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A DERIVATION OF KOHNERT'S ALGORITHM FROM MONK'S RULE

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Abstract. Kohnert's algorithm for the generation of Schubert polynomials is derived from Monk's rule for the multiplication of Schubert polynomials.

1. Introduction

The theory of Schubert polynomials is a beautiful blend of geometric, algebraic and combinatorial ideas: on one hand Schubert polynomials faithfully represent the cohomology calculus of Schubert varieties of flag manifolds; on the other hand Schubert polynomials are a generalization of the classical Schur polynomials and can be studied in a purely algebraic and combinatorial setting. The primary references for Schubert calculus and Schubert polynomials are the papers of Borel [Bo], Bernstein, Gelfand, and Gelfand [BGG], Demazure [D1,D2] and Lascoux and Schützenberger [LS] (see also [L]). Introductory and comprehensive accounts of both the "classical" theory and newer research are [F,FP] (for the algebraic-geometrical side) and [M1,M2,W3] (for the algebraic-combinatorial side).

The purpose of the present paper is to give a short and natural proof of a combinatorial rule for the generation of Schubert polynomials which was conjectured first by Kohnert in his Ph. D. dissertation [K] and subsequently proven in [W1]. The proof in [W1] has two advantages:

- It is the first and only one so far.
- It incorporates as an intermediary step the proof of two other (more complicated) rules for the generation of Schubert polynomials, namely the rules given by Bergeron [B] and Magyar [Ma].

The proof in [W1] has also two disadvantages:

- It is the first and only one so far — and first proofs are rarely the most simple and elegant ones.
- It incorporates as an intermediary step the proof of two other (more complicated) rules for the generation of Schubert polynomials, namely the rules given by Bergeron [B] and Magyar [Ma] — and (as we will make more precise below) these two rules do not provide a natural context for the proof of Kohnert's algorithm thus severely complicating matters.

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Subsequently in this introduction we collect the basic necessary information about Schubert polynomials, describe Kohnert's algorithm and introduce some useful notation. In Section 2 the proof is outlined and illustrated in two simple special cases, whereas Section 3 deals with the general case.

To every permutation π on numbers $1, \ldots, n$ contained in the symmetric group S_n one can associate a multivariate polynomial with non-negative integer coefficients $X_\pi \in \mathbb{Z}[x_1, \ldots, x_n]$ as follows: Let $\omega_n := n\,(n-1)\ldots 1 \in S_n$ be the permutation of maximal length, i.e., with the maximum number of inversions. Every other permutation $\pi \in S_n$ can then be reached from ω_n by going down in (right) *weak Bruhat order* where π' covers π in weak Bruhat order, if $\pi' = \pi \circ \sigma_k$ with σ_k the elementary transposition $(k, k+1)$ and $\pi(k) < \pi(k+1)$. In other words: π results from π' by removing the inversion of numbers on adjacent places k and $k+1$. If for all such pairs π, π' with $\pi' = \pi \circ \sigma_k$ one defines recursively $X_\pi := \partial_k X_{\pi'}$ where ∂_k is the divided difference operator acting on the variables x_k and x_{k+1} and if one starts at $X_{\omega_n} := x_1^{n-1} x_n^{n-2} \ldots x_{n-1}$ then this leads to well defined polynomials X_π for all $\pi \in S_n$, the Schubert polynomials. It is interesting to note that this (classical) approach to Schubert polynomials by divided difference operators reflects the relative geometric positions of Schubert varieties in a flag manifold where one variety lies on the boundary of another.

Alternatively, one can compute the Schubert polynomials without divided differences by going down from ω_n in (right strong) *Bruhat order* where π' covers π in Bruhat order, if $\pi' = \pi \circ (k, j)$ and $j \in J^{>k}(\pi)$ for any fixed k with $J^{>k}(\pi) :=$

(1) $\{ j \mid k < j,\ \pi(k) < \pi(j), \text{ and } |\{\nu \mid k < \nu < j,\ \pi(k) < \pi(\nu) < \pi(j)\}| = 0 \}$.

Then for every $\pi \neq \omega_n$ in S_n and $k := \pi^{-1}(1)$ one has

$$X_\pi = \frac{1}{x_k} \left(\sum_{j \in J^{>k}(\pi)} X_{\pi \circ (k,j)} \right) , \tag{2}$$

i.e., if k is chosen such that $\pi(k) = 1$ then $x_k X_\pi$ equals the sum of $X_{\pi'}$ where π' runs through certain π' covering π in Bruhat order. This formula has been noted first in [W2, Sec. 6] as a simple consequence of Monk's rule [Mo] for the multiplication of an arbitrary Schubert polynomial by any of the polynomials $X_{\sigma_k} = x_1 + \cdots + x_k$. Since (2) is essential for our new proof of Kohnert's algorithm we explain this point a bit more thoroughly: Monk's rule is

$$X_{\sigma_k} X_\pi = \sum_{(i,j) \in J(k,\pi)} X_{\pi \circ (i,j)}$$

where $J(k, \pi) :=$

$$\{(i,j) \mid i \leq k < j,\ \pi(i) < \pi(j), \text{ and } |\{\nu \mid i < \nu < j,\ \pi(i) < \pi(\nu) < \pi(j)\}| = 0 \} .$$

(Concise and simple proofs of Monk's rule can be found in each of [M1,M2,W2,W3].) From this formula one easily derives that the product $x_k X_\pi$ is of the form "sum on the r.h.s. of (2) minus a similar sum"; but the subtracted sum is zero, if for example $k = \pi^{-1}(1)$ (see [W2] or [W3] for details).

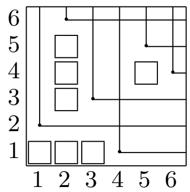
Note that formula (2) relates the calculation of Schubert polynomials directly to their multiplicative properties, respectively, the intersection calculus of Schubert varieties.

It is time now to describe the diagram rule for the generation of Schubert polynomials previously conjectured by Kohnert [K].

A (box) *diagram* is a finite collection of unit squares or boxes with vertices in the integer lattice $\mathbb{Z} \times \mathbb{Z}$; the diagram $D(\pi)$ of a permutation $\pi \in S_n$ is the diagram that originates from the set of boxes $\{[i,j] \mid 1 \leq i, j \leq n\}$ by cancellation of the 'hooks'

$$\{[\pi(j), j'] \mid j' \geq j\} \cup \{[i', j] \mid i' \geq \pi(j)\}$$

for $j = 1, \ldots, n$. For example, if $\pi = 263154 \in S_6$, then $D(\pi)$ =



where we have added hooks in the positions $(\pi(j), j)$, row numbers $i = 1, \ldots, 6$ at the left and column numbers $j = 1, \ldots, 6$ at the bottom of the diagram. Kohnert's algorithm says that all monomials occurring in a Schubert polynomial X_π can be found by looking at the set $K(D(\pi))$ of box diagrams derivable from $D(\pi)$ by (repeated) application of K-moves:

Definition 1. (K-moves) *Let $[i,j] \in D$ with $\{(i', j) \mid i' > i\} \cap D = \emptyset$, i.e., there is no box above $[i,j]$ in D, and assume that*

$$M_D(i,j) := \{(i,j') \mid j' < j,\ [i,j'] \notin D\} \neq \emptyset\ .$$

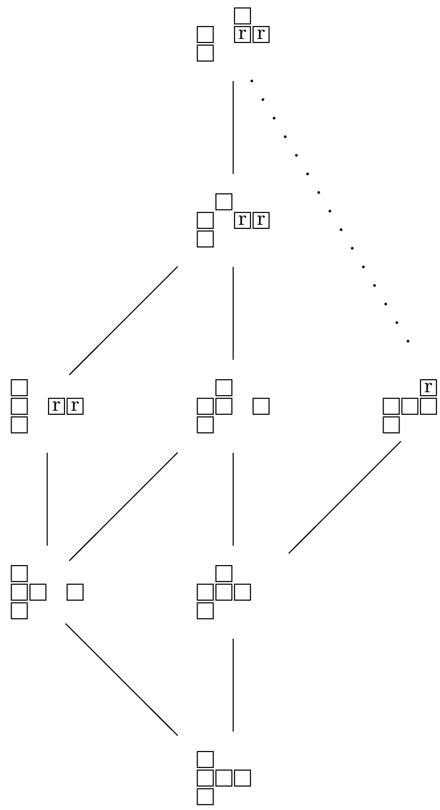
Then $[i,j]$ is allowed to move to the position in $M_D(i,j)$ with the greatest column number j', i.e., the closest empty position left to $[i,j]$ in row i of D. A K-move of the box $[i,j]$ will be called free, *if $(i, j-1) \in M_D(i,j)$, and a* tunnelling move *(through the boxes $[i,j-1], [i,j-2], \dots \in D$) otherwise.*

For any diagram D let $D^{[j,j']}$, $D_{[i,i']}$ and $D^{[j,j']}_{[i,i']}$ denote the sub diagrams ("minors") of D that contain only the boxes in the intervals of columns $[j,j']$, rows $[i,i']$ and in both columns $[j,j']$ and rows $[i,i']$, respectively. A special case of this notation can be used to associate a monomial to a diagram D by $x^D := x_1^{\beta_1} x_2^{\beta_2} \ldots$ where $\beta_j := |D^{[j]}|$ and $D^{[j]} := D^{[j,j]}$. The purpose of the present paper is to give a simple and natural proof of

Theorem 2. *For any natural number $n \in \mathbb{N}$ and permutation $\pi \in S_n$ let $K := K(D(\pi))$ be the set of diagrams derivable from $D(\pi)$ by repeated application of K-moves. Then*

$$X_\pi = \sum_{D \in K} x^D\ .$$

Example 3. For the permutation $\pi = 31542$ one computes algebraically $X_\pi = x_1^2 x_3^2 x_4 + x_1^2 x_2 x_3 x_4 + x_1^3 x_3 x_4 + x_1^2 x_2^2 x_4 + x_1^2 x_2 x_3^2 + x_1^3 x_2 x_4 + x_1^2 x_2^2 x_3 + x_1^3 x_2 x_3$. And indeed, the set K for this permutation contains all elements of the following poset:



where $D(\pi)$ is the top element and all K-derived diagrams are ordered such that D' covers D, if D originates from D' by an *irreducible* (or minimal) K-move. We have used straight lines for the free K-moves and a dotted line for the tunnelling move. More information about these posets can be found in [W1] where they form an integral part of the proof. Here we have exhibited the partial order only as a means to make the process of successive generation of diagrams more transparent. Note that the two boxes in the first column of $D(\pi)$ never move; such boxes are called *inertial*. (Why certain boxes are marked with an 'r' will become apparent later in Section 3.) $\square$

A "natural" proof of Kohnert's algorithm is one that "explains" the definition of K-moves and in particular the occurrence of tunnelling moves. Since Schur polynomials are special Schubert polynomials, the definition of K-moves must include moves which guarantee the realization of all monomials of Schur polynomials by box diagrams. But as already observed and proven by Kohnert in [K] the free K-moves are in fact in natural correspondence to the columnstrict numbering of semistandard Young tableaux which are used in the well known combinatorial definition of Schur polynomials (see also [W1, Section 4]). On the other hand free K-moves alone are obviously not sufficient to generate the monomials of general Schubert polynomials. Therefore Bergeron has included in his rule certain "backward" moves, i.e., moves of boxes from the left to the right; these backward moves were used to model in a combinatorial fashion the divided differences used in the classical approach to Schubert polynomials. Similarly Magyar's rule, which is a simplification of Bergeron's rule, has the divided difference approach as its starting point (cf. [W1, Section5]). Whereas the proof of these rules in [W1] with the help of the poset structure on the sets $K(D(\pi))$ and a suitable recursion argument is relatively transparent and easy, the subsequent derivation of Kohnert's rule is done by an exhaustive study of cases which establishes correctness without leading to a deeper understanding.

In the next section we will see that the generation of Schubert polynomials via formula (2) (without divided differences) makes the use of tunnelling moves virtually inevitable thereby suggesting the possibility of finding analogs of Kohnert's rule in more general settings: since the symmetric groups are Coxeter (and Weyl) groups of type A, one could use the generalization of Monk's (or "Pieri's") rule as the starting point for such an investigation.

2. Outline of the proof and two special examples

The proof proceeds by induction over n and k. It is not hard to verify Theorem 2 for $n = 1, 2$ and 3 and as induction hypothesis we assume that it is true for all permutations $\pi \in S_{n-1}$. Setting

$$S_{n,k} := \{\pi \in S_n \mid \pi(k) = 1\} \tag{3}$$

one can define a bijection between S_{n-1} and $S_{n,n}$ with the help of the mapping

$$\pi \mapsto 1_+(\pi)\,1 := (\pi(1)+1)\ldots(\pi(n-1)+1)\;1\;. \tag{4}$$

But because the diagram $D(1_+(\pi)\,1)_{[2,n]}$ has the same shape as $D(\pi)$ and the $n-1$ boxes in the row $D(1_+(\pi)\,1)_{[1]}$ are inertial, the formula

$$X_{1_+(\pi)\,1} = x_1 \cdot \cdots \cdot x_{n-1}\,X_\pi \tag{5}$$

from [W2, Prop. 3.3] (or [W3, Prop. 4.3.3]) shows that Theorem 2 is valid also for all $\pi \in S_{n,n}$.

Subsequently we will assume that π is contained in some $S_{n,k}$ with $1 \le k < n$ and that the assertion of Theorem 2 is true for all $\pi' \in S_{n,n}, \ldots, S_{n,k+1}$. Moreover, let

$$J^{>k}(\pi) = \{k_0, k_1, \ldots, k_q\} \quad \text{with} \quad k+1 = k_0 < k_1 < \cdots < k_q \le n \tag{6}$$

and for $\nu = 0, \ldots, q$ set $\pi_\nu := \pi \circ (k, k_\nu)$, $B_\nu := D(\pi_\nu)$, $K_\nu := K(B_\nu)$ and $i_\nu := \pi(k_\nu)$. Recall also that $K := K(D(\pi))$.

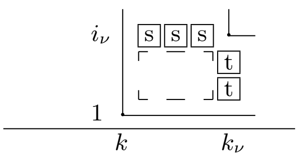
In view of formula (2) the following steps will constitute a proof of Theorem 2:

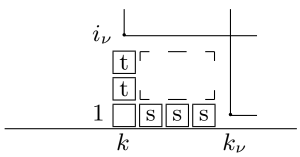
(1) There is a bijection between the sets K_ν and certain subsets H_ν of K that is *faithful with respect to columns*. By this we mean that the image $D \in H_\nu$ K of some diagram $B \in K_\nu$ has the same number of boxes in every column as B except for column k where $|D^{[k]}| = |B^{[k]} - 1|$.
(2) $H_\nu \cap H_{\nu'} = \emptyset$, if $\nu \neq \nu'$.
(3) $K \subset \bigcup_{\nu=0}^{q} H_\nu$.

Since $K \supset \bigcup_{\nu=0}^{q} H_\nu$ by 1, points 2 and 3 show that $K = \biguplus_{\nu=0}^{q} H_\nu$; and since the bijections of step 1 are faithful w.r.t. columns, one gets the desired conclusion by induction:

$$X_\pi \overset{(2)}{=} \frac{1}{x_k} \sum_{\nu=0}^{q} X_{\pi_\nu} = \frac{1}{x_k} \sum_{\nu=0}^{q} \sum_{B \in K_\nu} x^B = \sum_{\nu=0}^{q} \sum_{D \in H_\nu} x^D = \sum_{D \in K} x^D \ .$$

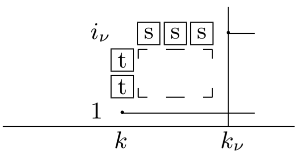
To carry out the above steps we compare first of all the diagrams of π and any of the π_ν:

$D(\pi) \ =$ 

$B_\nu := D(\pi_\nu) \ =$ 

The diagrams of $D(\pi)$ and B_ν are identical outside of the array of boxes formed by the intersection of rows $1, \ldots, i_\nu$ with columns $k, \ldots, k_\nu$ where the dashed rectangle is completely filled with boxes except for possibly empty rows on levels $\pi(1), \ldots, \pi(k-1)$. (These levels are of course also empty in the columns containing the t-boxes for both $D(\pi)$ and B_ν.) Note the additional box $[1, k] \in B_\nu$ corresponds to the fact that π_ν covers π in Bruhat order thereby having exactly one more inversion than π.

The diagram in K that corresponds faithfully w.r.t. columns and “naturally”, i.e., with a minimal number of differences to B_ν, is the diagram D_ν that results from $D(\pi)$ by “swapping” the boxes (marked with a ‘t’) in column k_ν to column k. By the recursiveness inherent in our induction this “swapping” is therefore a necessary supplement of the free moves. In fact, it is a sequence of valid tunnelling moves in the sense of Definition 1, because column k_ν is empty above level i_ν and because of the structure of the dashed rectangle.

$D_\nu \ =$ 

In Example 3 we have introduced already the notion of *inertial* boxes. They were defined as the boxes in $D(\pi)$ that can never K-move. Equivalently, they could be defined as the boxes that are not in the upper right area included by any of the hooks that were removed for the generation of the diagram $D(\pi)$. Clearly, the inertial boxes form a "partition" (with respect to columns or rows) in the lower left corner of $D(\pi)$ since there are no empty positions available. All other boxes can move (column wise from left to right) at least one position to the left. Below we will use the notion of inertial boxes in a slightly more general form, namely, a box is called *inertial* in an arbitrary diagram D, if it can not K-move relative to this diagram, and *movable* otherwise. Therefore the set of inertial boxes increases monotonically with every K-move executed on a diagram until all boxes are inertial (compare again the poset of Example 3). Furthermore, every rectangle of positions in D formed by the origin as lower left and an inertial box as upper right corner contains only inertial boxes (or empty positions).

Observe now that the s-boxes of B_ν above, i.e., the boxes marked with an 's', are inertial (in B_ν) so that all diagrams of $K_\nu = K(B_\nu)$ can be derived "as if the s-boxes were not there". Since we want to represent K_ν faithfully w.r.t. columns by a set $H_\nu \subset K$ the following definition is natural:

Definition 4. *For π, π_ν and D_ν as above let $S(D_\nu) := \{[i_\nu, k_0], [i_\nu, k_0+1], \ldots, [i_\nu, k_\nu - 1]\}$ denote the set of s-boxes of D_ν. Let $K_S(D_\nu)$ denote the set of all diagrams in $K(D_\nu)$ which can be derived without moving any s-box $S(D_\nu)$. Then D_ν is called* s-independent, *if*

$$K_S(D_\nu) = K(D_\nu \setminus S(D_\nu)) \cup S(D_\nu) \ ,$$

where the union with the set $S(D_\nu)$ is of course taken for every diagram in $K(D_\nu \setminus S(D_\nu))$. D_ν is called s-dependent, *if it is not s-independent.*

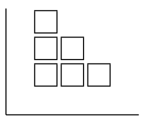
Note that (D_ν) is s-independent iff all boxes below $S(D_\nu)$ in D_ν are inertial.

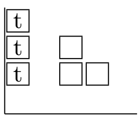
Moreover, we remark that similarly to the notation $S(D_\nu)$ of Definition 4 one defines the sets of s-boxes $S(B_\nu) := \{[1, k_0], \ldots, [1, k_\nu - 1]\}$ and the sets of t-boxes $T(B_\nu) := (B_\nu)^{[k]} \setminus \{[1, k]\}$, $T(D_\nu) := T(B_\nu)$, $T_\nu(D(\pi)) := D(\pi)^{[k_\nu]}$.

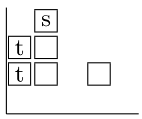
Lemma 5. *For every $\pi \in S_{n,k}$ with $k < n$ and $J^{>k}(\pi)$ as in (6) the diagrams D_0 and D_q are s-independent.*

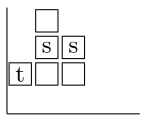
Proof. For D_0 this is trivial, because $S(D_0) = \emptyset$. For D_q one observes that $(D_q)^{[k_0,n]}_{[1,i_q-1]}$ and $(D_q)^{[k_q,n]}_{[i_q,i_q]}$ are empty so that no boxes in $D_q \setminus S(D_q)$ can be obstructed in their movability by the boxes in $S(D_q)$. (Compare D_3 in Examples 6 and 11.) $\square$

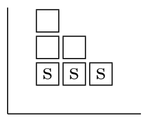
If for some π *all* diagrams D_ν are s-independent then it is easy to carry out the steps 1, 2 and 3 of the proof:

Example 6. Let $\pi = 15432 \in S_5$. Then $D(\pi) =$  and

$D_0 =$ 

, $D_1 =$ 

, $D_2 =$ 

, $D_3 =$ 

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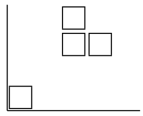
Clearly, all D_ν are s-independent, because the boxes below the sets $S(D_\nu)$ are inertial. Hence the faithful w.r.t. columns bijection of step 1 is simply given by setting $H_\nu := K(D_\nu \setminus S(D_\nu)) \cup S(D_\nu)$ (remember the removal of $[1, k]$).

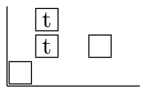
For any finite permutation π the number of boxes in $S(D_\nu)$ is $k_\nu - k - 1$ by definition. For any diagram $D \in K(D(15432))$ in our example (or more generally any D contained in a s-independent D_ν) set

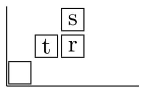
$$\mu(D) := \max \{\nu \mid S(D_\nu) \subset D\} .$$

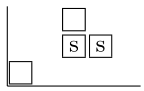
Since for all ν the s-boxes of D_ν are fixed for the $D \in H_\nu$ and since all boxes in rows below level i_ν are inertial, one has $D \in H_\nu \Leftrightarrow \mu(D) = \nu$. This shows both step 2 and step 3. $\square$

The case that for given π all D_ν are s-independent is of course a rare one. To get a first idea about how to deal with a D_ν that is s-dependent, we close this section with another simple example.

Example 7. Let $\pi = 21543 \in S_5$. Then $D(\pi) =$  and (omitting the empty second row)

$D_0 =$ 

, $D_1 =$ 

, $D_2 =$ 

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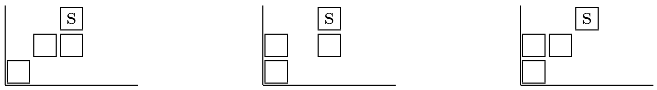
By Lemma 5 (or direct inspection) one sees that D_0 and D_2 are s-independent. But D_1 is s-dependent. If one K-moves the boxes of D_1 as if the s-box were not there, one would get



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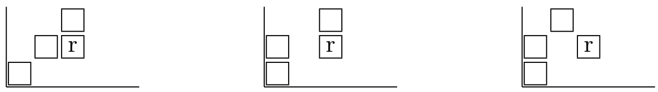
But then the second move would not be a valid K-move. The way out is to fix the r-box of D_1 (marked with an 'r') instead of the s-box. Then the above diagrams are faithfully w.r.t. columns represented by:



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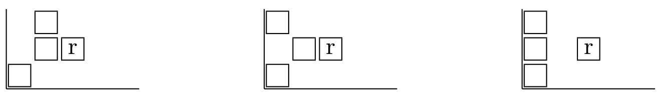
It is very important to note that simply fixing the r-box in D_1 allows for more K-derived diagrams than just the two depicted above, namely:



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But these three diagrams are already contained in $K_S(D_0) = K(D_0)$! The key to distinguish between the "right" three the "wrong" three diagrams which fix the r-box is to maintain the order of the two moving boxes "as if they were on the same level". The sub diagrams $\begin{smallmatrix} & \square \\ \square & \end{smallmatrix}$, $\begin{smallmatrix} & & \square \\ \square & & \end{smallmatrix}$, etc. where the boxes are in *ascending order* or form an *ascending step* are the "right" one's in $K_S(D_1)$. The subdiagrams $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square & \\ & \square \end{smallmatrix}$, etc. (the latter being in *descending order* or forming a *descending step*) are the "wrong" one's.

It is clear by the forgoing discussion that H_2 contains all diagrams derivable from D_2 with the two s-boxes fixed, that H_1 contains the diagrams derivable from D_1 with the r-box fixed and the two movable boxes maintaining ascending order, that D_0 contains all remaining diagrams of K and that the H_ν thus defined form a partition of K. (For a similar situation examine again the poset of Example 3.) $\square$

3. The general case

Since the discussion of Example 6 (and in particular the equivalence $D \in H_\nu \Leftrightarrow \mu(D) = \nu$) applies to all s-independent diagrams D_ν, the work in the general case has to focus on the s-dependent D_ν. Example 7 pointed out how to proceed: In case of an s-dependent diagram D_ν one must choose r-boxes as substitutes for the s-boxes. They must be below the s-boxes in the same columns (because of faithfulness w.r.t. columns) and the must guarantee optimal movability of the other boxes ("as if they were not there") without being inertial themselves (because the original s-boxes in B_ν are not inertial, too). These requirements solicit the following definition:

Definition 8. *Let the permutation π be element of some $S_{n,k}$ with $k < n$ and with $J^{>k}(\pi)$ as in (6). For any diagram D_ν the set of r-boxes $R(D_\nu) \subset (D_\nu)^{[k_0,k_\nu-1]}$ contains the unique box in every column in question that is directly above the highest inertial box.*

Lemma 9. *Let π, $J^{>k}(\pi)$ and the sets $R(D_\nu)$ be given as in Definition 8 above. Then:*
a) No r-box is above an s-box.
b) If a diagram D_ν is s-independent, then the r-boxes of D_ν are identical to the s-boxes: $R(D_\nu) = S(D_\nu)$. Otherwise the r-boxes are strictly below the s-boxes.
c) All r-boxes of an s-dependent D_ν are in the same row.
d) If not all D_ν are s-dependent, then

$$\rho := \max\, \{\nu \mid D_\nu \text{ is s-dependent }\} \tag{7}$$

exists and $D_0, D_{\rho+1}, \ldots, D_q$ are s-independent, whereas $D_1, \ldots, D_\rho$ are s-dependent. Moreover, the r-boxes of the s-dependent D_ν are all on level i_ρ.

Proof. a) follows from the definition of r-boxes, the fact that the s-boxes $S(D_\nu) \subset (D_\nu)^{[k_0,k_\nu-1]}$ are never inertial and that inertial boxes can not be above movable boxes.

b) After Definition 4 we remarked that D_ν is s-independent iff all boxes below $S(D_\nu)$ in D_ν are inertial. Hence b) is a consequence of a).

c) Suppose that $[i,j] \in R(D_\nu)$ has minimal column index j and that $[i',j']$ is a different box in $R(D_\nu)$, which means $j < j'$. Then by Definition 8 the next box $[i'',j']$ below $[i',j']$ is inertial and consequently every box in $(D_\nu)^{[1,j']}_{[1,i'']}$ is inertial, whence $i \geq i'$.

It remains to show that $i > i'$ is excluded, too. But since $[i',j']$ is movable, there must be an empty position (i',j'') in D_ν with $j'' < j'$. In fact, even $j'' < j$ because

$k < j < j' < k_\nu$ and all rows below $S(D_\nu)$ are either completely filled with boxes or completely empty. If now the position (i', j'') with $j'' < j$ is empty in D_ν, then the assumption $i > i'$ forces an empty position $(i-1, j'')$ in D_ν and $[i-1, j]$ would be movable instead of being inertial. Hence all r-boxes are in the same row.

d) Assume that D_ν is s-independent for some $\nu > 0$. Then the boxes below $S(D_\nu)$ in the interval of columns $[k_0, k_{\nu-1}]$ are inertial in D_ν. In particular the boxes below $[i_{\nu+1}, k_0], \dots, [i_{\nu+1}, k_{\nu-1}]$ are inertial D_ν. Comparison of D_ν and $D_{\nu+1}$ then shows that consequently the boxes below $[i_{\nu+1}, k_0], \dots, [i_{\nu+1}, k_\nu]$ are inertial in $D_{\nu+1}$. Hence $D_{\nu+1}$ is s-independent by Definition 4.

This proves the first part of d). That the r-boxes of the s-dependent D_ν are all on level i_ρ follows by an analogous argument that proceeds from the s-boxes of $D_{\rho+1}$ to the r-boxes of $D_\rho, D_{\rho-1}, \dots, D_1$ using Definition 8. $\square$

The next problem is to define the sets $H_\nu \subset K$ for the D_ν as required by step 1 in Section 2. For s-independent D_ν this is done by setting $H_\nu := K(D_\nu \setminus S(D_\nu)) \cup S(D_\nu)$. But for s-dependent D_ν it is not possible to simply substitute the s-boxes by the r-boxes. The K-derived set fixing the r-boxes would be too big in general. It is necessary to apply the notions of ascending and descending orders of boxes from Example 7 to suitable subdiagrams. This is done by generalizing the notion of a row or level by a "line":

Definition 10. *a) Let π, $J^{>k}(\pi)$, $S(D_\nu)$ and $R(D_\nu)$ as in Definition* 8 *and ρ as in* (7). *Then the* lines *of an s-independent D_ν are identical to the rows of D_ν; and the* lines *of an s-dependent D_ν are identical to the rows of D_ν outside of*

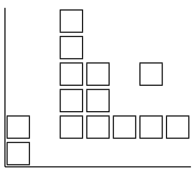
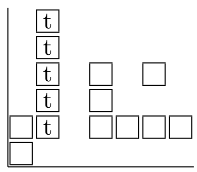
$$\mathcal{D}_\nu := (D_\nu)^{[k_0, k_\nu - 1]}_{[i_\rho + 1, i_\nu]} \tag{8}$$

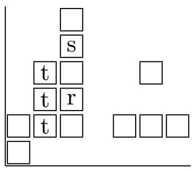
whereas each row i with $i_\rho \le i < i_0$ is continued inside $\mathcal{D}_\nu$ by the next higher non-empty row of boxes.

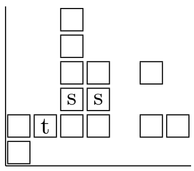
b) The lines that have a non-empty intersection with $\mathcal{D}_\nu$ are called proper lines. *The intersection with $\mathcal{D}_\nu$ is called the* middle part *of the line and the parts left and right of $\mathcal{D}_\nu$ the* left *and* right part, *respectively. The step from the left to the middle part of a proper line is called an* ascending step *and the step from the middle part to the right part of a proper line a* descending step. *Alternatively, one could say that the two boxes involved in both kinds of steps are in* ascending order *respective* descending order.

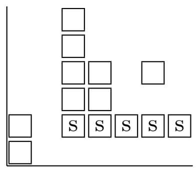
c) The number of boxes contained in the left, middle and right part of the l-th proper line (counted from bottom to top) of D_ν form the first, second and third entry of a vector $\chi^l_\nu \in \mathbb{N}^3_0$ called the characteristic *of that proper line.*

Example 11. Let $\pi = 31864752 \in S_8$. Then (without the empty third row)

$D(\pi) =$  and $D_0 =$  ,

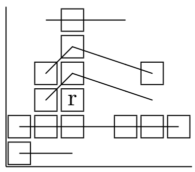
$D_1 =$ 

, $D_2 =$ 

, $D_3 =$ 

.

Therefore $\rho = 1$ and the lines of D_1 are

$D_1 =$ 

with the characteristics $\chi_1^1 = (1, 1, 0)$ and $\chi_1^2 = (1, 1, 1)$. $\square$

It is useful to think about the lines of an s-dependent D_ν as the rows of B_ν being "pushed up" locally by the upward move of the s-boxes of B_ν to the position of the r-boxes of D_ν while at the same time removing the s-boxes of D_ν. (Note that the removal of $S(D_\nu)$ in the process of pushing up guarantees that the lines modify the original rows only inside $\mathcal{D}_\nu$ and that all lines are disjoint.)

Definition 12. *Let π be as in Definition 8. For s-independent D_ν set $H_\nu := K(D_\nu \setminus S(D_\nu)) \cup S(D_\nu)$. For s-dependent D_ν let H_ν be the set of all diagrams in $K(D_\nu)$ that fix $R(D_\nu)$ and which respect lines in the following sense: there are never boxes of the same line on top of each other (except in a transitory way during tunnelling).*

In other words: The boxes in a proper line move as if they were on one level. Since all $D \in H_\nu$ are derived by K-moves that respect lines, it is possible to extend the notions of proper lines and characteristics of Definition 10 to all such D:

Definition 13. *Let π be as in Definition 8 and H_ν as in Definition 12 for any s-dependent D_ν and $D \in H_\nu$ arbitrary. Then the proper lines of some D can be derived recursively from the proper lines of D_ν by a sequence of K-move that respect lines.*

Accordingly, the notions of left, middle and right parts, of ascending and descending steps and of a characteristics from Definition 10 b)-c) apply to the proper lines in D. Again the middle part is exactly the elevated portion of the proper line and Again the number of boxes contained in the left, middle and right part of the l-th proper line (counted from bottom to top) of D form the characteristics *$\chi_\nu^l(D)$ of D.*

Note that $\chi_\nu^l(D)$ can be differ from χ_ν^l in a certain way: the number of boxes in the left part of a proper line can increase at the cost of the right part through tunnelling of a box.

We are now prepared to carry out the steps $1-3$ of the proof of Kohnert's algorithm (recall Section 2) also for the s-dependent D_ν where $1 \leq \nu \leq \rho$. For step 1 observe that for every diagram $D \in H_\nu$ there is clearly some counterpart $D' \in K_\nu = K(B_\nu)$ that is faithful w.r.t. columns, because the K-moves in H_ν respect lines. But conversely there is also for every $D' \in K_\nu$ a counterpart $D \in H_\nu$: any K-move K_ν can be imitated by a sequence of one or more K-moves in H_ν that respect lines. The "obstacle" $R(D_\nu)$ can be overcome by tunnelling.

For steps 2 and 3 it is sufficient to show that every diagram $D \in K$ is contained in exactly one H_ν. If $S(D_\nu) \subset D$ for any $\nu \in \{\rho+1,\dots,q\}$, then necessarily $D \in H_\nu$ for exactly that ν. (Also the reversal is true.) If this is not the case, then we have to show that $D \in H_\nu$ for exactly one $\nu \in \{0,1,\dots,\rho\}$. The idea how to proceed is as follows:

For all ν with $1 \le \nu \le \rho$ try to construct proper lines in D in a way that is consistent with the different characteristics χ_ν^l – we will describe an appropriate algorithm below –. If this is possible, we write $D \vDash H_\nu$, and $D \nvDash H_\nu$ otherwise. Clearly, $D \in H_\nu \Longrightarrow D \vDash H_\nu$ but not necessarily conversely. The desired conclusion "$D \in H_\nu$ for exactly one ν" is then implied by the two claims

a) $D \vDash H_\nu \implies D \notin H_{\nu'} \quad \forall \nu' < \nu$ and

b) $D \nvDash H_\nu,\ \forall \nu \ge 1 \implies D \in H_0$.

In fact a) and b) together say that the (existing) maximal $\nu \le \rho$ with $D \vDash H_\nu$ is the unique ν for which $D \in H_\nu$.

Note that for fixed $\nu \in \{1,\dots,\rho\}$

$$\chi_\nu^l(2) = r_\nu := |R(D_\nu)|,\ \forall\ l\ ,$$

i.e., the middle parts of all proper lines in D_ν have the same cardinality r_ν. Recall further that by tunnelling of boxes $\chi_\nu^l(1)$ may increase at the cost of $\chi_\nu^l(3)$ and that on levels i_ν no $D \in K$ has any boxes right to column k_ν.

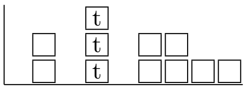
Lemma 14. (Non-crossing Condition) *Let $D \in K$ be a diagram with more than one proper line. Then no box from the middle part of a proper line can be left to (in the same row) any box from the left part of the next proper line above. (It is however possible that a box from the right part of a proper line is left to a box from the middle part of the next proper line below.)*

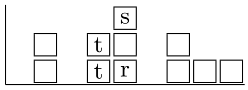
Proof. A box from the middle part of any proper line can be moved to the left only after the corresponding boxes from the middle parts of higher proper lines have been moved to the left. But Definition 12 says that moving boxes from the middle part of a proper line results in "pushing" the boxes from its left part father to the left. □

Given any $D \notin H_\nu$ for $\rho+1 \le \nu \le q$ we discuss now how to construct proper lines for given characteristics. Since $R(D_{\rho+1}) \nsubseteq D$ by assumption, there exists (recall Lemma 9 d)) a k': $k_0 \le k' < k_{\rho+1}$ with $\{[i_{\rho+1},k_0],\dots,[i_{\rho+1},k']\} \subset D$ but $[i_q,k'+1] \notin D$. Hence it is possible that $D \vDash H_\nu$ for some ν with $k_\nu \le k'$ and we begin by checking the maximal ν first, than the next smaller ν, etc. .

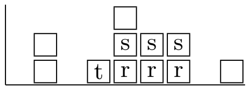
Assuming that $D \in H_\nu$ (for $1 \le \nu \le \rho$) the positions of the boxes $R(D_\nu) \subset D$ are known and the first proper line can be determined using the characteristic χ_ν^1. If there are less than $\chi_\nu^1(3)$ boxes right to $R(D_\nu)$ in the first proper line then the missing boxes must have tunnelled to the left part of the line (and possibly higher boxes must have tunnelled, too). Observing the non-crossing condition of Lemma 14 and using the characteristics $\chi_\nu^2, \chi_\nu^3, \ldots$ it can be checked one line after the other, whether it is possible to construct all proper lines bottom up in a consistent manner, i.e., whether $D \models H_\nu$.

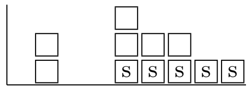
Example 15. Consider the permutation $\pi = 3\ 6\ 2\ 1\ 8\ 7\ 10\ 5\ 9\ 4 \in S_{10}$ with (relevant rows only)

$D_0 =$ 

, $D_1 =$ 

,

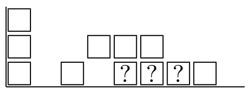
$D_2 =$ 

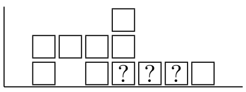
, $D_3 =$ 

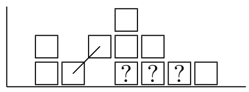
.

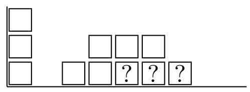
D_0 and D_3 have of course no proper lines whereas D_1 has two of them with characteristics $\chi_1^1 = (2, 1, 3)$, $\chi_1^2 = (2, 1, 1)$ and D_2 has one with characteristic $\chi_2^1 = (2, 3, 1)$. Accordingly, $r_0 = 0$, $r_1 = 1$, $r_2 = 3$ and $r_3 = 5$.

Investigation of the following four diagrams

$D =$ 

, $D' =$ 

$D'' =$ 

, $D''' =$ 

shows that none of them is an element of H_3, but all of them contain boxes in the positions $R(D_1)$ and $R(D_2)$ indicated by the question marks. Moreover, it is not hard to verify that $D \models H_2$ and $D \not\models H_1$, $D' \models H_1$ (with tunnelling in the second proper line) and $D' \not\models H_2$, $D'' \models H_1$ and $D'' \models H_2$ whereas $D''' \not\models H_1$ and $D''' \not\models H_2$ (watch the necessity of a ascending step!).

Indeed, $D \in H_2$, $D' \in H_1$, $D'' \in H_2$ and $D''' \in H_0$ in accordance with the claims a) and b) to be shown next. $\square$

Proof. (of a)) Let $D \models H_\nu$ with $\nu \leq \rho$ maximal. Then all middle parts of the proper lines may be in their original position (forming a "rectangle" of boxes with possibly some empty rows) or may not.

For the first case recall that $\nu' < \nu$ implies $k_{\nu'} < k_\nu$, $i_{\nu'} > i_\nu$ and $r_{\nu'} < r_\nu$ whence row i_ν contains more boxes right to column k than possible for a $D \in H_{\nu'}$.

For the second case Definition 12 shows that moving a box from the middle part of a proper lines causes the boxes of its left part to move, too, such that the ascending step from left to middle part is retained. But since $i_{\nu'} > i_\nu$ for $\nu' < \nu$ in case of $D \in H_{\nu'}$ this would mean an ascending step among the t-boxes of D which is impossible. (For an illustration examine again D'' of Example 15 where the crucial ascending step is marked by a line segment). $\square$

Proof. (of b)) Assume $D \not\models H_\nu$, $\forall \nu \geq 1$. Then for all $\nu \geq 1$ the number of boxes right to column k in row i_ν must be less than r_ν by the arguments in the proof of a). Clearly, D_0 obeys this condition and at the same time it is the maximal element of the sub poset H_0 of K. In other words: every $D \in K$ obeying the mentioned condition can be derived by a sequence of K-moves from D_0 whence $D \in H_0$. (For an illustration examine again D''' of Example 15.) □

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References

[B] N. Bergeron, A combinatorial construction of the Schubert polynomials, *J. Comb. Th. A* **60** (1992), 168-182.

[BGG] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Schubert Cells and cohomology of the spaces G/B, *Russian Math. Surveys* **28** (1973), 1-26.

[Bo] A. Borel, Sur la cohomologie des espaces fibré principaux et des espaces homogènes des groupes Lie compacts, *Ann. Math.* **57** (1953), 115 - 207.

[D1] M. Demazure, Invariants symétriques des groups de Weyl et torsion, *inventiones math.* **21** (1973), 287 - 301.

[D2] M. Demazure, Désingularisation des variétés de Schubert généralisés, *Ann. Scient. Ec. Norm. Sup.* **7** (1974), 53 - 88.

[F] W. Fulton, "Young tableaux", Cambridge University Press, Cambridge, 1997.

[FP] W. Fulton, P. Pragacz, "Schubert Varieties and Degeneracy Loci", *Lecture Notes in Mathematics* **1689**, Springer, Berlin, 1998.

[K] A. Kohnert, "Weintrauben, Polynome, Tableaux", *Bayreuther Mathematische Schriften* **38** (1991), 1 - 97.

[L] A. Lascoux, Polynômes de Schubert. Une approache historique, *Discrete Math.* **139** (1995), 303 - 317.

[LS] A. Lascoux, M.-P. Schützenberger, Polynomes de Schubert, *C.R.Acad.Sci.Paris* **294** (1982), 447 - 450.

[M1] I.G. Macdonald, Schubert polynomials, *in*: A.D. Kendwell (ed.), "Surveys in Combinatorics", *London Math. Soc. Lec. Notes Ser.* **166**, 73 - 99, Cambridge University Press, Cambridge 1991.

[M2] I.G. Macdonald, "Notes on Schubert polynomials", Publications du L.A.C.I.M., vol. 6, Université du Québec, Montréal, 1991.

[Ma] P. Magyar, Four new formulas for Schubert polynomials, preprint (1995).

[Mo] D. Monk, The geometry of flag manifolds, *Proc. London Math. Soc. (3)* **9** (1959), 253-286.

[W1] R. Winkel, Diagram rules for the generation of Schubert polynomials, *J. Combin. Theory A* **86** (1999), 14 - 48.

[W2] R. Winkel, Recursive and combinatorial properties of Schubert polynomials, *Sem. Loth. Comb.* **B38c** (1996), 29 pp.

[W3] R. Winkel, "On algebraic and combinatorial properties of Schur and Schubert polynomials", *Bayreuther Mathematische Schriften* **59** (2000), 225 pp.

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