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ON THE SYMMETRY CLASSES OF THE FIRST COVARIANT DERIVATIVES OF TENSOR FIELDS

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ABSTRACT. We show that the symmetry classes of torsion-free covariant derivatives ∇T of r-times covariant tensor fields T can be characterized by Littlewood-Richardson products $\sigma[1]$ where σ is a representation of the symmetric group $\mathcal{S}_r$ which is connected with the symmetry class of T. If $\sigma \sim [\lambda]$ is irreducible then $\sigma[1]$ has a multiplicity free reduction $[\lambda][1] \sim \sum_{\lambda \subset \mu}[\mu]$ and all primitive idempotents belonging to that sum can be calculated from a generating idempotent e of the symmetry class of T by means of the irreducible characters or of a discrete Fourier transform of $\mathcal{S}_{r+1}$. We apply these facts to derivatives ∇S, ∇A of symmetric or alternating tensor fields. The symmetry classes of the differences $\nabla S - \mathrm{sym}(\nabla S)$ and $\nabla A - \mathrm{alt}(\nabla A) = \nabla A - dA$ are characterized by Young frames $(r, 1) \vdash r + 1$ and $(2, 1^{r-1}) \vdash r + 1$, respectively. However, while the symmetry class of $\nabla A - \mathrm{alt}(\nabla A)$ can be generated by Young symmetrizers of $(2, 1^{r-1})$, no Young symmetrizer of $(r, 1)$ generates the symmetry class of $\nabla S - \mathrm{sym}(\nabla S)$. Furthermore we show in the case $r = 2$ that $\nabla S - \mathrm{sym}(\nabla S)$ and $\nabla A - \mathrm{alt}(\nabla A)$ can be applied in generator formulas of algebraic covariant derivative curvature tensors. For certain symbolic calculations we used the Mathematica packages Ricci and PERMS.

1. INTRODUCTION

The present paper continues our investigations of tensors by methods of Algebraic Combinatorics in [12, 13, 14]. The starting point are some questions concerning *algebraic covariant derivative curvature tensors* which arise from [13]. Many considerations which are necessary for an answer of these questions can be carried out for arbitrary tensor fields and lead to results about a connection of torsion-free covariant derivatives of differentiable tensor fields and Littlewood-Richardson products.

Algebraic curvature tensors are covariant tensors of order 4 which have the same algebraic properties as the *Riemannian curvature tensor*.

Definition 1.1. A covariant tensor $\mathfrak{R}$ of order 4 is called an *algebraic curvature tensor* iff its coordinates satisfy the conditions

$$\mathfrak{R}_{ijkl} = -\mathfrak{R}_{jikl} = -\mathfrak{R}_{ijlk} = \mathfrak{R}_{klij} \tag{1.1}$$

$$\mathfrak{R}_{ijkl} + \mathfrak{R}_{iklj} + \mathfrak{R}_{iljk} = 0\,. \tag{1.2}$$

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A covariant tensor $\mathfrak{R}'$ of order 5 is called an *algebraic covariant derivative curvature tensor* iff its coordinates fulfil

$$\mathfrak{R}'_{ijklm} \;=\; -\mathfrak{R}'_{jiklm} \;=\; -\mathfrak{R}'_{ijlkm} \;=\; \mathfrak{R}'_{klijm} \tag{1.3}$$

$$\mathfrak{R}'_{ijklm} + \mathfrak{R}'_{ikljm} + \mathfrak{R}'_{iljkm} \;=\; 0 \tag{1.4}$$

$$\mathfrak{R}'_{ijklm} + \mathfrak{R}'_{ijlmk} + \mathfrak{R}'_{ijmkl} \;=\; 0\,. \tag{1.5}$$

Relation (1.1) represents the index commutation symmetry of the Riemannian curvature tensor R whereas relations (1.2) and (1.5) correspond to the first and second Bianchi identity for the Riemann tensor

$$\begin{aligned} R_{ijkl} + R_{iklj} + R_{iljk} \;&=\; 0 \\ R_{ijkl\,;m} + R_{ijlm\,;k} + R_{ijmk\,;l} \;&=\; 0\,. \end{aligned}$$

Investigations of algebraic curvature tensors were carried out by many authors. (See the extensive bibliography in the book [17] by P. B. Gilkey.) One of the problems which are considered in connection with algebraic curvature tensors is the *search for generators* of algebraic curvature tensors. In [12, 13, 14] we applied tools of Algebraic Combinatorics to treat this problem. In particular we used symmetry operators for tensors, given by elements a of the group ring $\mathbb{K}[\mathcal{S}_r]$ of a symmetric group $\mathcal{S}_r$, Young tableaux t, Young symmetrizers y_t and symmetry classes of tensors, defined by right ideals $\mathfrak{r}$ of $\mathbb{K}[\mathcal{S}_r]$. Details about these concepts can be found in Sections 2 and 4 and in [12, 13, 14].

P. B. Gilkey [17, pp.41-44] and B. Fiedler [12] gave different proofs that the vector space of all algebraic curvature tensors is spanned by each of the following types of tensors

$$y_t^*(S \otimes S) \quad , \quad y_t^*(A \otimes A) \tag{1.6}$$

which are defined by symmetric or alternating covariant tensors S or A of order 2. The vector space of algebraic covariant derivative curvature tensors is generated by each of the following tensor types

$$\begin{array}{ccc} y_{t'}^*(S \otimes S') & , & y_{t'}^*(S' \otimes S) \\ y_{t'}^*(U \otimes S) & , & y_{t'}^*(S \otimes U) \\ y_{t'}^*(U \otimes A) & , & y_{t'}^*(A \otimes U) \end{array} \tag{1.7}$$

(see[1] B. Fiedler [13]). Here S, A are again symmetric or alternating tensors of order 2, S' is a symmetric tensor of order 3 and U is a covariant tensor of order 3 from an irreducible symmetry class that belongs to the partition $(2,1) \vdash 3$ and is defined by a minimal right ideal different from the right ideal $f \cdot \mathbb{K}[\mathcal{S}_3]$ with the generating idempotent

$$f \;:=\; \left\{ \frac{1}{2}\,(\mathrm{id} - (1\,3)) - \frac{1}{6}\,y \right\} \quad , \quad y \;:=\; \sum_{p \in \mathcal{S}_3} \mathrm{sign}(p)\,p\,. \tag{1.8}$$

[1]A first proof that $y_{t'}^*(S \otimes S')$ and $y_{t'}^*(S' \otimes S)$ are generators for $\mathfrak{R}'$ was given by P. B. Gilkey [17, p.236].

In (1.6) and (1.7) y_t and $y_{t'}$ denote the Young symmetrizers of the Young tableaux

$$t \;=\; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad , \quad t' \;=\; \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \;. \tag{1.9}$$

The generators (1.7) lead to the question whether there exist typical tensorial quantities of differential geometry which possess a symmetry of the same type as the above tensors U. It can be shown (see Section 3) that the differences

$$\nabla S \,-\, \mathrm{sym}(\nabla S) \quad , \quad \nabla A \,-\, \mathrm{alt}(\nabla A) \;=\; \nabla A \,-\, \mathrm{d}A \tag{1.10}$$

between the covariant derivatives ∇S, ∇A of symmetric/alternating covariant tensor fields S, A of order 2 and their symmetrized/anti-symmetrized covariant derivatives $\mathrm{sym}(\nabla S)$, $\mathrm{alt}(\nabla A)$ have such a symmetry[2]. Furthermore computer calculations by means of the Mathematica packages Ricci [21] and PERMS [9] showed that the symmetry class of $\nabla A \,-\, \mathrm{alt}(\nabla A)$ is generated by certain Young symmetrizers, for instance by the Young symmetrizer of the standard tableau

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \;.$$

However, the same computation yields the surprising result that no Young symmetrizer of a Young frame $(2\,1) \vdash 3$ generates the symmetry class of $\nabla S \,-\, \mathrm{sym}(\nabla S)$. One goal of the present paper is to find out whether the covariant derivatives of symmetric or alternating tensor fields of order $r > 2$ have such a behaviour, too.

In Section 2 we collect some basic facts about the connection between symmetry classes of covariant tensors of order r and left or right ideals of the group ring $\mathbb{K}[\mathcal{S}_r]$ of the symmetric group $\mathcal{S}_r$. In Section 3 we show that the symmetry class of a torsion-free covariant derivative ∇T of a differentiable tensor field T of order r is defined by a left ideal $\mathfrak{l} \subseteq \mathbb{K}[\mathcal{S}_{r+1}]$ which is the representation space of a Littlewood-Richardson product $\sigma[1]$, where σ is a representation of $\mathcal{S}_r$ connected with the symmetry class of T. If $\sigma \sim [\lambda]$, $\lambda \vdash r$, is irreducible then the Littlewood-Richardson rule yields a multiplicity-free decomposition

$$[\lambda][1] \;\sim\; \sum_{\substack{\mu \vdash r+1 \\ \lambda \subset \mu}} [\mu]\,. \tag{1.11}$$

In particular the symmetry classes of the covariant derivatives ∇S, ∇A of symmetric/alternating tensor fields S, A of order r are characterized by Littlewood-Richardson products

$$[r][1] \;\sim\; [r+1] + [r,1] \quad , \quad [1^r][1] \;\sim\; [1^{r+1}] + [2, 1^{r-1}]\,.$$

If we know a primitive generating idempotent $e \in \mathbb{K}[\mathcal{S}_r]$ of the symmetry class of T, then all unique primitive idempotents $h_\mu \in \mathbb{K}[\mathcal{S}_{r+1}]$ corresponding to (1.11) can be calculated from e by means of the symmetrizers of the irreducible characters of $\mathcal{S}_{r+1}$ or, more efficiently, by a discrete Fourier transform.

[2]Note that $\mathrm{alt}(\nabla A)$ is equal to the exterior derivative $\mathrm{d}A$ of the alternating tensor field A.

In Section 4 we investigate the parts $[r,1]$, $[2,1^{r-1}]$ of ∇S, ∇A for arbitrary order $r \geq 2$. We show for ∇S that no Young symmetrizer with a Young frame $(r,1) \vdash r+1$ is a generator of the $[r,1]$-part of ∇S. The $[2,1^{r-1}]$-part of ∇A, however, is generated by the Young symmetrizer of the lexicographically greatest standard tableau of $(2,1^{r-1}) \vdash r+1$ (see (4.2)) and every other standard tableau of $(2,1^{r-1}) \vdash r+1$ annihilates the $[2,1^{r-1}]$-part. Furthermore that $[2,1^{r-1}]$-part is generated or annihilated by many other Young symmetrizers of non-standard tableaux of $(2,1^{r-1}) \vdash r+1$. We present complete computer generated lists of such Young symmetrizers for $r = 2,3,4$ in an Appendix.

The last Section of the paper deals with the question whether tensors (1.10) can be used as generators U of algebraic covariant derivative curvature tensors in formulas (1.7). Both S and A satisfy the condition that the symmetry classes of the tensors (1.10) are not generated by the above right ideal $f \cdot \mathbb{K}[\mathcal{S}_3]$ with generating idempotent (1.8). Thus all tensors (1.10) can play the role of U in (1.7).

2. Symmetry classes of tensors

Let $\mathbb{K}$ be the field of real or complex numbers $\mathbb{R}$, $\mathbb{C}$. We denote by $\mathbb{K}[\mathcal{S}_r]$ the *group ring* of a symmetric group $\mathcal{S}_r$. Furthermore we consider the $\mathbb{K}$-vector space $\mathcal{T}_r V$ of r-times covariant $\mathbb{K}$-valued tensors T over a finite dimensional $\mathbb{K}$-vector space V. Every group ring element $a = \sum_{p \in \mathcal{S}_r} a(p)\, p \in \mathbb{K}[\mathcal{S}_r]$ acts as so-called *symmetry operator* on tensors $T \in \mathcal{T}_r V$ according to the definition

$$(aT)(v_1, \ldots, v_r) \;:=\; \sum_{p \in \mathcal{S}_r} a(p)\, T(v_{p(1)}, \ldots, v_{p(r)}) \quad , \quad v_i \in V \,. \tag{2.1}$$

Equation (2.1) is equivalent to

$$(aT)_{i_1 \ldots i_r} \;=\; \sum_{p \in \mathcal{S}_r} a(p)\, T_{i_{p(1)} \ldots i_{p(r)}} \,. \tag{2.2}$$

Definition 2.1. Let $\mathfrak{r} \subseteq \mathbb{K}[\mathcal{S}_r]$ be a right ideal of $\mathbb{K}[\mathcal{S}_r]$ for which an $a \in \mathfrak{r}$ and a $T \in \mathcal{T}_r V$ exist such that $aT \neq 0$. Then the tensor set

$$\mathcal{T}_{\mathfrak{r}} \;:=\; \{ aT \mid a \in \mathfrak{r} \,,\; T \in \mathcal{T}_r V \} \tag{2.3}$$

is called the *symmetry class* of tensors defined by $\mathfrak{r}$. If $\mathfrak{r}$ is a minimal right ideal, then $\mathcal{T}_{\mathfrak{r}}$ is called *irreducible.*

Since $\mathbb{K}[\mathcal{S}_r]$ is semisimple for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, every right ideal $\mathfrak{r} \subseteq \mathbb{K}[\mathcal{S}_r]$ possesses a generating idempotent e, i.e. $\mathfrak{r}$ fulfils $\mathfrak{r} = e \cdot \mathbb{K}[\mathcal{S}_r]$.

Lemma 2.2. [3] *If e is a generating idempotent of $\mathfrak{r}$, then a tensor $T \in \mathcal{T}_r V$ belongs to $\mathcal{T}_{\mathfrak{r}}$ iff*

$$eT \;=\; T \,. \tag{2.4}$$

[3] See H. Boerner [1, p.127] or B. Fiedler [11], [8, p.110].

Thus we have

$$\mathcal{T}_{\mathfrak{r}} \;=\; \{eT \mid T \in \mathcal{T}_r V\}\,. \tag{2.5}$$

Symmetry classes can be characterized by left ideals of $\mathbb{K}[\mathcal{S}_r]$, too. To see this, we construct group ring elements from tensors.

Definition 2.3. Every tensor $T \in \mathcal{T}_r V$ and every r-tuple $b = (v_1, \ldots, v_r) \in V^r$ of vectors from V induce a group ring element

$$T_b \;:=\; \sum_{p \in \mathcal{S}_r} T(v_{p(1)}, \ldots, v_{p(r)})\, p \quad \in \mathbb{K}[\mathcal{S}r]\,. \tag{2.6}$$

A connection between (2.1) and (2.6) is given by the formula[4]

$$\forall\, T \in \mathcal{T}_r V \;,\; \forall\, a \in \mathbb{K}[\mathcal{S}_r] \;,\; \forall\, b \in V^r \;:\quad (aT)_b \;=\; T_b \cdot a^* \tag{2.7}$$

where the star '$*$' denotes the mapping

$$* : a \;=\; \sum_{p \in \mathcal{S}_r} a(p)\, p \;\;\mapsto\;\; a^* \;:=\; \sum_{p \in \mathcal{S}_r} a(p)\, p^{-1}\,. \tag{2.8}$$

Now, if a tensor T belongs to a certain symmetry class, then its T_b lie in a certain left ideal.

Proposition 2.4. [5] *Let $e \in \mathbb{K}[\mathcal{S}_r]$ be an idempotent. Then a $T \in \mathcal{T}_r V$ fulfils the condition $eT = T$ iff $T_b \in \mathfrak{l} := \mathbb{K}[\mathcal{S}_r] \cdot e^*$ for all $b \in V^r$, i.e. all T_b of T lie in the left ideal $\mathfrak{l}$ generated by e^*. Moreover, if $\dim V \geq r$ then $\mathfrak{l}$ is spanned by all group ring elements T_b of tensors $T \in \mathcal{T}_{\mathfrak{l}^*}$, i.e. $\mathfrak{l} = \mathcal{L}\{T_b \mid T \in \mathcal{T}_{\mathfrak{l}^*}\,,\, b \in V^r\}$ where $\mathcal{L}$ denotes the linear closure.*

Thus we can use the left ideal $\mathfrak{l} = \mathfrak{r}^*$ instead of the right ideal $\mathfrak{r}$ to characterize the symmetry class of $\mathfrak{r}$.

3. Symmetry classes of the first covariant derivatives of tensor fields

Now we determine results about the symmetry classes of the first covariant derivatives of tensor fields. In particular, we are interested in symmetric or alternating covariant tensor fields.

We consider only differentiable objects of class C^∞. Let M be an m-dimensional *differentiable manifold* equipped with a *linear connection* or *covariant derivative* ∇. We denote by $\mathcal{T}_r M$, $r \geq 0$, the set of *differentiable covariant tensor fields* of order r on M. If $T \in \mathcal{T}_r M$, $r \geq 1$, then its *covariant derivative* ∇T has a coordinate

[4] See B. Fiedler [11] or [8, p.110].

[5] See B. Fiedler [8, Sec.III.3.1] and [5, 6, 10].

representation[6]

$$\nabla_{i_{r+1}} T_{i_1\ldots i_r} \;=\; \partial_{i_{r+1}} T_{i_1\ldots i_r} - \sum_{k=1}^{r} \Gamma^{s_k}_{i_{r+1} i_k}\, T_{i_1\ldots i_{k-1} s_k i_{k+1}\ldots i_r} \tag{3.1}$$

where ∂ is a partial derivative of the coordinates of T and Γ^k_{ij} are the connection coefficients of ∇. Instead of (3.1) we write also

$$T_{i_1\ldots i_r\,;\,i_{r+1}} \;=\; T_{i_1\ldots i_r\,,\,i_{r+1}} - \sum_{k=1}^{r} \Gamma^{s_k}_{i_{r+1} i_k}\, T_{i_1\ldots i_{k-1} s_k i_{k+1}\ldots i_r}\,. \tag{3.2}$$

Every tensor in a fixed point p of M can be gained as covariant derivative of a suitable tensor field.

Lemma 3.1. *Let M_p be the tangent space of M in a point $p \in M$ of M and $W \in \mathcal{T}_{r+1} M_p$, $r \geq 0$, be a covariant tensor of order $r+1$ over M_p. Then we can find a covariant tensor field $T \in \mathcal{T}_r M$ such that $(\nabla T)|_p = W$.*

Proof. First we consider the case $r \geq 1$. In a suitable open neighbourhood U of p we can choose a chart x such that $x(p) = 0$. If $W_{i_1\ldots i_{r+1}}$ are the coordinates of W with respect to x and x^i are the coordinate functions of x then $\tilde{T}_{i_1\ldots i_r} := W_{i_1\ldots i_r k} x^k$ yields a differentiable tensor field on U. Further we can consider a function ϕ of class C^∞ on U for which open neighbourhoods U_1 and U_2 of p exist such that $p \in U_1 \subset U_2 \subset U$ and $\phi|_{U_1} \equiv 1$, $\phi|_{U \setminus U_2} \equiv 0$. By means of ϕ we obtain a differentiable tensor field $T \in \mathcal{T}_r M$ if we set $T|_U := \phi \tilde{T}$ and $T|_{M\setminus U} := 0$. But T fulfils $(\nabla T)|_p = W$ since we can write

$$T_{i_1\ldots i_r\,;\,i_{r+1}}(p) \;=\; T_{i_1\ldots i_r\,,\,i_{r+1}}(p) - \sum_{k=1}^{r} \Gamma^{s_k}_{i_{r+1} i_k}(p)\, T_{i_1\ldots i_{k-1} s_k i_{k+1}\ldots i_r}(p) \;=\; W_{i_i\ldots i_r i_{r+1}}\,.$$

The last equality follows from $T_{i_1\ldots i_r}(p) = 0$ and $T_{i_1\ldots i_r\,,\,i_{r+1}}(p) = W_{i_i\ldots i_r i_{r+1}}$.

In the case $r = 0$ the tensor W has order 1 and $\tilde{T} = W_k x^k$ is a tensor field of order 0, i.e. a differentiable function. Obviously we can form the tensor field (function) $T \in \mathcal{T}_0 M$ in the same way as in the case $r \geq 1$ and we obtain $T_{;i} = T_{,i} = W_i$ on the neighbourhood U_1 of p. □

Lemma 3.1 leads to the consequence that every symmetry class can be generated by covariant derivatives of suitable tensor fields.

Corollary 3.2. *Let $\mathfrak{r} \subseteq \mathbb{K}[\mathcal{S}_{r+1}]$, $r \geq 0$, be a right ideal with generating idempotent $e \in \mathbb{K}[\mathcal{S}_{r+1}]$ for which an $a \in \mathfrak{r}$ and a $W \in \mathcal{T}_{r+1} M_p$ exist such that $aW \neq 0$. Then the symmetry class $\mathcal{T}_\mathfrak{r}$ of tensors of order $r+1$ over M_p fulfils*

$$\mathcal{T}_\mathfrak{r} \;=\; \{ e(\nabla T)|_p \mid T \in \mathcal{T}_r M \}\,. \tag{3.3}$$

[6] We use the Einstein summation convention, i.e. a symbol such as $T_{i_1\ldots s\ldots}{}^{s}{}_{\ldots i_r}$ means $T_{i_1\ldots s\ldots}{}^{s}{}_{\ldots i_r} := \sum_{s=1}^{m} T_{i_1\ldots s\ldots}{}^{s}{}_{\ldots i_r}$.

Proof. According to (2.5) we have $\mathcal{T}_{\mathfrak{r}} = \{eW \mid W \in \mathcal{T}_{r+1}M_p\}$. But for every $W \in \mathcal{T}_{r+1}M_p$ there exists a $T \in \mathcal{T}_r M$ such that $W = (\nabla T)|_p$. Thus Corollary 3.2 follows. □

The *symmetrization* '$_{(\ldots)}$' and *anti-symmetrization* '$_{[\ldots]}$' of a tensor field T of order r is defined by

$$T_{(i_1 \ldots i_r)} := \frac{1}{r!} \sum_{p \in \mathcal{S}_r} T_{i_{p(1)} \ldots i_{p(r)}} \tag{3.4}$$

$$T_{[i_1 \ldots i_r]} := \frac{1}{r!} \sum_{p \in \mathcal{S}_r} \mathrm{sign}(p)\, T_{i_{p(1)} \ldots i_{p(r)}}\,. \tag{3.5}$$

From now on we consider only covariant derivatives ∇ which are *torsion-free*, i.e. $\Gamma^k_{[ij]} = 0$. It is well-known for such ∇ that the operations '$_{(\ldots)}$' or '$_{[\ldots]}$' and the operator ∇ are permutable. However, this statement is correct for arbitrary symmetry operators $a \in \mathbb{K}[\mathcal{S}_r]$, too.

Lemma 3.3. *Let ∇ be torsion-free and $a = \sum_{p \in \mathcal{S}_r} a(p)\, p \in \mathbb{K}[\mathcal{S}_r]$, $r \geq 2$. If we consider a $T \in \mathcal{T}_r M$ and set $H_{i_1 \ldots i_r} := (aT)_{i_1 \ldots i_r}$ and $W_{i_1 \ldots i_r i_{r+1}} := T_{i_1 \ldots i_r\,;\,i_{r+1}}$ with respect to arbitrary local coordinates, then it holds*

$$\sum_{p \in \mathcal{S}_r} a(p)\, W_{i_{p(1)} \ldots i_{p(r)}\, i_{r+1}} = H_{i_1 \ldots i_r\,;\,i_{r+1}}\,. \tag{3.6}$$

Proof. Let $q \in M$ be an arbitrary point of M. We can choose such coordinates arround q that all Γ^k_{ij} vanish in q, i.e. $\Gamma^k_{ij}(q) = 0$. Then we have $W_{i_1 \ldots i_r i_{r+1}}(q) = T_{i_1 \ldots i_r\,,\,i_{r+1}}(q)$ and $H_{i_1 \ldots i_r\,;\,i_{r+1}}(q) = H_{i_1 \ldots i_r\,,\,i_{r+1}}(q)$. But since obviously

$$\sum_{p \in \mathcal{S}_r} a(p)\, T_{i_{p(1)} \ldots i_{p(r)}\,,\,i_{r+1}}(q) = H_{i_1 \ldots i_r\,,\,i_{r+1}}(q)\,,$$

we obtain Lemma 3.3. □

The version of Lemma 3.3 for the symmetry operators '$_{(\ldots)}$' and '$_{[\ldots]}$' reads

Lemma 3.4. *Let ∇ be torsion-free and $r \geq 2$. If we set $S_{i_1 \ldots i_r} := T_{(i_1 \ldots i_r)}$, $A_{i_1 \ldots i_r} := T_{[i_1 \ldots i_r]}$ and $W_{i_1 \ldots i_r i_{r+1}} := T_{i_1 \ldots i_r\,;\,i_{r+1}}$, then it holds*

$$W_{(i_1 \ldots i_r)\, i_{r+1}} = S_{i_1 \ldots i_r\,;\,i_{r+1}} \tag{3.7}$$

$$W_{[i_1 \ldots i_r]\, i_{r+1}} = A_{i_1 \ldots i_r\,;\,i_{r+1}}\,. \tag{3.8}$$

Proposition 3.5. *Consider the case $r \geq 2$. Let $\tilde{\mathcal{S}}_r := \{p \in \mathcal{S}_{r+1} \mid p(r+1) = r+1\}$ be the subgroup of those permutations of $\mathcal{S}_{r+1}$ which have $r+1$ as fixed point. Then*

$$e_s := \frac{1}{r!} \sum_{p \in \tilde{\mathcal{S}}_r} p \quad , \quad e_a := \frac{1}{r!} \sum_{p \in \tilde{\mathcal{S}}_r} \mathrm{sign}(p)\, p \tag{3.9}$$

are idempotents of the group ring $\mathbb{K}[\mathcal{S}_{r+1}]$ *which fulfil*

$$e_s^* \;=\; e_s \quad , \quad e_a^* \;=\; e_a \,. \tag{3.10}$$

If ∇ *is a torsion-free covariant derivative and* S, A *are symmetric or alternating differentiable tensor fields of order* r*, respectively, then it holds*[7]

$$e_s^*\nabla S \;=\; \nabla S \quad , \quad e_a^*\nabla A \;=\; \nabla A \,. \tag{3.11}$$

Proof. Taking into account Lemma 3.4 and $\mathrm{sign}(p\cdot q) = \mathrm{sign}(p)\mathrm{sign}(q)$, $\mathrm{sign}(p^{-1}) = \mathrm{sign}(p)$, we can prove Proposition 3.5 by simple calculations. □

Now we will show that the symmetry classes of covariant derivatives of tensor fields are characterized by *Littlewood-Richardson products*. Let us denote by $\breve{\omega}$ the *regular representation* of the symmetric group $\mathcal{S}_{r+1}$:

$$\breve{\omega} : \mathcal{S}_{r+1} \to \mathrm{Gl}(\mathbb{K}[\mathcal{S}_{r+1}]) \quad , \quad \breve{\omega}_p(a) \;=\; p\cdot a \;,\; p\in\mathcal{S}_{r+1} \;,\; a\in\mathbb{K}[\mathcal{S}_{r+1}] \,. \tag{3.12}$$

The left ideals $\mathfrak{l} \subseteq \mathbb{K}[\mathcal{S}_{r+1}]$ of $\mathbb{K}[\mathcal{S}_{r+1}]$ can be considered representation spaces of subrepresentations $\rho = \breve{\omega}|_{\mathfrak{l}}$ of $\breve{\omega}$:

$$\rho : \mathcal{S}_{r+1} \to \mathrm{Gl}(\mathfrak{l}) \quad , \quad \rho_p(a) \;=\; p\cdot a \;,\; p\in\mathcal{S}_{r+1} \;,\; a\in\mathfrak{l} \,. \tag{3.13}$$

We see from a generalization of Proposition 3.5 that investigations of covariant derivatives of tensor fields can be based on following

Setting 3.6. Let $T\in\mathcal{T}_rM$ be a differentiable tensor field of order r on M that lies everywhere in a symmetry class defined by a left ideal $\mathbb{K}[\mathcal{S}_r]\cdot e$ with generating idempotent $e\in\mathbb{K}[\mathcal{S}_r]$, i.e. $e^*T = T$ on every tangent space M_p of M. We identify $\mathcal{S}_r$ with $\tilde{\mathcal{S}}_r$ by means of $[i_1,\ldots,i_r] \mapsto [i_1,\ldots,i_r,r+1]$ (where $[i_1,\ldots,i_r]$ is the list representation of a permutation) and denote by $\tilde{e}\in\mathbb{K}[\tilde{\mathcal{S}}_r]$ the corresponding embedding of $e\in\mathbb{K}[\mathcal{S}_r]$ into $\mathbb{K}[\mathcal{S}_{r+1}]$. If ∇ is a torsion-free covariant derivative on M then the left ideal $\mathfrak{l} := \mathbb{K}[\mathcal{S}_{r+1}]\cdot\tilde{e}$ defines symmetry classes in the $\mathcal{T}_{r+1}M_p$ that contain ∇T in every point of M, i.e. $\tilde{e}^*\nabla T = \nabla T$ on every tangent space M_p. Furthermore, we can consider the left ideal $\tilde{\mathfrak{l}} := \mathbb{K}[\tilde{\mathcal{S}}_r]\cdot\tilde{e}$ of $\mathbb{K}[\tilde{\mathcal{S}}_r]$. The left ideals $\mathfrak{l}$ and $\tilde{\mathfrak{l}}$ are the representation spaces of the representations[8]

$$\mathfrak{l} \iff \rho \;:=\; \breve{\omega}|_{\mathfrak{l}} \tag{3.14}$$

$$\tilde{\mathfrak{l}} \iff \sigma \;:=\; (\breve{\omega}\downarrow\tilde{\mathcal{S}}_r)|_{\tilde{\mathfrak{l}}} \,. \tag{3.15}$$

A simple consequence of Corollary 3.2, Lemma 3.3 and Setting 3.6 is that the symmetry class of the above ∇T is generated by the covariant derivatives of the symmetizations e^*W of arbitrary tensor fields $W\in\mathcal{T}_rM$.

[7] Here we assume that the tensor indices are numbered in the manner of (3.2).

[8] If α is a representation of a group G and $H\subseteq G$ is a subgroup of G, then $\alpha\downarrow H$ denotes the restriction of α to H.

Proposition 3.7. *Assume that Setting* 3.6 *and* $r \geq 1$ *are valid and* $p \in M$. *Then we have*

$$\mathcal{T}_{\mathfrak{l}^*} \;=\; \{\tilde{e}^*(\nabla W)|_p \mid W \in \mathcal{T}_r M\} \;=\; \{(\nabla e^* W)|_p \mid W \in \mathcal{T}_r M\}\,.$$

Now we determine Littlewood-Richardson products describing covariant derivatives.

Theorem 3.8. *Assume that Setting* 3.6 *and* $r \geq 1$ *are valid. Then the representation* ρ *is a Littlewood-Richardson product*

$$\rho \;\sim\; \sigma\,[1]\,. \tag{3.16}$$

If σ *is an irreducible representation* $\sigma \sim [\lambda]$, $\lambda \vdash r$, *then the Littlewood-Richardson rule*[9] *yields a multiplicity-free decomposition*

$$\rho \;\sim\; [\lambda][1] \;\sim\; \sum_{\substack{\mu \vdash r+1 \\ \lambda \subset \mu}} [\mu]\,, \tag{3.17}$$

i.e. $[\lambda][1]$ *decomposes into a sum of all Young frames* $[\mu]$ *which can be formed from* $[\lambda]$ *by adding one box to* $[\lambda]$.

Proof. When we introduce the notation $\tilde{\mathcal{S}}_1 := \{\mathrm{id}\}$ for the trivial subgroup of $\mathcal{S}_{r+1}$, then the set product $\tilde{\mathcal{S}}_r = \tilde{\mathcal{S}}_r \cdot \tilde{\mathcal{S}}_1$ is a direct product $\tilde{\mathcal{S}}_r = \tilde{\mathcal{S}}_r \times \tilde{\mathcal{S}}_1$. (It is even a Young subgroup.) We consider the representations

$$\iota : \tilde{\mathcal{S}}_1 \to \mathrm{Gl}(\mathbb{K}[\tilde{\mathcal{S}}_1]) \quad , \quad \iota_{\mathrm{id}}(u) = u \;\;,\;\; u \in \mathbb{K}[\tilde{\mathcal{S}}_1] \tag{3.18}$$

$$\sigma : \tilde{\mathcal{S}}_r \to \mathrm{Gl}(\mathbb{K}[\tilde{\mathcal{S}}_r] \cdot \tilde{e}) \quad , \quad \sigma_p(v) = p \cdot v \;\;,\;\; p \in \tilde{\mathcal{S}}_r \;,\; v \in \mathbb{K}[\tilde{\mathcal{S}}_r] \cdot \tilde{e}\,. \tag{3.19}$$

Obviously we can regard σ as an outer tensor product of representations

$$\sigma \;=\; \sigma \,\#\, \iota\,.$$

Thus the left ideal $\mathfrak{l} = \mathbb{K}[\mathcal{S}_{r+1}] \cdot \tilde{e}$ is the representation space of the induced representation $(\sigma \,\#\, \iota) \uparrow \mathcal{S}_{r+1}$ which has the structure of a Littlewood-Richardson product, i.e.

$$\rho \;=\; (\sigma \,\#\, \iota) \uparrow \mathcal{S}_{r+1} \;\sim\; \sigma[1]\,.$$

If σ is irreducible, i.e. $\sigma \sim [\lambda]$, $\lambda \vdash r$, then the Littlewood-Richardson rule yields (3.17) (see also Figure 1). □

Remark 3.9. If we restrict us to irreducible representations σ, then the proof of Theorem 3.8 is a repetition of a part of the proof of the *branching theorem* for irreducible representations of symmetric groups (see A. Kerber [19, Vol.240/p.85]).

[9]See D. E. Littlewood [22, pp.94-96], A. Kerber [19, Vol.240/p.84], G. D. James and A. Kerber [18, p.93], A. Kerber [20, Sec.5.5], I. G. Macdonald [23, Chap.I,Sec.9], R. Merris [24, p.100], W. Fulton and J. Harris [16, pp.455-456], S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins [15]. See also B. Fiedler [8, Sec.II.5].

Remark 3.10. If σ is a reducible representation and we know a decomposition $\sigma = \bigoplus_i \sigma_i$ into subrepresentations σ_i (irreducible or reducible), then we can use the formula

$$\sigma[1] \;\sim\; \sum_i \sigma_i[1] \tag{3.20}$$

to determine the structure of the decomposition of $\sigma[1]$ into irreducible subrepresentations.

Remark 3.11. Additional information about a tensor field considered can lead to a further reduction of the sum (3.17). For instance it is well-known that the symmetry classes of the *Riemannian curvature tensor* R and its covariant derivative[10] ∇R are defined by the *Young symmetrizers*[11] y_t and $y_{t'}$ of the Young tableaux[12]

$$t \;=\; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \quad , \quad t' \;=\; \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \quad . \tag{3.21}$$

However, if we apply (3.17) to the tableau t we obtain $[2^2][1] \sim [3,2]+[2^2,1] \not\sim [3,2]$. The difference results from the fact that ∇R fulfils the second Bianchi identity

$$R_{ijkl\,;m} \;+\; R_{ijlm\,;k} \;+\; R_{ijmk\,;l} \;=\; 0$$

which is not satisfied by other tensor fields from $\mathcal{T}_4 M$ in general.

A second example which shows such effects is the case of higher covariant derivatives of tensor fields. If we apply Theorem 3.8 to covariant derivatives[13] of second order $T_{i_1\ldots i_r\,;\,i_{r+1}i_{r+2}}$ of a tensor field $T \in \mathcal{T}_r M$ then Theorem 3.8 yields a result in which the so-called *Ricci identity*

$$T_{i_1\ldots i_r\,;\,[i_{r+1}i_{r+2}]} \;=\; \frac{1}{2}\sum_{k=1}^{r} R_{i_{r+1}i_{r+2}i_k}{}^{s_k}\, T_{i_1\ldots i_{k-1}s_k i_{k+1}\ldots i_r} \tag{3.22}$$

was left out of account. Thus the set of Young frames determined by multiple application of (3.17) will be "too large". (3.17) produces a set of Young frames which is correct also for covariant derivatives $T_{i_1\ldots i_r i_{r+1}\,;\,i_{r+2}}$ of tensor fields $T \in \mathcal{T}_{r+1}M$ of order $r+1$ to which an identity (3.22) is irrelevant.

Now we present a version of Theorem 3.8 for the special case of symmetric or alternating tensor fields.

Theorem 3.12. *Assume that $r \geq 2$. Let $\mathfrak{l}_s := \mathbb{K}[\mathcal{S}_{r+1}]\cdot e_s$, $\mathfrak{l}_a := \mathbb{K}[\mathcal{S}_{r+1}]\cdot e_a$ be the left ideals generated by the idempotents e_s, e_a. Then the subrepresentations $\rho_s := \breve{\omega}|_{\mathfrak{l}_s}$, $\rho_a := \breve{\omega}|_{\mathfrak{l}_a}$ are Littlewood-Richardson products $\rho_s \sim [r][1]$, $\rho_a \sim [1^r][1]$ for which the Littlewood-Richardson rule yields decompositions*

$$[r][1] \;\sim\; [r+1]+[r,1] \quad , \quad [1^r][1] \;\sim\; [1^{r+1}]+[2,1^{r-1}]\,. \tag{3.23}$$

[10] Here we assume that ∇ is the Levi-Civita connection of a pseudi-Riemannian fundamental tensor $g \in \mathcal{T}_2 M$ and R is the curvature tensor of ∇.

[11] See Section 4 for some details about Young symmetrizers.

[12] See S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins [15]. See also B. Fiedler [7].

[13] Again we assume that ∇ is a Levi-Civita connection.

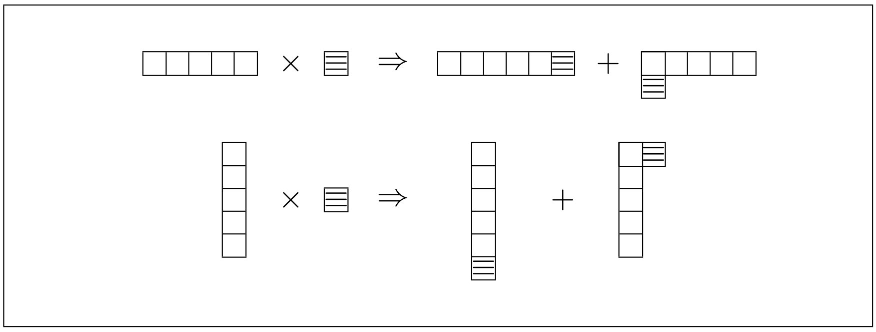

FIGURE 1. The Littlewood-Richardson products $[r][1]$ and $[1^r][1]$.

The idempotents e_s, e_a *have unique decompositions corresponding to* (3.23) *into primitive orthogonal idempotents*

$$e_s \;=\; f_s + h_s \quad , \quad e_a \;=\; f_a + h_a \tag{3.24}$$

which generate the minimal left ideals that are the representation spaces of the irreducible representations in (3.23)*:*

$$\begin{array}{rcl} \mathbb{K}[\mathcal{S}_{r+1}]\cdot f_s & \Longleftrightarrow & [r+1] \\ \mathbb{K}[\mathcal{S}_{r+1}]\cdot h_s & \Longleftrightarrow & [r,1] \\ \mathbb{K}[\mathcal{S}_{r+1}]\cdot f_a & \Longleftrightarrow & [1^{r+1}] \\ \mathbb{K}[\mathcal{S}_{r+1}]\cdot h_a & \Longleftrightarrow & [2,1^{r-1}]\,. \end{array}$$

We know all idempotents in (3.24) *since*

$$f_s \;=\; \frac{1}{(r+1)!}\sum_{p\in\mathcal{S}_{r+1}} p \quad , \quad f_a \;=\; \frac{1}{(r+1)!}\sum_{p\in\mathcal{S}_{r+1}} \mathrm{sign}(p)\,p\,. \tag{3.25}$$

Proof. If we apply Theorem 3.8 to the representations

$$\sigma_s : \tilde{\mathcal{S}}_r \to \mathrm{Gl}(\mathbb{K}[\tilde{\mathcal{S}}_r]\cdot e_s) \quad , \quad (\sigma_s)_p(v) = p\cdot v \;\; , \;\; p\in\tilde{\mathcal{S}}_r \; , \; v\in\mathbb{K}[\tilde{\mathcal{S}}_r]\cdot e_s \tag{3.26}$$

$$\sigma_a : \tilde{\mathcal{S}}_r \to \mathrm{Gl}(\mathbb{K}[\tilde{\mathcal{S}}_r]\cdot e_a) \quad , \quad (\sigma_a)_p(v) = p\cdot v \;\; , \;\; p\in\tilde{\mathcal{S}}_r \; , \; v\in\mathbb{K}[\tilde{\mathcal{S}}_r]\cdot e_a \tag{3.27}$$

then we obtain (3.23) since σ_s and σ_a are irreducible and fulfil $\sigma_s \sim [r]$, $\sigma_a \sim [1^r]$. (See Figure 1 for the use of the *Littlewood-Richardson rule*.)

The relations (3.23) tell us that every of the representations $(\sigma_s \,\#\, \iota) \uparrow \mathcal{S}_{r+1}$, $(\sigma_a \,\#\, \iota) \uparrow \mathcal{S}_{r+1}$ decomposes into two irreducible representations which have multiplicities 1. Thus the generating idempotents e_s, e_a of the representation spaces possess unique corresponding decompositions (3.24) into primitive othogonal idempotents. Since we know that the idempotents for representations $[r+1]$ and $[1^{r+1}]$ are given by (3.25) we can calculate the remaining idempotents h_s and h_a, too. □

Remark 3.13. In the case of an alternating tensor field $A \in \mathcal{T}_r M$ the symmetry operator f_a transforms ∇A into the exterior derivative dA of A, i.e. $f_a^*(\nabla A) =$

$f_a(\nabla A) = dA$. Thus the symmetry operator h_a yields the difference of ∇A and dA, i.e. $h_a^*(\nabla A) = \nabla A - dA$.

Now let us consider the more general case of an arbitrary tensor field $T \in \mathcal{T}_r M$ whose symmetry class is defined by a known primitive idempotent $e \in \mathbb{K}[\mathcal{S}_r]$. Also in this case, there is a simple possibility to calculate all primitive idempotents which belong to a decomposition (3.17) for the covariant derivatives of T. A starting point is the well-known

Lemma 3.14. [14] *Let $\lambda \vdash r$ be a partition of $r \geq 1$ and χ_λ be the irreducible character of $\mathcal{S}_r$ which belongs to that equivalence class of irreducible representations of $\mathcal{S}_r$ which contains the irreducible representations*[15] $\breve{\omega}|_{\mathbb{K}[\mathcal{S}_r]\cdot y_t}$ *defined by Young symmetrizers y_t with Young frame λ. Then the group ring element*

$$e_\lambda \;:=\; \frac{\chi_\lambda(id)}{r!} \sum_{p \in \mathcal{S}_r} \chi_\lambda(p)\, p^{-1} \tag{3.28}$$

is the unique centrally primitive idempotent that generates the minimal two-sided ideal $\mathfrak{a}_\lambda := \mathbb{K}[\mathcal{S}_r] \cdot e_\lambda$ from the isotypic decomposition $\mathbb{K}[\mathcal{S}_r] = \bigoplus_{\mu \vdash r} \mathfrak{a}_\mu$ of the group ring $\mathbb{K}[\mathcal{S}_r]$ into minimal two-sided ideals $\mathfrak{a}_\mu$. $\mathfrak{a}_\lambda$ is that minimal two-sided ideal which contains all left ideals $\mathbb{K}[\mathcal{S}_r] \cdot y_t$ generated by Young symmetrizers y_t with Young frame λ. The idempotents e_λ fulfil

$$\sum_{\lambda \vdash r} e_\lambda \;=\; \mathrm{id} \qquad ; \qquad e_\lambda \cdot e_{\lambda'} \;=\; 0 \quad \text{if} \quad \lambda \neq \lambda' \,. \tag{3.29}$$

Theorem 3.15. *Assume, that the symmetry class of a tensor field $T \in \mathcal{T}_r M$, $r \geq 1$, is defined by a primitive idempotent $e \in \mathbb{K}[\mathcal{S}_r]$ whose representation σ according to Setting 3.6 satisfies $\sigma \sim [\lambda]$, $\lambda \vdash r$. Then*

$$\tilde{e} \;=\; \sum_{\substack{\mu \vdash r+1 \\ \lambda \subset \mu}} h_\mu \qquad , \qquad h_\mu \;:=\; \tilde{e} \cdot e_\mu \tag{3.30}$$

yields the decomposition of $\tilde{e}$ into primitive idempotents h_μ, which corresponds to relation (3.17)*.*

Proof. Because (3.17) is multiplicity-free, a decomposition of $\tilde{e}$ according to (3.17) into primitive idempotents contains exactly one primitive idempotent h_μ for every $[\mu]$ in (3.17). Every such h_μ lies in the corresponding two-sided ideal $\mathfrak{a}_\mu$, i.e. $h_\mu \in \mathfrak{a}_\mu$. On the other hand, we can write $\tilde{e} = \tilde{e} \cdot \mathrm{id} = \sum_{\mu \vdash r+1} \tilde{e} \cdot e_\mu$. Since $\tilde{e} \cdot e_\mu \in \mathfrak{a}_\mu$, we obtain (3.30). □

If we carry out calculations in large $\mathcal{S}_r$, then a use of formula (3.30) leads to very high costs in calculation time and computer memory. However, fast discrete Fourier transforms can help to solve this problem.

[14] See H. Boerner [1, Sec.III.3, III.4] and R. Merris [24, Sec.4], in particular Exercise 41 in [24, p.117]. See also B. Fiedler [8, Prop.II.1.47, Prop.I.1.8].

[15] Here $\breve{\omega}$ denotes the regular representation of $\mathcal{S}_r$.

Definition 3.16. A *discrete Fourier transform* for $\mathcal{S}_r$ is an isomorphism

$$D : \mathbb{K}[\mathcal{S}_r] \;\rightarrow\; \bigotimes_{\lambda\vdash r} \mathbb{K}^{d_\lambda\times d_\lambda} \tag{3.31}$$

according to Wedderburn's theorem which maps the group ring $\mathbb{K}[\mathcal{S}_r]$ onto an outer direct product $\bigotimes_{\lambda\vdash r} \mathbb{K}^{d_\lambda\times d_\lambda}$ of full matrix rings $\mathbb{K}^{d_\lambda\times d_\lambda}$. We denote by D_λ the *natural projections* $D_\lambda : \mathbb{K}[\mathcal{S}_r] \rightarrow \mathbb{K}^{d_\lambda\times d_\lambda}$.

Since the subrings $(0,\ldots,0,\mathbb{K}^{d_\lambda\times d_\lambda},0,\ldots,0)$ correspond to the two-sided ideals $\mathfrak{a}_\lambda$ under a discrete Fourier transform we obtain

Corollary 3.17. *If a discrete Fourier transform D is known for $\mathbb{K}[\mathcal{S}_{r+1}]$, then the idempotents h_μ in* (3.30) *can be calculated by*

$$h_\mu \;=\; D^{-1}\left((0,\ldots,D_\mu(\tilde{e}),\ldots,0)\right) . \tag{3.32}$$

A use of (3.32) in computer calculations is much more efficient than an application of (3.30). A very efficient algorithm of a fast Fourier transform for $\mathcal{S}_r$ was developped by M. Clausen and U. Baum (see [2, 3]). It is based on *Young's seminormal representation* of $\mathcal{S}_r$. Our Mathematica package PERMS [9] uses *Young's natural representation* of $\mathcal{S}_r$ as discrete Fourier transform.

4. Do Young symmetrizers describe the symmetry classes of ∇S or ∇A?

Now we turn to the question wheter the symmetry classes of ∇S and ∇A can be characterized by *Young symmetrizers.* According to Proposition 3.5 and Theorem 3.12 the symmetry classes of ∇S and ∇A are defined by the idempotents

$$e_s \;=\; f_s + h_s \quad , \quad e_a \;=\; f_a + h_a \,.$$

The idempotents f_s and f_a are proportional to the Young symmetrizers of the Young frames $(r+1) \vdash r+1$ or $(1^{r+1}) \vdash r+1$, respectively. Now we investigate the

Problem 4.1. Can we find Young tableaux t_s, t_a with frame $(r,1) \vdash r+1$ or $(2,1^{r-1}) \vdash r+1$, respectively, such that the idempotents $e_{t_s} = \mu_{t_s} y_{t_s}$, $e_{t_a} = \mu_{t_a} y_{t_a}$ are generating idempotents of the minimal left ideals $\mathfrak{l}'_s := \mathbb{K}[\mathcal{S}_{r+1}]\cdot h_s$, $\mathfrak{l}'_a := \mathbb{K}[\mathcal{S}_{r+1}]\cdot h_a$? Here y_{t_s} and y_{t_a} are the *Young symmetrizers*[16] of the Young tableaux t_s, t_a and $\mu_{t_s}, \mu_{t_a} \neq 0$ are constants.

If the answer is "yes", then the symmetry classes of ∇S or ∇A are determined by new idempotents

$$\tilde{e}_s \;=\; f_s + e_{t_s} \quad , \quad \tilde{e}_a \;=\; f_a + e_{t_a} \,, \tag{4.1}$$

which are completely built from Young symmetrizers.

[16] We define a Young symmetrizer y_t of a Young tableau t by the formula $y_t := \sum_{p\in\mathcal{H}_t}\sum_{q\in\mathcal{V}_t} \mathrm{sign}(q)\, p\cdot q$, where $\mathcal{H}_t$, $\mathcal{V}_t$ are the groups of horizontal or vertical permutations of t, respectively.

We investigated this problem by computer calculations by means of the Mathematica package PERMS [9] in the groups $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{S}_5$, i.e.in the cases $r = 2, 3, 4$. We obtained the following results:

- No Young symmetrizer idempotent e_t of a Young frame $(r,1) \vdash r+1$ reproduces or annihilates $h_s \in \mathbb{K}[\mathcal{S}_{r+1}]$, i.e. no relation $h_s \cdot e_t = h_s$ or $h_s \cdot e_t = 0$ is satisfied.
- For $h_a \in \mathbb{K}[\mathcal{S}_{r+1}]$ there are many Young symmetrizer idempotents e_t with Young frame $(2, 1^{r-1}) \vdash r+1$ which reproduce or annihilate h_a, i.e. which fulfil $h_a \cdot e_t = h_a$ or $h_a \cdot e_t = 0$. In particular, the idempotent e_t of the lexicographically greatest standard tableau

$$t \quad = \quad \begin{array}{|c|c|} \hline 1 & r+1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{4.2}$$

  reproduces h_a whereas the idempotents e_t of all other standard tableaux of $(2, 1^{r-1})$ annihilate h_a.

For the $\mathcal{S}_3$ and $r = 2$, we also verified these results by calculations by means of the packages PERMS [9] and Ricci [21] in which we checked the action of idempotents e_t^* onto tensors with a symmetry given by h_s or h_a. Mathematica notebooks of all above calculations can be downloaded from my internet page [4]

`http://home.t-online.de/home/Bernd.Fiedler.RoschStr.Leipzig/pnbks.htm`

We present tables of all Young tableaux, whose idempotents e_t reproduce or annihilate h_a, in the Appendix.

Now we present theorems which tell us that essential parts of the above computer results are valid for all $r \geq 2$.

It is well-known that two idempotents $e, f \in \mathbb{K}[\mathcal{S}_r]$ generate the same left ideal iff $e \cdot f = e$ and $f \cdot e = f$. In the case of primitive idempotents e, f we have

Lemma 4.2. *If $e, f \in \mathbb{K}[\mathcal{S}_r]$ are primitive idempotents, then the equations $e \cdot f = e$ and $f \cdot e = f$ are equivalent.*

Proof. Assume that $e \cdot f = e$ is valid. Then e is an element of the left ideal $\mathfrak{l} = \mathbb{K}[\mathcal{S}_r] \cdot f$ and generates a non-vanishing subideal $\mathfrak{l}' = \mathbb{K}[\mathcal{S}_r] \cdot e$ of $\mathfrak{l}$. Since e and f are primitive, the left ideals $\mathfrak{l}$ and $\mathfrak{l}'$ are minimal. Thus we obtain $\mathfrak{l} = \mathfrak{l}'$. But then, it follows $f \cdot e = f$ because f belongs to the left ideal $\mathfrak{l}'$ generated by e. $\square$

Theorem 4.3. *Consider the idempotent h_a for an arbitrary $r \geq 2$. Then the Young symmetrizer idempotent e_t of the lexicographically greatest standard tableau (4.2) with Young frame $(2, 1^{r-1}) \vdash r+1$ reproduces h_a, i.e. $h_a \cdot e_t = h_a$. Furthermore the idempotents e_t of all other standard tableaux t with Young frame $(2, 1^{r-1}) \vdash r+1$ annihilate h_a, i.e. $h_a \cdot e_t = 0$.*

Proof. The groups $\mathcal{H}_t$ and $\mathcal{V}_t$ of horizontal/vertical permutations of the Young tableau t according to (4.2) fulfil $\mathcal{H}_t = \langle(1\,,\,r+1)\rangle$ and $\mathcal{V}_t = \tilde{\mathcal{S}}_r$. Thus we can write

$$e_t \;=\; \mu_t \sum_{p\in\mathcal{H}_t}\sum_{q\in\mathcal{V}_t} \mathrm{sign}(q)\, p\cdot q \;=\; \mu_t\cdot r!\,\{\mathrm{id}+(1\,,\,r+1)\}\cdot e_a\,. \tag{4.3}$$

It holds[17] $e_t\cdot f_a = 0$ since e_t and f_a are proportional to Young symmetrizers of the different Young frames $(2,1^{r-1})$ and (1^{r+1}). From this and (4.3) we obtain

$$e_t\cdot h_a \;=\; e_t\cdot(e_a - f_a) \;=\; e_t\cdot e_a \;=\; e_t\,.$$

Because e_t and h_a are primitive idempotents, Lemma 4.2 yields $h_a\cdot e_t = h_a$.

It is well-known[18]: If y_{t_1} and y_{t_2} are Young symmetrizers of two standard tableaux t_1, t_2 which possess the same Young frame, and t_1 is lexicographically smaller[19] than t_2, then they satisfy $y_{t_2}\cdot y_{t_1} = 0$. Because (4.3) is built from the lexicographically greatest standard tableau t of $(2,1^{r-1})$, we obtain

$$h_a\cdot e_{t'} \;=\; h_a\cdot e_t\cdot e_{t'} \;=\; 0$$

for the idempotent $e_{t'}$ of every other standard tableau of $(2,1^{r-1})$ □

Theorem 4.4. *Consider the idempotent h_s for an arbitrary $r\geq 2$. Then it holds $h_s\cdot e_t \neq h_s$ for all Young tableaux t with Young frame $(r,1)\vdash r+1$.*

Proof. A Young tableau t with a Young frame $(r,1)\vdash r+1$ has a form

$$t \;=\; \begin{array}{|c|c|c|c|c|}\hline k & \star & \star & \ldots & \star \\ \hline l \\ \cline{1-1}\end{array}\;. \tag{4.4}$$

If we assume that the first column of such a tableau contains the numbers k and l as in (4.4), then the groups of horizontal or vertical permutations of t read

$$\mathcal{H}_t \;=\; \{p\in\mathcal{S}_{r+1}\mid p(l)=l\} \;=:\; (\mathcal{S}_{r+1})_l \tag{4.5}$$

$$\mathcal{V}_t \;=\; \langle(k\,,\,l)\rangle\,. \tag{4.6}$$

First we consider the case, that the first column of (4.4) does not contain $r+1$. In this case we have

$$y_t \;=\; \Big(\sum_{p\in(\mathcal{S}_{r+1})_l} p\Big)\cdot\{\mathrm{id}-(k\,,\,l)\}\,.$$

But because $\mathcal{V}_t \subseteq \tilde{\mathcal{S}}_r$, we obtain $\{\mathrm{id}-(k\,,\,l)\}\cdot e_s = 0$ and $y_t\cdot e_s = 0$. Thus $e_t = \mu_t y_t$ does not lie in the left ideal $\mathfrak{l}_s := \mathbb{K}[\mathcal{S}_{r+1}]\cdot e_s$ and can not play the role of a generating idempotent of $\mathfrak{l}'_s := \mathbb{K}[\mathcal{S}_{r+1}]\cdot h_s$. Consequently $h_s\cdot e_t \neq h_s$.

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[17] See e.g. H. Boerner [1, p.98] or W. Müller [25, p.73]. See also B. Fiedler [8, Sec.I.3.1].

[18] See A. Kerber [19, Vol.240/p.73] or H. Boerner [1, p.101]. See also B. Fiedler [8, Sec.I.3.1].

[19] A tableau t_2 is regarded as greater than a tableau t_1 (of the same Young frame), if the simultaneous run through the rows of both tableaux from left to right and from top to bottom reaches earlier in t_2 a number which is greater than the number on the corresponding place in t_1.

Netxt we investigate the case $l = r+1$. In this case we have $\mathcal{H}_t = \tilde{\mathcal{S}}_r$, $\mathcal{V}_t = \langle (k\,,\, r+1)\rangle$,

$$(4.7)\quad e_t \;=\; \mu_t \Big(\sum_{p\in\tilde{\mathcal{S}}_r} p\Big)\cdot\{\mathrm{id} - (k\,,\, r+1)\} \;=\; \mu_t\cdot r!\, e_s\cdot\{\mathrm{id} - (k\,,\, r+1)\}\,,$$

which leads to

$$h_s\cdot e_t \;=\; (e_s - f_s)\cdot e_t \;=\; e_s\cdot e_t \;=\; e_t\,.$$

We decompose h_s and e_t into parts that correspond to the right cosets of $\mathcal{S}_{r+1}$ relative to $\tilde{\mathcal{S}}_r$. Obviously $\mathfrak{R} := \{(i\,,\, r+1)\mid i = 1,\ldots,r+1\}$ is a complete set of representatives of those right cosets. If we arrange the summands of f_s and h_s according to the decomposition of $\mathcal{S}_{r+1}$ into cosets we obtain

$$f_s \;=\; \frac{1}{(r+1)!}\sum_{s\in\mathfrak{R}}\Big(\sum_{p\in\tilde{\mathcal{S}}_r} p\Big)\cdot s$$

and

$$h_s \;=\; e_s - f_s \;=\; \frac{r}{(r+1)!}\Big(\sum_{p\in\tilde{\mathcal{S}}_r} p\Big) \;-\; \frac{1}{(r+1)!}\sum_{\substack{s\in\mathfrak{R}\\ s\neq\mathrm{id}}}\Big(\sum_{p\in\tilde{\mathcal{S}}_r} p\Big)\cdot s\,.$$

Thus h_s has a non-vanishing part in every right coset of $\mathcal{S}_{r+1}$ relative to $\tilde{\mathcal{S}}_r$. From (4.7) we see that e_t has non-vanishing parts only in the right cosets $\tilde{\mathcal{S}}_r\cdot(k\,,\, r+1)$ and $\tilde{\mathcal{S}}_r$. This leads to $h_s \neq e_t$ and $h_s\cdot e_t = e_t \neq h_s$.

Finally, we consider the case $k = r+1$. If t is the tableau (4.4) with $k = r+1$, then $t' := (l, r+1)\circ t$ is a Young tableau[20] (4.4) with $l = r+1$. A relation $t' = p\circ t$, $p\in\mathcal{S}_{r+1}$, between Young tableaux leads to

$$\mathcal{H}_{t'} = p\circ\mathcal{H}_t\circ p^{-1}\quad,\quad \mathcal{V}_{t'} = p\circ\mathcal{V}_t\circ p^{-1}\quad,\quad y_{t'} = p\cdot y_t\cdot p^{-1}\,.$$

For the above tableaux $t' := (l, r+1)\circ t$ we obtain $y_{t'} = -(l, r+1)\cdot y_t$ since $y_t\cdot(l, r+1)^{-1} = -y_t$.

Now, if we assume $h_s\cdot e_t = h_s$, then it follows from Lemma 4.2

$$e_t\cdot h_s = e_t \;\Rightarrow\; y_{t'}\cdot h_s = y_{t'} \;\Rightarrow\; e_{t'}\cdot h_s = e_{t'} \;\Rightarrow\; h_s\cdot e_{t'} = h_s\,.$$

However, the last equation is a contradiction to our proof in the case $l = r+1$. $\square$

[20]A Young tableau t of $\mathcal{S}_r$ can be regarded a one-to-one mapping of the boxes of the Young frame of t onto the set $\{1,\ldots,r\}$. t maps every box onto that number which was placed into the box. Then the composition $p\circ t$ of a Young tableau t and a permutation $p\in\mathcal{S}_r$ is a Young tableau again.

5. Use of ∇S and ∇A in generator formulas of algebraic covariant derivative curvature tensors

Now we return to the question whether tensors (1.10) can be used as generators U for algebraic covariant derivative curvature tensors in formulas (1.7). In [13] we proved

Theorem 5.1. *Let us denote by $S, A \in \mathcal{T}_2 V$ symmetric or alternating tensors of order 2 and by $U \in \mathcal{T}_3 V$ covariant tensors of order 3 whose symmetry class $\mathcal{T}_{\mathfrak{r}}$ is defined by a fixed minimal right ideal $\mathfrak{r}$ from the equivalence class characterized by $(2,1) \vdash 3$. We consider the following types τ of tensors*

$$\tau : \quad \begin{array}{ccc} y_{t'}^*(S \otimes U) & , & y_{t'}^*(U \otimes S) \quad , \\ y_{t'}^*(A \otimes U) & , & y_{t'}^*(U \otimes A) \quad , \end{array} \tag{5.1}$$

where $y_{t'} \in \mathbb{K}[\mathcal{S}_5]$ is the Young symmetrizer of the standard tableau

$$t' \;=\; \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2} \end{array} \quad .$$

Then for each of the above types τ the following assertions are equivalent:

1. *The vector space of algebraic covariant derivative curvature tensors $\mathfrak{R}' \in \mathcal{T}_5 V$ is the set of all finite sums of tensors of the type τ considered.*
2. *The right ideal $\mathfrak{r}$ is different from the right ideal $\mathfrak{r}_0 := f \cdot \mathbb{K}[\mathcal{S}_3]$ with generating idempotent*

$$f \;:=\; \left\{ \frac{1}{2}\,(\mathrm{id} - (1\,3)) - \frac{1}{6}\,y \right\} \quad , \quad y \;:=\; \sum_{p \in \mathcal{S}_3} \mathrm{sign}(p)\, p\,. \tag{5.2}$$

From Theorem 5.1 we obtain easily

Theorem 5.2. *Let ∇ be a torsion-free covariant derivative on the mannifold M and $p \in M$. Then Statement (1) of Theorem 5.1 holds for the algebraic covariant derivative curvature tensors $\mathfrak{R}' \in \mathcal{T}_5 M_p$ if we take the tensors U from one of the tensor sets*

$$U \;=\; h_s^*(\nabla S)|_p \;=\; \nabla S|_p - \mathrm{sym}(\nabla S)|_p \tag{5.3}$$

or

$$U \;=\; h_a^*(\nabla A)|_p \;=\; \nabla A|_p - \mathrm{alt}(\nabla A)|_p \;=\; \nabla A|_p - \mathrm{d}A|_p \tag{5.4}$$

formed from the whole of symmetric or alternating tensor fields $S, A \in \mathcal{T}_2 M$.

Proof. Let us denote by $\mathcal{T}_e$ the symmetry class $\mathcal{T}_{\mathfrak{r}}$ that is defined by a right ideal $\mathfrak{r} = e \cdot \mathbb{K}[\mathcal{S}_r]$ with generating idempotent $e \in \mathbb{K}[\mathcal{S}_r]$. Then Proposition 3.7 yields

$$\begin{aligned} \mathcal{T}_{e_s} &= \{(\nabla S)|_p \mid S \in \mathcal{T}_2 M \ \text{ symmetric}\} \\ \mathcal{T}_{e_a} &= \{(\nabla A)|_p \mid A \in \mathcal{T}_2 M \ \text{ alternating}\} \end{aligned}$$

from which we obtain[21]

$$\begin{array}{rcl} \mathcal{T}_{h_s} & = & \{h_s(\nabla S)|_p \mid S \in \mathcal{T}_2 M \ \text{ sym.}\} \ = \ \{\nabla S|_p \, - \, \text{sym}(\nabla S)|_p \mid S \in \mathcal{T}_2 M \ \text{ sym.}\} \\ \mathcal{T}_{h_a} & = & \{h_a(\nabla A)|_p \mid A \in \mathcal{T}_2 M \ \text{ alt.}\} \ = \ \{\nabla A|_p \, - \, \text{alt}(\nabla A)|_p \mid A \in \mathcal{T}_2 M \ \text{ alt.}\} \, . \end{array}$$

Now we have only to check that the idempotents

$$h_s \ = \ e_s \, - \, f_s \qquad , \qquad h_a \ = \ e_a \, - \, f_a$$

do not generate the right ideal $\mathfrak{r}_0$. We do this by verifying

$$f \cdot h_s \ \neq \ h_s \qquad , \qquad f \cdot h_a \ \neq \ h_a \, . \tag{5.5}$$

The fastes way would be a computer calculation by means of PERMS [9]. A calculation by hand has the starting point (3.11), (3.25), (5.2) and $z := \frac{1}{2}(\mathrm{id} - (1\,3))$. From the rules

- "symmetrization + alternation = 0"
- "alternation + alternation = alternation"

we obtain immediately

$$\begin{array}{lllll} y \cdot f_s \ = \ 0 & , & y \cdot e_s \ = \ 0 & , & z \cdot f_s \ = \ 0 \\ y \cdot f_a \ = \ y & , & y \cdot e_a \ = \ y & , & z \cdot f_a \ = \ f_a \end{array}$$

Furthermore we have the products

$$\begin{array}{rcl} z \cdot e_s & = & \frac{1}{4}\,\{[1,2,3] + [2,1,3] - [2,3,1] - [3,2,1]\} \\ z \cdot e_a & = & \frac{1}{4}\,\{[1,2,3] - [2,1,3] + [2,3,1] - [3,2,1]\} \, . \end{array}$$

This leads to

$$f \cdot h_s \ = \ z \cdot e_s \ = \ \frac{1}{4}\,\{[1,2,3] + [2,1,3] - [2,3,1] - [3,2,1]\}$$

and

$$\begin{array}{rcl} f \cdot h_a & = & z \cdot e_a - z \cdot f_a - \frac{1}{6}\, y \cdot e_a + \frac{1}{6}\, y \cdot f_a \\ & = & z \cdot e_a - f_a - \frac{1}{6}\, y + \frac{1}{6}\, y \\ & = & z \cdot e_a - f_a \, . \end{array}$$

But we see from these results that (5.5) is valid because

- h_s is a linear combination of 6 permutations and $z \cdot e_s$ has only 4 summands,
- $z \cdot e_a \neq e_a$. □

[21] Note that the idempotents $e_s, f_s, h_s, e_a, f_a, h_a$ fulfil $e_s^* = e_s$, $f_s^* = f_s$, $h_s^* = h_s$, $e_a^* = e_a$, $f_a^* = f_a$, $h_a^* = h_a$.

Appendix

Using the Mathematica package PERMS [9] for $r = 2, 3, 4$ we found Young symmetrizer idempotents $e_t = \mu_t y_t$ with Young frames $(2, 1^{r-1}) \vdash r+1$ which reproduce $(h_a \cdot e_t = h_a)$ or annihilate $(h_a \cdot e_t = 0)$ the idempotent $h_a \in \mathbb{K}[\mathcal{S}_{r+1}]$ of the symmetric group $\mathcal{S}_{r+1}$ considered. Here we present complete lists of the Young tableaux t for which the e_t possess such a property.

2 *tableaux for* $r = 2$ *such that* $h_a \cdot e_t = h_a$.

```
{1, 3}, {2, 3}
{2}     {1}
```

2 *tableaux for* $r = 2$ *such that* $h_a \cdot e_t = 0$.

```
{1, 2}, {2, 1}
{3}     {3}
```

6 *tableaux for* $r = 3$ *such that* $h_a \cdot e_t = h_a$.

```
{1, 4}, {1, 4}, {2, 4}, {2, 4}, {3, 4}, {3, 4}
{2}     {3}     {1}     {3}     {1}     {2}
{3}     {2}     {3}     {1}     {2}     {1}
```

12 *tableaux for* $r = 3$ *such that* $h_a \cdot e_t = 0$.

```
{1, 2}, {1, 2}, {1, 3}, {1, 3}, {2, 1}, {2, 1}, {2, 3}, {2, 3},
{3}     {4}     {2}     {4}     {3}     {4}     {1}     {4}
{4}     {3}     {4}     {2}     {4}     {3}     {4}     {1}

{3, 1}, {3, 1}, {3, 2}, {3, 2}
{2}     {4}     {1}     {4}
{4}     {2}     {4}     {1}
```

24 *tableaux for* $r = 4$ *such that* $h_a \cdot e_t = h_a$.

```
{1, 5}, {1, 5}, {1, 5}, {1, 5}, {1, 5}, {1, 5}, {2, 5}, {2, 5},
{2}     {2}     {3}     {3}     {4}     {4}     {1}     {1}
{3}     {4}     {2}     {4}     {2}     {3}     {3}     {4}
{4}     {3}     {4}     {2}     {3}     {2}     {4}     {3}

{2, 5}, {2, 5}, {2, 5}, {2, 5}, {3, 5}, {3, 5}, {3, 5}, {3, 5},
{3}     {3}     {4}     {4}     {1}     {1}     {2}     {2}
{1}     {4}     {1}     {3}     {2}     {4}     {1}     {4}
{4}     {1}     {3}     {1}     {4}     {2}     {4}     {1}

{3, 5}, {3, 5}, {4, 5}, {4, 5}, {4, 5}, {4, 5}, {4, 5}, {4, 5}
{4}     {4}     {1}     {1}     {2}     {2}     {3}     {3}
{1}     {2}     {2}     {3}     {1}     {3}     {1}     {2}
{2}     {1}     {3}     {2}     {3}     {1}     {2}     {1}
```

72 *tableaux for* $r = 4$ *such that* $h_a \cdot e_t = 0$.

```
{1, 2}, {1, 2}, {1, 2}, {1, 2}, {1, 2}, {1, 2}, {1, 3}, {1, 3},
{3}     {3}     {4}     {4}     {5}     {5}     {2}     {2}
{4}     {5}     {3}     {5}     {3}     {4}     {4}     {5}
{5}     {4}     {5}     {3}     {4}     {3}     {5}     {4}

{1, 3}, {1, 3}, {1, 3}, {1, 3}, {1, 4}, {1, 4}, {1, 4}, {1, 4},
{4}     {4}     {5}     {5}     {2}     {2}     {3}     {3}
{2}     {5}     {2}     {4}     {3}     {5}     {2}     {5}
{5}     {2}     {4}     {2}     {5}     {3}     {5}     {2}

{1, 4}, {1, 4}, {2, 1}, {2, 1}, {2, 1}, {2, 1}, {2, 1}, {2, 1},
{5}     {5}     {3}     {3}     {4}     {4}     {5}     {5}
{2}     {3}     {4}     {5}     {3}     {5}     {3}     {4}
{3}     {2}     {5}     {4}     {5}     {3}     {4}     {3}

{2, 3}, {2, 3}, {2, 3}, {2, 3}, {2, 3}, {2, 3}, {2, 4}, {2, 4},
{1}     {1}     {4}     {4}     {5}     {5}     {1}     {1}
{4}     {5}     {1}     {5}     {1}     {4}     {3}     {5}
{5}     {4}     {5}     {1}     {4}     {1}     {5}     {3}

{2, 4}, {2, 4}, {2, 4}, {2, 4}, {3, 1}, {3, 1}, {3, 1}, {3, 1},
{3}     {3}     {5}     {5}     {2}     {2}     {4}     {4}
{1}     {5}     {1}     {3}     {4}     {5}     {2}     {5}
{5}     {1}     {3}     {1}     {5}     {4}     {5}     {2}
{3, 1}, {3, 1}, {3, 2}, {3, 2}, {3, 2}, {3, 2}, {3, 2}, {3, 2},
{5}     {5}     {1}     {1}     {4}     {4}     {5}     {5}
{2}     {4}     {4}     {5}     {1}     {5}     {1}     {4}
{4}     {2}     {5}     {4}     {5}     {1}     {4}     {1}

{3, 4}, {3, 4}, {3, 4}, {3, 4}, {3, 4}, {3, 4}, {4, 1}, {4, 1},
{1}     {1}     {2}     {2}     {5}     {5}     {2}     {2}
{2}     {5}     {1}     {5}     {1}     {2}     {3}     {5}
{5}     {2}     {5}     {1}     {2}     {1}     {5}     {3}

{4, 1}, {4, 1}, {4, 1}, {4, 1}, {4, 2}, {4, 2}, {4, 2}, {4, 2},
{3}     {3}     {5}     {5}     {1}     {1}     {3}     {3}
{2}     {5}     {2}     {3}     {3}     {5}     {1}     {5}
{5}     {2}     {3}     {2}     {5}     {3}     {5}     {1}

{4, 2}, {4, 2}, {4, 3}, {4, 3}, {4, 3}, {4, 3}, {4, 3}, {4, 3}
{5}     {5}     {1}     {1}     {2}     {2}     {5}     {5}
{1}     {3}     {2}     {5}     {1}     {5}     {1}     {2}
{3}     {1}     {5}     {2}     {5}     {1}     {2}     {1}
```

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