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TWO INVOLUTIONS FOR SIGNED EXCEDANCE NUMBERS

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Abstract

Two involutions on permutations and derangements are constructed. They lead to a direct evaluation of some signed Eulerian polynomials on permutations and derangements. The latter formulas, combined with the exponential formula, give a new derivation of the corresponding generating functions.

1 Introduction

Let n be a positive integer. A permutation σ of the set $[n] := \{1, \ldots, n\}$ has an *excedance* at i if $\sigma(i) > i$, for $1 \leq i \leq n$. As usual, we denote the numbers of excedances, cycles and fixed points of σ by $\operatorname{exc}\sigma$, $\operatorname{cyc}\sigma$ and $\operatorname{fix}\sigma$, respectively. A fixed-point-free permutation is called a *derangement*. Let $\mathcal{S}_n$ (resp. $\mathcal{D}_n$) be the set of *permutations* (resp. *derangements*) of $[n]$. For example, the permutation $\sigma = \sigma(1)\ldots\sigma(5)\sigma(6) = 3\,2\,6\,5\,4\,1 \in \mathcal{S}_6$ has the cycle decomposition $(1,3,6)(2)(4,5)$, so 2 is the only fixed point and σ has an excedance at 1, 3 and 4; finally $\operatorname{cyc}\sigma = 3$, $\operatorname{exc}\sigma = 3$ and $\operatorname{fix}\sigma = 1$. Consider the following enumerative polynomial of $\mathcal{S}_n$:

$$P_n(x,y,\beta) = \sum_{\sigma\in\mathcal{S}_n} x^{\operatorname{exc}\sigma} y^{\operatorname{fix}\sigma} \beta^{\operatorname{cyc}\sigma}.$$

Then $P_n(x,1,1)$ is the classical *Eulerian polynomial* (cf. [8, Chap. 1]), and $P_n(x,0,1)$ the counterpart for the derangements. For a combinatorial treatment of Eulerian polynomials the reader is referred to the seminal book of Foata and Schützenberger [5].

It is remarkable that when $\beta = -1$, these two polynomials have simple closed formulas :

$$\begin{aligned} P_n(x,1,-1) &= -(x-1)^{n-1}, &(1)\\ P_n(x,0,-1) &= -x - x^2 - \cdots - x^{n-1}. &(2) \end{aligned}$$

Although $P_n(x,0,-1)$ is a special case of $P_n(x,y,\beta)$, its evaluation does imply the exponential generating function of $P_n(x,y,\beta)$. Actually, weighting each permutation σ by the monomial $x^{\operatorname{exc}\sigma}y^{\operatorname{fix}\sigma}(-1)^{\operatorname{cyc}\sigma}$, the counting polynomial for permutations in $\mathcal{S}_n$ with k chosen fixed points for $0 \le k \le n$ is $(-y)^k P_{n-k}(x,0,-1)$, hence we deduce from (2) that

$$P_n(x,y,-1) = \sum_{k=0}^{n} \binom{n}{k} (-y)^k P_{n-k}(x,0,-1) = \frac{(x-y)^n - x(1-y)^n}{1-x}. \tag{3}$$

Now, according to the *exponential formula* (cf. [5], [8, Chap. 5]) and the evident identity $e^x = (e^{-x})^{-1}$, we have

$$1 + \sum_{n\ge 1} P_n(x,y,\beta)\frac{t^n}{n!} = \left[\exp\left(-\sum_{n\ge1}\sum_{\sigma\in\mathcal{C}_n} x^{\operatorname{exc}\sigma} y^{\operatorname{fix}\sigma}\frac{t^n}{n!}\right)\right]^{-\beta}, \tag{4}$$

where $\mathcal{C}_n$ denotes the set of cyclic permutations in $\mathcal{S}_n$. Setting $\beta=-1$ in (4) we see that the expression under brackets in (4) is equal to $1+\sum_{n\ge1} P_n(x,y,-1)t^n/n!$, hence

$$1 + \sum_{n\ge 1} P_n(x,y,\beta)\frac{t^n}{n!} = \left(\sum_{n\ge0} \frac{(x-y)^n - x(1-y)^n}{1-x}\frac{t^n}{n!}\right)^{-\beta}. \tag{5}$$

Conversely, it is a simple matter to derive (2) from (5) with $\beta=1$ and $y=0$, so (2) is actually equivalent to (5), of which the $\beta=1$ case was already given by Foata and Schützenberger [5].

Note that the polynomial $P_n(x,y,\beta)$ can also be interpreted through some *linear statistics* on permutations. Identify each permutation $\sigma \in \mathcal{S}_n$ with the word $\sigma(1)\sigma(2)\ldots\sigma(n)$. An integer i, $1 \le i \le n-1$, is a *descent* (resp. *succession*) of σ if $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i+1) = \sigma(i)+1$), and an integer $j=\sigma(i)$ for some $i \in [n]$ is a *left-to-right maximum* of σ if $\sigma(k) < j$ for all $k < i$. Let $\operatorname{des}\sigma$, $\operatorname{suc}\sigma$ and $\operatorname{lrm}\sigma$ be the numbers of descents, successions and left-to-right maxima of σ, respectively. Then, thanks to the *fundamental transformation* of Foata and Schützenberger [5], we have also

$$P_n(x,y,\beta) = \sum_{\sigma\in\mathcal{S}_n} x^{\operatorname{des}\sigma} y^{\operatorname{suc}\sigma} \beta^{\operatorname{lrm}\sigma}.$$

For some latest developments of the above fundamental transformation on words we refer the reader to a recent paper by Foata and Han [4].

Our purpose is to provide bijective proofs to the identities (1) and (2). In particular, this bijective approach leads to the following refinement of (2), which seems to be new.

Theorem 1 *Let $D_{n,j} = \{\sigma \in \mathcal{D}_n \mid \sigma(n) = j\}$ for $j \in [n-1]$. Then*

$$\sum_{\sigma\in D_{n,j}} (-1)^{\operatorname{cyc}\sigma} x^{\operatorname{exc}\sigma} = -x^{n-j}. \tag{6}$$

This paper was motivated by several recent papers: after deriving (1) from its generating function, which is equivalent to formula (5) with $y = 1$, Brenti [1, p.163] asked for a bijective proof, Chapman [2] gave a bijective proof of (2) when $x = 1$, Gessel [6] gave a combinatorial proof of (5) when $\beta = x = 1$ and $y = 0$, and Kim and Zeng [7] have given a direct combinatorial proof of (5) when $\beta = 1$ and $y = 1$ or 0, see also [3, 9] for some related problems.

In Section 2 we first give a bijective proof of the following recurrences:

$$\begin{aligned} P_n(x,1,-1) &= (x-1)P_{n-1}(x,1,-1), \qquad n \geq 3, & (7)\\ P_n(x,0,-1) &= P_{n-1}(x,0,-1) - x^{n-1}, \qquad n \geq 3. & (8) \end{aligned}$$

Since $P_2(x,1,-1) = 1-x$ and $P_2(x,0,-1) = -x$, (7) implies (1) and (8) implies (2). The reason for including such recursive proofs of (1) and (2) is that they use simpler involutions than the bijective ones. In Section 3 we then give a second proof of (1) and (2) (in fact (6)) based on a direct interpretation of their right-sides.

All our four involutions are constructed by composing (*from right to left*) a permutation σ with a *transposition* (i,j). It is easy to see that the number of the cycles of the resulting permutation will be increased by *one* if i and j are in the same cycle of σ, and decreased by one otherwise.

The *weight* of a permutation $\sigma \in \mathcal{S}_n$ is $w(\sigma) = (-1)^{\mathrm{cyc}\,\sigma} x^{\mathrm{exc}\,\sigma}$, and that of a subset $E \subseteq \mathcal{S}_n$ is the sum of the weights of its elements, i.e., $w(E) = \sum_{\sigma \in E} w(\sigma)$. Therefore, the basic idea of proving an identity is then to construct a subset E of $\mathcal{S}_n$ whose weight is the right-side of the identity; and a *killing involution* on $\mathcal{S}_n \setminus E$, i.e. an involution which preserves the number of excedances but changes the sign of the weight of each element.

2 Involutions for recursive proofs

2.1 Permutations

Define the involution ϕ_1 on $\mathcal{S}_n$ by

$$\sigma \mapsto \sigma' = (\sigma(n-1), \sigma(n)) \circ \sigma.$$

Clearly $\sigma'(j) = \sigma(j)$ if $j \neq n-1, n$, $\sigma'(n-1) = \sigma(n)$, and $\sigma'(n) = \sigma(n-1)$. Let's partition $\mathcal{S}_n$ into three subsets:

$$\begin{aligned} \mathcal{S}_n^1 &= \{\sigma \in \mathcal{S}_n \mid \sigma(n-1) \neq n,\, \sigma(n) \neq n\},\\ \mathcal{S}_n^2 &= \{\sigma \in \mathcal{S}_n \mid \sigma(n) = n\},\\ \mathcal{S}_n^3 &= \{\sigma \in \mathcal{S}_n \mid \sigma(n-1) = n\}. \end{aligned}$$

It is easy to check that the restriction of ϕ_1 to $\mathcal{S}_n^1$ is a killing involution, and the mapping $\sigma \mapsto \sigma'' = \sigma(1)\sigma(2)\cdots\sigma(n-1)$ is a bijection from $\mathcal{S}_n^2$ to $\mathcal{S}_{n-1}$ such that $w(\sigma) = -w(\sigma'')$, so the weight of $\mathcal{S}_n^2$ is $-P_{n-1}(x,1,-1)$. Finally the restriction of ϕ_1 to $\mathcal{S}_n^3$ is a bijection from $\mathcal{S}_n^3$ to $\mathcal{S}_n^2$ such that $w(\sigma) = -x\,w(\sigma')$. So the weight of $\mathcal{S}_n^3$ is $xP_{n-1}(x,1,-1)$. Summarizing the above three cases yields (7).

2.2 Derangements

Let $U_n = \{\sigma \in \mathcal{D}_n \mid \sigma(2) = 1 \text{ and } \sigma(1) \neq 2\}$. Clearly the mapping $\sigma \mapsto \sigma'' = \bar{\sigma}(1)\bar{\sigma}(3)\cdots\bar{\sigma}(n)$, where $\bar{\sigma}(i) = \sigma(i) - 1$, is a weight preserving bijection from U_n to D_{n-1}, so $w(U_n) = P_{n-1}(x, 0, -1)$. Note also that the weight of the cyclic permutation $\sigma_0 = (1, 2, \ldots, n)$ is $-x^{n-1}$. Hence, to prove (8) it remains to construct a killing involution on $\overline{\mathcal{D}}_n = \mathcal{D}_n \setminus (U_n \cup \{\sigma_0\})$. We claim that $\varphi_1 : \sigma \mapsto (\sigma(i_\sigma), \sigma(i_\sigma + 1)) \circ \sigma$ is such an involution on $\overline{D}_n$, where i_σ is the smallest integer $i \in [n-1]$ such that $\sigma(i) \neq i + 1$.

We first show that φ_1 is an involution on $\overline{D}_n$. Let $\sigma \in \overline{D}_n$.

- $\varphi_1(\sigma) \in \mathcal{D}_n$: if $i_\sigma = 1$, then $\sigma(1) \neq 2$ and by definition of σ, $\sigma(2) \neq 1$; if $i_\sigma \neq 1$, then $\sigma(i_\sigma) \neq i_\sigma + 1$ and $\sigma(i_\sigma + 1) \neq i_\sigma$ since i_σ is the image of $i_\sigma - 1$ by definition of i_σ.

- $\varphi_1(\sigma) \notin U_n \cup \{\sigma_0\}$: If $i_\sigma = 1$, $\varphi_1(\sigma)(2) = \sigma(1) \neq 1$ since $\sigma \in \mathcal{D}_n$; otherwise $\varphi_1(\sigma)(1) = \sigma(1) = 2$. So $\varphi_1(\sigma) \notin U_n$, moreover $\varphi_1(\sigma) \neq \sigma_0$ because
$$\varphi_1(\sigma)(i_\sigma) = \sigma(i_\sigma + 1) \neq i_\sigma + 1, \tag{9}$$

- φ_1 is an involution : for all $j < i_\sigma$, $\varphi_1(\sigma)(j) = \sigma(j) = j+1$, so $\varphi_1(\sigma) = i_\sigma$ in view of (9).

Next, we check that $\operatorname{exc}\varphi_1(\sigma) = \operatorname{exc}\sigma$: if $\sigma(i_\sigma) > i_\sigma$ (resp. $< i_\sigma$), then $\varphi_1(\sigma)(i_\sigma + 1) = \sigma(i_\sigma) > i_\sigma + 1$ (resp. $< i_\sigma + 1$); if $\sigma(i_\sigma + 1) < i_\sigma + 1$ (resp. $> i_\sigma + 1$), then $\varphi_1(\sigma)(i_\sigma) = \sigma(i_\sigma + 1) < i_\sigma$ (resp. $> i_\sigma$) since i_σ is the image of $i_\sigma - 1$. Note that the number of cycles changes by one, and so our involution reverses sign, and hence is a killing involution on $\overline{D}_n$.

3 Involutions for bijective proofs

3.1 Permutations

Let Ω_n denote the set of permutations σ in $\mathcal{S}_n$, whose cycle decomposition consists of a k-cycle, $k \geq 1$, such that
$$\sigma(1) < \sigma^2(1) < \cdots < \sigma^{k-1}(1), \tag{10}$$
and $n-k$ singletons. Set $C_\sigma(1) = \{1, \sigma(1), \ldots, \sigma^{k-1}(1)\}$, the orbit of σ containing 1. It is not hard to see that the weight of Ω_n is $-(x-1)^{n-1}$. Therefore, to prove (2), it remains to construct a killing involution on $\overline{\Omega}_n = \mathcal{S}_n \setminus \Omega_n$. Each permutation σ in $\overline{\Omega}_n$ can be characterized by the condition that either the k-cycle containing 1 does not satisfy the condition (10) or there exists an m-cycle ($m > 1$) disjoint with $C_\sigma(1)$; in other words, there exists $i \in [n]$ such that
$$i \in C_\sigma(1) \quad \text{and} \quad \sigma^{-1}(i) > i > 1, \qquad \text{or} \quad i \notin C_\sigma(1) \quad \text{and} \quad \sigma(i) \neq i. \tag{11}$$

σ with $i_\sigma \in C_\sigma(1)$	i_σ	j_σ	$(\sigma(j_\sigma), \sigma(i_\sigma))$	$\phi_2(\sigma)$
3 4 2 1=(1,3,2,4)	2	1	(3,4)	4 3 2 1=(1,4)(2,3)
3 1 4 2=(1,3,4,2)	2	1	(3,1)	1 3 4 2=(1)(2,3,4)
4 3 1 2=(1,4,2,3)	2	1	(4,3)	3 4 1 2=(1,3)(2,4)
4 1 2 3=(1,4,3,2)	2	1	(4,1)	1 4 2 3=(1)(2,4,3)
2 4 1 3=(1,2,4,3)	3	2	(4,1)	2 1 4 3=(1,2)(3,4)
3 1 2 4=(1,3,2)(4)	2	1	(3,1)	1 3 2 4=(1)(2,3)(4)
4 1 3 2=(1,4,2)(3)	2	1	(4,1)	1 4 3 2=(1)(2,4)(3)
4 2 1 3=(1,4,3)(2)	3	1	(4,1)	1 2 4 3=(1)(2)(3,4)

Table 1: Involution ϕ_2 for $n = 4$

Let i_σ denote the *smallest* integer satisfying (11) and j_σ the *largest* integer $< i_\sigma$ in $C_\sigma(1)$. Then we claim that the mapping $\phi_2 : \sigma \mapsto \sigma \circ (i_\sigma, j_\sigma)$ is a killing involution on $\overline{\Omega}_n$.

- Assume that $i_\sigma \in C_\sigma(1)$, then there exist integers $p, q \geq 0$ such that $q < p$, $i_\sigma = \sigma^p(1)$, $j_\sigma = \sigma^q(1)$ and the only elements $< i_\sigma$ in $C_\sigma(1)$ are $1, \sigma(1), \ldots, \sigma^q(1) = j_\sigma$. It follows that $i_{\phi_2(\sigma)} = i_\sigma$ and ϕ_2 is an involution. It remains to check the corresponding values of σ and $\phi_2(\sigma)$ at i_σ and j_σ. Clearly $\sigma(j_\sigma) > i_\sigma$ and $\sigma(i_\sigma) < i_\sigma$ (resp. $> i_\sigma$) if $\sigma(i_\sigma) = 1$ (resp. otherwise). On the other hand, $\phi_2(\sigma)(i_\sigma) = \sigma(j_\sigma) > i_\sigma$ and
$$\phi_2(\sigma)(j_\sigma) = \begin{cases} \sigma(i_\sigma) = 1 \leq j_\sigma & \text{if} \quad \sigma(i_\sigma) = 1; \\ \sigma(i_\sigma) > i_\sigma > j_\sigma & \text{otherwise.} \end{cases}$$
So $\operatorname{exc} \phi_2(\sigma) = \operatorname{exc} \sigma$.
- The case where $i_\sigma \notin C_\sigma(1)$ can be proved similarly. Note that if $i_\sigma \notin C_\sigma(1)$, then i_σ is the smallest element in $C_\sigma(i_\sigma)$, and the only elements $< i_\sigma$ in $C_\sigma(1)$ are $1, \sigma(1), \ldots, j_\sigma$, arranged in increasing order. The rest of the verification is left to the reader.

Remark. 1. Since $n \notin \{i_\sigma, j_\sigma\}$ for $\sigma \in \overline{\Omega}_n$, we see that ϕ_2 is actually a killing involution on $\overline{\Omega}_{n,j} := \{\sigma \in \overline{\Omega}_n \mid \sigma(n) = j\}$ for $j \in [n]$.
2. It is also easy to see that ϕ_2 is a bijection between permutations $\sigma \in \overline{\Omega}_n$ such that $i_\sigma \in C_\sigma(1)$ and those $\sigma \in \overline{\Omega}_n$ such that $i_\sigma \notin C_\sigma(1)$.

We end with an example of ϕ_2 for $n = 4$ in Table 1. As observed above, we need only to apply ϕ_2 to permutations $\sigma \in \overline{\Omega}_n$ with $i_\sigma \in C_\sigma(1)$.

3.2 Derangements

Fix $j \in [n-1]$ and set $D_{n,j} = \{\sigma \in \mathcal{D}_n \mid \sigma(n) = j\}$. By definition, the weight of each $\sigma \in D_{n,j}$ is $(-1)^{\operatorname{cyc} \sigma} x^{\operatorname{exc} \sigma}$, hence the weight of the cyclic permutation
$$\sigma_j = (n, j, j+1, \ldots, n-1, j-1, j-2, \ldots, 2, 1) \in D_{n,j}$$

is $-x^{n-j}$, to prove (6) it remains to construct a killing involution on $\overline{D}_{n,j} = D_{n,j} \setminus \{\sigma_j\}$.

For a, $b \in [n]$, we say that a is *smaller* than b, denoted $a \prec b$, if a is to the left of b in the word:

$$w_j = n\,1\,2\,\ldots\,(j-2)\,(j-1)\,(n-1)\,\ldots\,(j+1)\,j.$$

For $\sigma \in \overline{D}_{n,j} = D_{n,j} \setminus \{\sigma_j\}$ denote by i_σ the *smallest* integer i in the above word such that $\sigma(i) \neq \sigma_j(i)$. We notice that $i_\sigma \neq n$ because σ and σ_j have the same value j at n, and also $i_\sigma \neq j$ because, otherwise, σ and σ_j differ only at j and coincide at all the other $k \neq j$.

We claim that $\varphi_2 : \sigma \longrightarrow (i_\sigma, \sigma_j(i_\sigma)) \circ \sigma$ is a killing involution on $\overline{D}_{n,j}$. It suffices to check the following points :

- $\varphi_2(\sigma) \in \mathcal{D}_n$: First, by definition of i_σ, $\sigma(i_\sigma) \neq \sigma_j(i_\sigma)$. Secondly, if $i_\sigma = 1$ then $\sigma(\sigma_j(i_\sigma)) = j \neq i_\sigma$; otherwise $\sigma(\sigma_j(i_\sigma)) = \sigma_j^2(i_\sigma) \prec i_\sigma$ since σ_j maps each $k \neq n$ to his left in w_j. Thus $\sigma(\sigma_j(i_\sigma)) \neq i_\sigma$.

- $\varphi_2(\sigma) \in D_{n,j}$: Since $i_\sigma \neq j$ and $i_\sigma \neq n$ then $j \notin \{i_\sigma, \sigma_j(i_\sigma)\}$. It follows that $\varphi_2(\sigma)(n) = j$.

- $\varphi_2(\sigma) \neq \sigma_j$ and $i_{\varphi_2(\sigma)} = i_\sigma$: For $k \prec i_\sigma$ and $k \neq n$, since $\sigma(k) = \sigma_j(k) \neq \sigma_j(i_\sigma)$ and $\sigma(k) = \sigma_j(k) \prec k \prec i_\sigma$, then $\sigma(k) \notin \{i_\sigma, \sigma_j(i_\sigma)\}$, which implies that $\varphi_2(\sigma)(k) = \sigma_j(k)$. Since $\varphi_2(\sigma)(i_\sigma) = \sigma(i_\sigma) \neq \sigma_j(i_\sigma)$, then $\varphi_2(\sigma) \neq \sigma_j$ and $i_{\varphi_2(\sigma)} = i_\sigma$.

- $\operatorname{exc} \varphi_2(\sigma) = \operatorname{exc} \sigma$: It suffices to check the values of $\varphi_2(\sigma)$ and σ at $k \in [n-1]$ such that $\sigma(k) = i_\sigma$ or $\sigma(k) = \sigma_j(i_\sigma)$. If $i_\sigma \leq j-1$ then σ maps $1, 2, \ldots, i_\sigma - 1$ respectively to n, $1, \ldots, i_\sigma - 2$, so $\sigma^{-1}(i_\sigma) > i_\sigma$ and $\sigma^{-1}(i_\sigma - 1) > i_\sigma$; if $i_\sigma = n-1$ then σ maps $1, 2, \ldots, j-1$ respectively to n, $1, \ldots, j-2$, moreover $\sigma^{-1}(j) = n$, so $j-1 < \sigma^{-1}(n-1) < n-1$ and $j-1 < \sigma^{-1}(j-1) < n-1$; if $i_\sigma \in \{n-2, \ldots, j+1\}$ then σ maps $1, 2, \ldots, j-2, j-1, n-1, \ldots, i_\sigma+1$ respectively to $n, 1, \ldots, j-1, n-1, \ldots, i_\sigma+2$, so $\sigma^{-1}(i_\sigma) < i_\sigma$ and $\sigma^{-1}(i_\sigma+1) < i_\sigma$. Summarizing $\phi_2(\sigma)$ and σ have the same number of excedances at $\sigma^{-1}(i_\sigma)$ and $\sigma^{-1}(\sigma_j(i_\sigma))$.

As an example, we illustrate φ_2 for $n = 5$. There are four subsets $\overline{D}_{5,j}$ $(1 \leq j \leq 4)$:

- subset $\overline{D}_{5,4}$: $\sigma_4 = 5\,1\,2\,3\,4$, $w_4 = 5\,1\,2\,3\,4$,

σ	i_σ	$\sigma_4(i_\sigma)$	$(i_\sigma, \sigma_4(i_\sigma)) \circ \sigma$
2 1 5 3 4=(1,2)(3,5,4)	1	5	2 5 1 3 4=(1,2,5,4,3)
2 3 1 5 4=(1,2,3)(4,5)	1	5	2 3 5 1 4=(1,2,3,5,4)
3 1 2 5 4=(1,3,2)(4,5)	1	5	3 5 2 1 4=(1,3,2,5,4)
3 5 1 2 4=(1,3)(2,5,4)	1	5	3 1 5 2 4=(1,3,5,4,2)
5 3 2 1 4=(1,5,4)(2,3)	2	1	5 3 1 2 4=(1,5,4,2,3)

- subset $\overline{D}_{5,3}$: $\sigma_3 = 5\,1\,4\,2\,3$, $w_3 = 5\,1\,2\,4\,3$,

σ	i_σ	$\sigma_3(i_\sigma)$	$(i_\sigma, \sigma_3(i_\sigma)) \circ \sigma$
2 1 4 5 3=(1,2)(3,4,5)	1	5	2 5 4 1 3=(1,2,5,3,4)
2 4 5 1 3=(1,2,4)(3,5)	1	5	2 4 1 5 3=(1,2,4,5,3)
4 1 5 2 3=(1,4,2)(3,5)	1	5	4 5 1 2 3=(1,4,2,5,3)
4 5 2 1 3=(1,4)(2,5,3)	1	5	4 1 2 5 3=(1,4,5,3,2)
5 4 1 2 3=(1,5,3)(2,4)	2	1	5 4 2 1 3=(1,5,3,2,4)

- subset $\overline{D}_{5,2}$: $\sigma_2 = 5\,3\,4\,1\,2$, $w_2 = 5\,1\,4\,3\,2$,

σ	i_σ	$\sigma_2(i_\sigma)$	$(i_\sigma, \sigma_2(i_\sigma)) \circ \sigma$
3 4 1 5 2=(1,3)(2,4,5)	1	5	3 4 5 1 2=(1,3,5,2,4)
3 5 4 1 2=(1,3,4)(2,5)	1	5	3 1 4 5 2=(1,3,4,5,2)
4 3 5 1 2=(1,4)(2,3,5)	1	5	4 3 1 5 2=(1,4,5,2,3)
4 5 1 3 2=(1,4,3)(2,5)	1	5	4 1 5 3 2=(1,4,3,5,2)
5 1 4 3 2=(1,5,2)(3,4)	4	1	5 4 1 3 2=(1,5,2,4,3)

- subset $\overline{D}_{5,1}$: $\sigma_1 = 2\,3\,4\,5\,1$, $w_1 = 5\,4\,3\,2\,1$,

σ	i_σ	$\sigma_1(i_\sigma)$	$(i_\sigma, \sigma_1(i_\sigma)) \circ \sigma$
2 5 4 3 1=(1,2,5)(3,4)	4	5	2 4 5 3 1=(1,2,4,3,5)
3 4 5 2 1=(1,3,5)(2,4)	4	5	3 5 4 2 1=(1,3,4,2,5)
5 3 4 2 1=(1,5)(2,3,4)	4	5	4 3 5 2 1=(1,4,2,3,5)
5 4 2 3 1=(1,5)(2,4,3)	4	5	4 5 2 3 1=(1,4,3,2,5)
4 3 2 5 1=(1,4,5)(2,3)	3	4	3 4 2 5 1=(1,3,2,4,5)

Remark. Chapman [2] has given a bijective proof of (6) with $x = 1$, but his involution does not work for our purpose.

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