Lagrange inversion formula)

$$\begin{gathered} g_n(x) = [t^n]\frac{1}{1-y}, \quad \text{where} \quad y = t\Phi(y), \\ \text{with} \quad \Phi(y) = \frac{(\alpha-\beta)yw+\beta}{1-yw} \quad \text{and} \quad w = \frac{\beta-x}{\beta-\alpha}. \end{gathered} \tag{5}$$

Hence[2]

$$
\begin{aligned}
g_n(x) &= \frac{1}{n}[z^{n-1}]\frac{1}{(1-z)^2}\left(\frac{(\alpha-\beta)zw+\beta}{1-zw}\right)^n\\
&= \beta^n + [z^{n-1}]\frac{1}{(1-z)^2}\frac{1}{n}\sum_{j=1}^{n}\binom{n}{j}\alpha^j\beta^{n-j}\frac{(zw)^j}{(1-zw)^j}\\
&= \beta^n + \sum_{j=1}^{n}\frac{1}{j}\binom{n-1}{j-1}\alpha^j\beta^{n-j}[z^{n-1}]\frac{(zw)^j}{(1-z)^2(1-zw)^j}\\
&= \beta^n + \sum_{j=1}^{n}\frac{1}{j}\binom{n-1}{j-1}\alpha^j\beta^{n-j}\sum_{k=j}^{n}(n-k)\binom{k-1}{j-1}w^k\\
&= \beta^n + \sum_{k=1}^{n}(n-k)\sum_{j=1}^{k}\frac{1}{j}\binom{n-1}{j-1}\binom{k-1}{j-1}\alpha^j\beta^{n-j}w^k\\
&= \beta^n + \sum_{k=1}^{n-1}A_k^{(n)}w^k.
\end{aligned}
$$

Remark. As one referee has pointed out, an early application of the kernel method (but not as early as Knuth's!) was in queuing models, see [4].

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[2]This form (first line) of the polynomials $g_n(x)$ was not observed before, although it is quite appealing. Maple V. 4 computed the inner sum in the fourth line incorrectly, which cost me several hours!