Séminaire Lotharingien de Combinatoire **59** (2008), Article B59c

NUMBER OF "*udu*"S OF A DYCK PATH AND *ad*-NILPOTENT IDEALS OF PARABOLIC SUBALGEBRAS OF $sl_{\ell+1}(\mathbb{C})$

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Abstract. For an ad-nilpotent ideal $\mathfrak{i}$ of a Borel subalgebra of $sl_{\ell+1}(\mathbb{C})$, we denote by $I_{\mathfrak{i}}$ the maximal subset I of the set of simple roots such that $\mathfrak{i}$ is an ad-nilpotent ideal of the standard parabolic subalgebra $\mathfrak{p}_I$. We use the bijection of Andrews, Krattenthaler, Orsina and Papi [*Trans. Amer. Math. Soc.* **354** (2002), 3835–3853] between the set of ad-nilpotent ideals of a Borel subalgebra in $sl_{\ell+1}(\mathbb{C})$ and the set of Dyck paths of length $2\ell+2$, to exhibit a bijection between ad-nilpotent ideals $\mathfrak{i}$ of the Borel subalgebra such that $\sharp I_{\mathfrak{i}} = r$ and the Dyck paths of length $2\ell+2$ having r occurrences of "*udu*". We obtain also a duality between antichains of cardinality p and $\ell - p$ in the set of positive roots.

1. Introduction

Let $M_{\ell+1}(\mathbb{C})$ be the set of $(\ell+1)$-by-$(\ell+1)$ matrices with coefficients in $\mathbb{C}$, and $\mathfrak{g}$ be the simple Lie algebra $sl_{\ell+1}(\mathbb{C})$ consisting of elements of $M_{\ell+1}(\mathbb{C})$ whose trace is equal to zero. Let $\mathfrak{h}$ be the maximal toral subalgebra of $\mathfrak{g}$ consisting of trace zero diagonal matrices. Let $(E_{i,j})$ be the canonical basis of $M_{\ell+1}(\mathbb{C})$ and $(E^*_{i,j})$ be its dual basis. For $1 \leqslant i \leqslant \ell+1$, set $\epsilon_i = E^*_{i,i}$. Then $\Delta = \{\epsilon_i - \epsilon_j; 1 \leqslant i, j \leqslant \ell+1, i \neq j\}$ is the root system associated to $(\mathfrak{g}, \mathfrak{h})$, and $\Delta^+ = \{\epsilon_i - \epsilon_j; 1 \leqslant i < j \leqslant \ell+1\}$ is a system of positive roots. Denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i = 1, \dots, \ell$. Then $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ is the corresponding set of simple roots. For each $\alpha \in \Delta$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}; [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ be the root space of $\mathfrak{g}$ relative to α.

For $I \subset \Pi$, set $\Delta_I = \mathbb{Z}I \cap \Delta$. We fix the corresponding standard parabolic subalgebra,

$$\mathfrak{p}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I \cup \Delta^+} \mathfrak{g}_\alpha \right).$$

Note that $\mathfrak{p}_\emptyset$ is a Borel subalgebra $\mathfrak{b}$ associated to the choice of Δ^+.

An ideal $\mathfrak{i}$ of $\mathfrak{p}_I$ is ad-nilpotent if and only if for all $x \in \mathfrak{i}$, $ad_{\mathfrak{p}_I} x$ is nilpotent. Since any ideal of $\mathfrak{p}_I$ is $\mathfrak{h}$-stable, we can deduce easily that an ideal is ad-nilpotent if and only if it is nilpotent. Moreover, we have $\mathfrak{i} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, for some subset $\Phi \subset \Delta^+ \setminus \Delta_I$.

A Dyck path of length $2n$ can be defined as a word of $2n$ letters u or d, having the same number of u and d, and such that there is always more u's than d's to the left of a letter.

Andrews, Krattenthaler, Orsina and Papi established in [AKOP] a bijection between the set of ad-nilpotent ideals of the Borel subalgebra $\mathfrak{p}_\emptyset$ and the set of Dyck paths of length $2\ell+2$ which allows them to enumerate ad-nilpotent ideals of a fixed class of nilpotence. The purpose of this paper is to explain some applications of this correspondence for the ad-nilpotent ideals of parabolic subalgebras.

More precisely, let $\mathfrak{i}$ be an ad-nilpotent ideal of the Borel subalgebra $\mathfrak{p}_\emptyset$. Denote by $I_\mathfrak{i}$ the maximal subset $I \subset \Pi$ such that $\mathfrak{i}$ is an ad-nilpotent ideal of $\mathfrak{p}_I$. The main result we prove here is the following theorem.

Theorem 1. *There is a bijection between the ad-nilpotent ideals* $\mathfrak{i}$ *of* $\mathfrak{b}$ *such that* $\sharp I_\mathfrak{i} = r$ *and the Dyck paths of length* $2\ell + 2$ *having* r *occurrences of "udu".*

We can deduce a formula for the desired number of ideals since the number of Dyck paths having r occurrences of "*udu*" have been calculated in [Sun].

This paper is organized as follows: we first recall the natural bijection between ℓ-partitions and Dyck paths of length $2\ell + 2$, as in [Pa]. In Section 3, we recall the iterative construction of the bijection of [AKOP]. Then, in Section 4, we explain how to calculate the number of occurrences of "*udu*" of a Dyck path obtained by the previous construction. In Section 5, we recall some facts of [R] and [CP] on ad-nilpotent ideals and we prove Theorem 1. Finally, in Section 6, we establish a duality between ad-nilpotent ideals of $\mathfrak{p}_\emptyset$. Such a duality has already been constructed by Panyushev in [Pa], however, it is not the same as the one we have here.

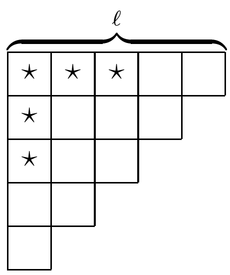
Acknowledgment. This work was realized while I was visiting the Istituto Guido Castelnuovo di Matematica (Roma). I would like to thank the European program Liegrits for offering me the possibility to go there and the institute for its hospitality.

2. Partitions and Dyck paths

In this section, we shall see how to generate a Dyck path from a partition.

Recall that a partition is an ℓ-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{N}^\ell$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell$. A partition will be called an ℓ-partition if $\lambda_i \leqslant i$ for $i = 1, \ldots, l$.

Partitions are usually represented by their Ferrers diagrams. Let T_ℓ be the Ferrers diagram of the ℓ-partition $(\ell, \ell-1, \ldots, 1)$. Then the Ferrers diagram F of any ℓ-partition λ can be viewed as a subdiagram of T_ℓ. For example, for $\ell = 5$, the Ferrers diagram of $\lambda = (3, 1, 1, 0, 0)$ is the subdiagram of T_ℓ, whose boxes are denoted by some $\star$:



Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an ℓ-partition and let F be its Ferrers diagram. We draw a dotted horizontal line from the top of the line $x + y = \ell + 1$ to F and a dotted vertical line from F to the bottom of the line $x + y = \ell + 1$. For example, when $\lambda = (5, 3, 1, 1, 1, 0, 0)$, we have:

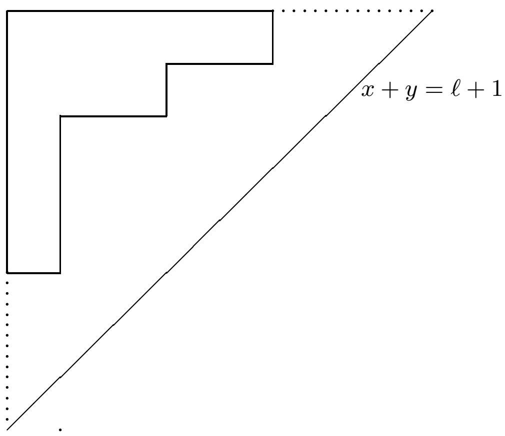
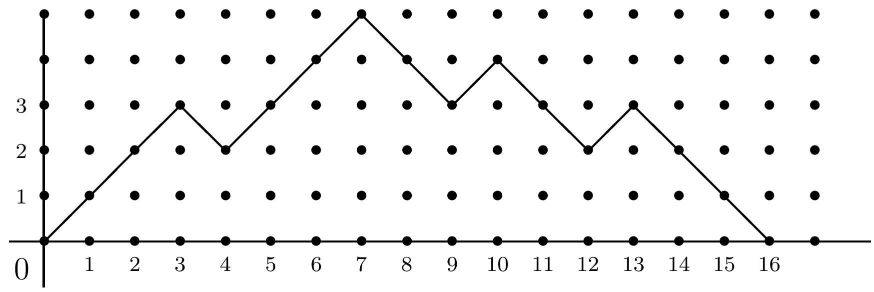


FIGURE 1

If we rotate the figure clockwise by 45 degrees, we can easily see that we obtain a Dyck path of length $2\ell + 2$ called $P(\lambda)$ as in [Pa]. This construction defines clearly a bijection $P : \lambda \mapsto P(\lambda)$ between ℓ-partitions and Dyck paths of length $2\ell + 2$. In the above example, the Dyck path $P(\lambda)$ is:



3. AKOP-bijection

Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be an ℓ-partition whose Ferrers diagram is F. We shall draw a dotted line associated to λ. We start at the top of the line $x + y = \ell + 1$. We go left until we meet F. Then, we continue downwards until we reach $x + y = \ell + 1$. Then we iterate the procedure until we reach the bottom. For example, for $\ell = 13$ and $\lambda = (10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0)$:

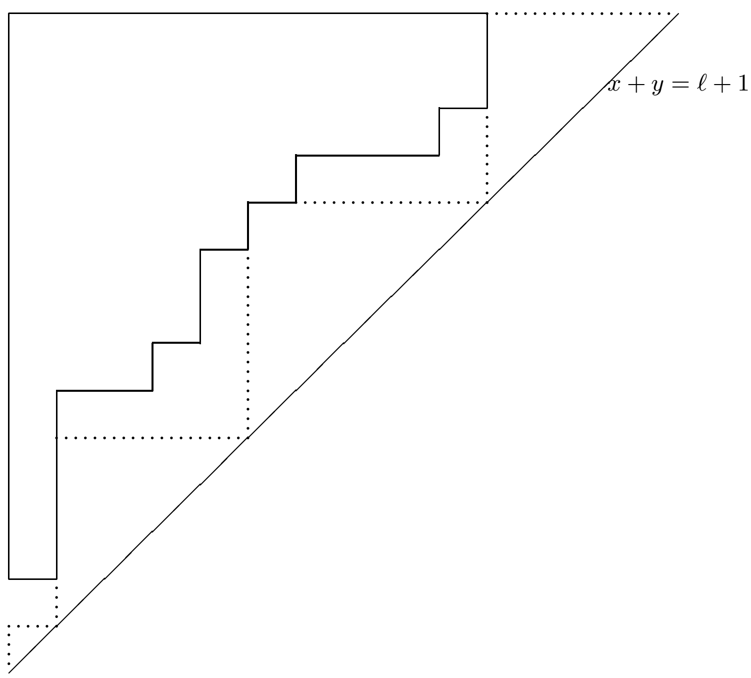


Figure 2

Let $n(\lambda)$ be the number of points of the dotted line on $x + y = \ell + 1$, which are not at the top or bottom. For example, we have

$n((0,\ldots,0)) = 0$, and for the ℓ-partition λ of Figure 2, we have $n(\lambda) = 3$.

We shall describe the construction of this line in a more formal way. Let $k = n(\lambda)$. Set $i_n = \ell + 1$ for all $n > k$, $i_k = \lambda_1$, $i_{k-1} = \lambda_{\ell - i_k + 2}$, $i_{k-2} = \lambda_{\ell - i_{k-1} + 2}$, $\ldots$, $i_1 = \lambda_{\ell - i_2 + 2}$ and $i_p = 0$ for all $p \leqslant 0$. We have $0 < i_1 < \cdots < i_k < \ell + 1$. The dotted line describes the shape of an ℓ-partition

$$\lambda^M = (i_k^{\ell - i_k + 1}, i_{k-1}^{i_k - i_{k-1}}, \ldots, i_1^{i_2 - i_1}, 0^{i_1 - 1}). \tag{1}$$

Any ℓ-partition λ whose associated dotted line gives the partition λ^M must necessarily contain the cells

$$(1, i_k), (\ell - i_k + 2, i_{k-1}), (\ell - i_{k-1} + 2, i_{k-2}), \ldots, (\ell - i_2 + 2, i_1).$$

The "minimal" ℓ-partition in the sense of inclusion of diagrams that contains these cells is

$$\lambda^m = (i_k, i_{k-1}^{\ell - i_k + 1}, i_{k-2}^{i_k - i_{k-1}}, \ldots, i_1^{i_3 - i_2}, 0^{i_2 - 2}). \tag{2}$$

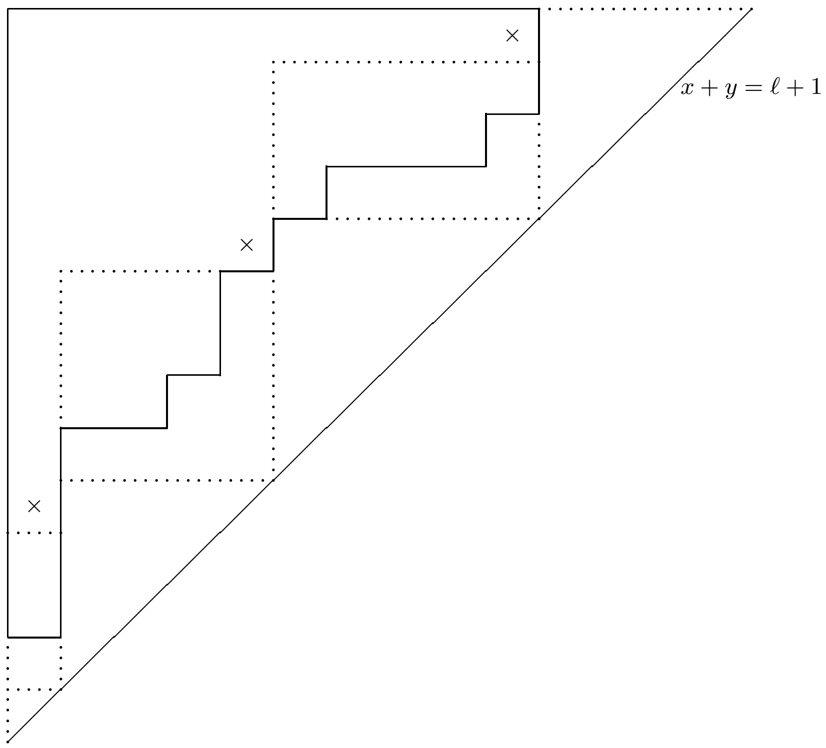
For example, take $\ell = 13$ and $\lambda = (10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0)$, as above, we have $n(\lambda) = k = 3$, $i_3 = 10, i_2 = 5, i_1 = 1$. The three distinguished cells above are

$$(1, 10), (5, 5), (10, 1).$$

So we have

$$\lambda^M = (10, 10, 10, 10, 5, 5, 5, 5, 5, 1, 1, 1, 1), \text{ and}$$
$$\lambda^m = (10, 5, 5, 5, 5, 1, 1, 1, 1, 1, 0, 0, 0).$$

These partitions are illustrated in the figure below, where the distinguished cells are marked with $\times$, and λ^M is the partition corresponding to the dotted line outside λ, while λ^m is the one which corresponds to the dotted line inside λ.



Observe that the difference $\lambda^M \setminus \lambda^m$ is a disjoint union of k rectangles, denoted by $R_k, \ldots, R_1$ from the top to the bottom. More precisely,

$$R_j = \{(s,t); \ell - i_{p+1} + 2 < s < \ell - i_p + 2 \text{ and } i_{p-1} < t \leqslant i_p\}.$$

Inside each rectangle R_j, the shape of λ could be described by a word M_j, whose letters are d and l, where d indicates a down step and l indicates a left step.

Let h_j be the number of d's in M_j, which is at most the height of R_j and let l_j be the number of l's in M_j, which is the length of R_j. Then we have

$$h_j = i_{j+1} - i_j - 1 \text{ if } j \neq 1, \text{ and } h_j \leqslant i_{j+1} - i_j - 1 \text{ if } j = 1,$$
$$l_j = i_j - i_{j-1},$$

so $h_j \leqslant l_{j+1} - 1$ and the equality holds if $j \neq 1$. Furthermore the shape of M_j is $l^{a_{j,0}} d l^{a_{j,1}} d \ldots d l^{a_{j,h_j}}$, where $a_{j,i} \in \mathbb{N}$, $0 \leqslant i \leqslant h_j$. We then have that

$$l_j = \sum_{i=0}^{h_j} a_{j,i}. \tag{3}$$

In the above example, we have $M_3 = dldl^3dl$, $M_2 = lddldl^2d$ and $M_1 = ddl$.

We shall now generate a Dyck path step by step from the M_j. We call a peak of a Dyck path, an occurrence of ud in the corresponding Dyck word.

First, let D_{k+1} be the Dyck path of length $2(\ell + 1 - i_k)$ containing $\ell + 1 - i_k$ peaks. Next, we have $M_k = l^{a_{k,0}} d l^{a_{k,1}} d \dots d l^{a_{k,h_k}}$. We insert $a_{k,0}$ peaks on the first peak of the already existing Dyck path D_{k+1}, then $a_{k,1}$ peaks on the second peak, and so on. We call D_k the new Dyck path obtained. Observed that the highest peaks of D_k are exactly those newly inserted, so there are exactly l_k. Since $h_{k-1} \leqslant l_k - 1$, the procedure can then be iterated by inserting peaks only on highest peaks. Each intermediate Dyck path obtained after using the word M_j is denoted by D_j. At the end, we obtain a Dyck path D_λ of length $2\ell + 2$.

For example, let us consider $\ell = 7$ and $\lambda = (5, 3, 1, 1, 1, 0, 0)$:

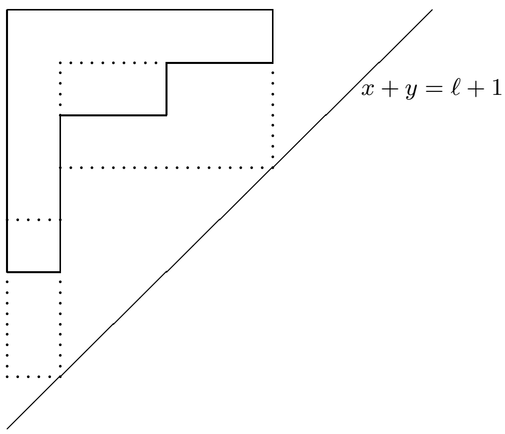
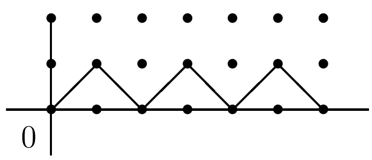
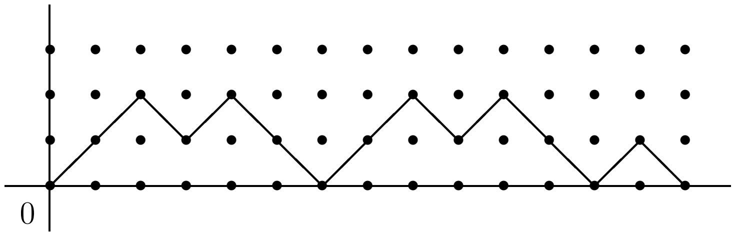


FIGURE 3

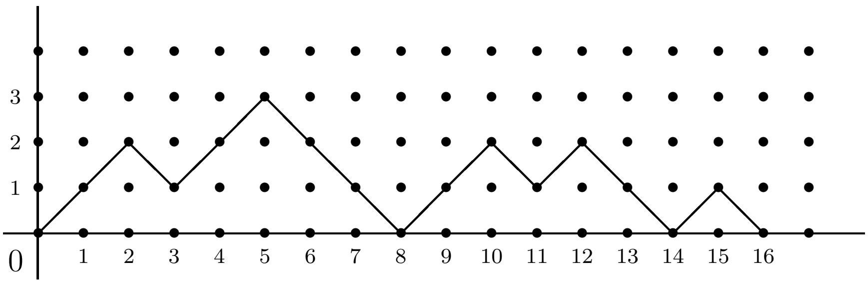
We have $n(\lambda) = k = 2$, $i_2 = 5$ and $i_1 = 1$. Then D_3 is the following Dyck path:



We have $M_2 = l^2 d l^2 d$, so we first insert 2 peaks on the first peak of D_3, then again two peaks on the second one. We obtain D_2:



Finally, $M_1 = dl$ so we insert $a_{1,0} = 0$ peak on the first highest peak of D_2 and $a_{1,1} = 1$ peak on the second highest peak. We obtain D_λ:



By [AKOP], we have the following proposition.

Proposition 3.1. *The map $D : \lambda \mapsto D_\lambda$ defines a bijection between the set of ℓ-partitions and the set of Dyck paths of length $2\ell + 2$.*

4. Dyck path and number of occurrences of "*udu*"

Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be an ℓ-partition such that $n(\lambda) = k$. Let D_λ be the Dyck path obtained from λ as described in Section 3. We shall see how to count the number of occurrences of "*udu*" contained in D_λ.

A peak could be followed by a "*u*", a "*d*" or nothing in the Dyck word. If it is followed by a "*u*", we call it a *u*-peak. Each *u*-peak will give an "*udu*" and vice versa.

Let $1 \leqslant j \leqslant k + 1$. Let u_j be the number of *u*-peaks in the Dyck path D_j. For example, D_{k+1} contains $\ell - \lambda_1 + 1 = \ell - i_k + 1$ peaks, so it is easy to see that $u_{k+1} = \ell - \lambda_1$.

To construct D_{j-1} from D_j, we add some peaks on the highest peaks of D_j. Then, one must understand how the insertion of p peaks on a highest peak modifies the number of occurrences of "*udu*". Consider a peak P of maximal height on a Dyck path. If we add p peaks, the part of the Dyck word which corresponds to P (which was ud) becomes $uudud \dots udd$ (with p ud), so we obtain $p - 1$ occurrences of udu. If P is a u-peak, then we also "destroy" the udu given by P. So at the end, we only add $p-2$ occurrences of udu. For example, let us consider the following Dyck path which contains 2 occurrences of udu:

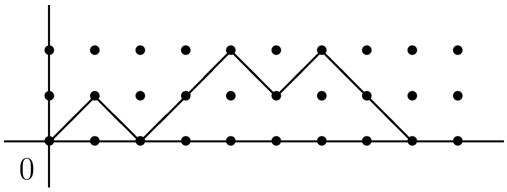
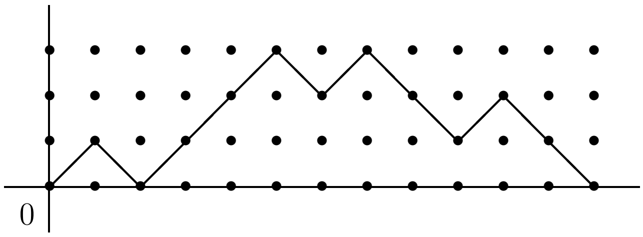
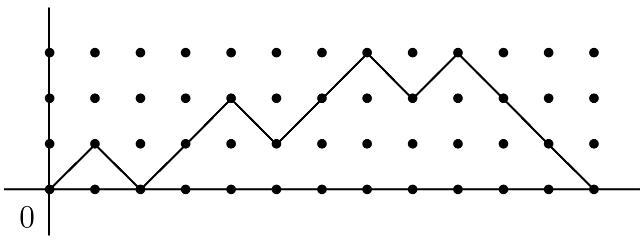


FIGURE 4

If we add 2 peaks on the first highest peak, we add $2-2=0$ occurrences of udu. So we obtain the following Dyck path with still 2 occurrences of udu:



If P is not a u-peak, then we do not "destroy" a udu, so we indeed add $p-1$ occurrences of "udu". For example, if we add 2 peaks on the second highest peak of Figure 4, we add $2-1=1$ occurrence of udu, so we obtain 3 occurrences of udu at the end:



Set $a_{k+1,0} = \ell - i_k + 1$, $M_{k+1} = l^{a_{k+1,0}}$, and $h_{k+1} = 0$. We have seen that each word M_j is in the form $l^{a_{j,0}} d l^{a_{j,1}} d \dots d l^{a_{j,h_j}}$. Let

$$\mathcal{A}_j = \{(j,t); t \in \{0, \dots, h_j\}; a_{j,t} \neq 0\},$$

$$\mathcal{A} = \bigcup_{j=1}^{k} \mathcal{A}_j.$$

Recall from the construction that the number of highest peaks in D_j is

$$\sum_{t=0}^{h_j} a_{j,i} = l_j. \tag{4}$$

Observe that a highest peak is a u-peak if it is not the last one of a consecutive group of highest peaks. Hence, the q-th peak of D_j is not a u-peak if and only if there exists $r \in \{0, \dots, h_j\}$ such that $q = \sum_{s=0}^{r} a_{j,s}$. Set

$$\mathcal{L}_p = \left\{ (p,t); \text{there exists } \ 0 \leqslant r \leqslant h_{p+1}; t+1 = \sum_{q=0}^{r} a_{p+1,q} \right\},$$

$$\mathcal{U}_p = \mathcal{A}_p \setminus \mathcal{L}_p, \quad \mathcal{L} = \bigcup_{p=1}^{k} \mathcal{L}_p, \quad \mathcal{U} = \bigcup_{p=1}^{k} \mathcal{U}_p.$$

Thus $\mathcal{L}_j$ corresponds exactly to the set of highest peaks in D_j which are not u-peaks and where we insert new peaks. It follows that

$$u_{j-1} = u_j + \sum_{(j-1,t)\in\mathcal{U}_{j-1}} (a_{j-1,t} - 2) + \sum_{(j-1,t)\in\mathcal{L}_{j-1}} (a_{j-1,t} - 1).$$

At the end of the construction, the number of occurrences of "udu" in D_λ is u_1. By induction, we have

$$u_1 = \ell - \lambda_1 + \sum_{(j,t)\in\mathcal{U}} (a_{j,t} - 2) + \sum_{(j,t)\in\mathcal{L}} (a_{j,t} - 1).$$

Since $\sum_{(j,t)\in\mathcal{A}} a_{j,t} = \lambda_1$, we obtain the following proposition.

Proposition 4.1. *Let λ be an ℓ-partition. Then, the number of occurrences of "udu" in D_λ is $\ell - 2\sharp\mathcal{U} - \sharp\mathcal{L}$.*

To illustrate this, we could follow again the construction of the Dyck path which corresponds to $\lambda = (5,3,1,1,1,0,0)$. We first have the Dyck path D_3 in Section 3, with $n - \lambda_1 + 1 = 3$ peaks, and $u_3 = 2$. Then we use the word $M_2 = l^2dl^2d = l^{a_{2,0}}dl^{a_{2,1}}d$, where $a_{2,0}, a_{2,1} \in \mathcal{L}_2$, so we add $a_{2,0} - 2 + a_{2,1} - 2 = 0$ peak. So $u_2 = 2$. Then we use the word $M_1 = dl = l^{a_{1,0}}dl^{a_{1,1}}$, where $a_{1,1} \in \mathcal{U}_1$, so we add $a_{1,1} - 1 = 0$ peak. Hence, $u_1 = 2$.

5. Ad-nilpotent ideals of a parabolic subalgebra and Dyck paths

Let $I \subset \Pi$ and $\mathfrak{i}$ be an ad-nilpotent ideal of $\mathfrak{p}_I$. We set

$$\Phi_{\mathfrak{i}} = \{\alpha \in \Delta^+ \setminus \Delta_I; \ \mathfrak{g}_\alpha \subseteq \mathfrak{i}\}.$$

Then $\mathfrak{i} = \bigoplus_{\alpha\in\Phi_\mathfrak{i}} \mathfrak{g}_\alpha$ and if $\alpha \in \Phi_\mathfrak{i}$, $\beta \in \Delta^+ \cup \Delta_I$ are such that $\alpha+\beta \in \Delta^+$, then $\alpha+\beta \in \Phi_\mathfrak{i}$.

Conversely, set

$$\mathcal{F}_I = \{\Phi \subset \Delta^+\backslash\Delta_I; \text{if } \alpha \in \Phi, \beta \in \Delta^+\cup\Delta_I, \alpha+\beta \in \Delta^+, \text{then } \alpha+\beta \in \Phi\}.$$

Then for $\Phi \in \mathcal{F}_I$, $\mathfrak{i}_\Phi = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is an ad-nilpotent ideal of $\mathfrak{p}_I$.

We obtain therefore a bijection

$$\{\text{ad-nilpotent ideals of } \mathfrak{p}_I\} \to \mathcal{F}_I,\ \mathfrak{i} \mapsto \Phi_\mathfrak{i}.$$

Recall the following partial order on Δ^+: $\alpha < \beta$ if $\beta - \alpha$ is a sum of positive roots. Then it is easy to see that $\Phi \in \mathcal{F}_\emptyset$ if and only if for all $\alpha \in \Phi, \beta \in \Delta^+$, such that $\alpha < \beta$, then $\beta \in \Phi$.

Let $\Phi \in \mathcal{F}_\emptyset$. Set

$$\Phi_{min} = \{\beta \in \Phi; \beta - \alpha \notin \Phi, \text{ for all } \alpha \in \Delta^+\}.$$

Then, Φ_{min} is an antichain of Δ^+ with respect to the above partial order. Conversely, if we consider an antichain Γ, then, the set of roots which are bigger than any one of the elements of Γ is an element of $\mathcal{F}_\emptyset$.

As in [CP], we display the positive roots Δ^+ in the Ferrers diagram T_ℓ of $(\ell, \ell-1, \dots, 1)$ as follows: we assign to each box in the i-th row and the j-th column, labelled (i,j) in T_ℓ, a positive root $t_{i,j} = \alpha_i + \dots + \alpha_{\ell-j+1}$, $1 \leqslant i, j \leqslant \ell$.

For example, for $\ell = 5$, we have

$$\overbrace{\begin{array}{|c|c|c|c|c|}\hline t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} \\ \hline t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ \cline{1-4} t_{3,1} & t_{3,2} & t_{3,3} \\ \cline{1-3} t_{4,1} & t_{4,2} \\ \cline{1-2} t_{5,1} \\ \cline{1-1}\end{array}}^{\ell}$$

Observe that given two positive roots α and β, α is bigger than or equal to β if the box corresponding to α is in the quadrant north-west of the box corresponding to β. It follows easily that the map which sends an element $\Phi \in \mathcal{F}_\emptyset$ to the subdiagram of T_ℓ consisting of the boxes corresponding to the roots of Φ defines a bijection between $\mathcal{F}_\emptyset$ and the set of northwest flushed subdiagrams of T_ℓ, i.e with the set of subdiagrams which contain the quadrant north-west of their boxes. Hence, by Section 2, we obtain a bijection σ from $\mathcal{F}_\emptyset$ to the set of ℓ-partitions.

By Proposition 3.1, $D \circ \sigma$ is a bijection from $\mathcal{F}_\emptyset$ to the set of Dyck paths of length $2\ell + 2$.

For $\Phi \in \mathcal{F}_\emptyset$, set

$$I_\Phi = \{\alpha \in \Pi; \Phi \in \mathcal{F}_{\{\alpha\}}\}.$$

It is the maximal element of $\{I \subset \Pi; \Phi \in \mathcal{F}_I\}$ with respect to inclusion order. We shall see how to link the number of occurrences of "*udu*" of the Dyck path $(D \circ \sigma)(\Phi)$ and the cardinality of I_Φ.

Set $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$, for all $1 \leqslant i \leqslant j \leqslant \ell$. We have easily the following lemma.

Lemma 5.1. *Let $I \subset \Pi$. An element $\Phi \in \mathcal{F}_\emptyset$ is an element of $\mathcal{F}_I$ if and only if for all $\alpha_{i,j} \in \Phi_{min}$, we have $\alpha_i, \alpha_j \notin I$.*

It follows from Lemma 5.1 that

$$I_\Phi = \Pi \setminus \{\alpha_i \in \Pi; \text{there exists } \alpha_{i,j} \text{ or } \alpha_{k,i} \in \Phi_{min}\}.$$

The problem is not to count the same root twice. For example, in A_7, for $\Phi_{min} = \{\alpha_{1,3}, \alpha_{2,5}, \alpha_{5,7}\}$, we have $\Pi \setminus I_\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7\}$ but we find α_5 in the beginning or in the end of the support of two roots in Φ_{min}. So if we set

$$L = \{\alpha_{i,j} \in \Phi_{min};\ \text{there exists a root of shape } \alpha_{p,i} \in \Phi_{min}\},$$

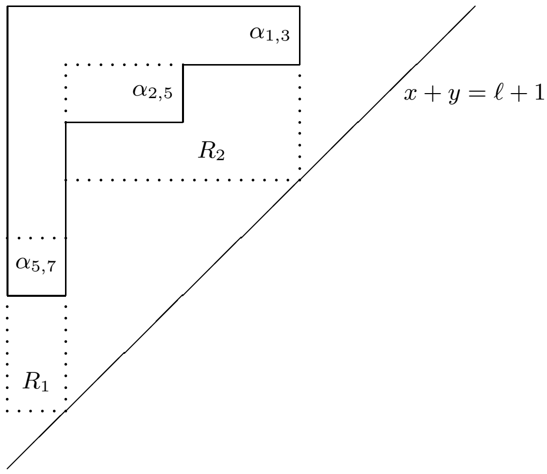
$$U = \Phi_{min} \setminus L,$$

we obtain that

$$\sharp I_\Phi = l - 2\sharp U - \sharp L. \tag{5}$$

Let $\lambda = \sigma(\Phi)$, F its Ferrers diagram and $D_\lambda = D(\lambda)$ be the Dyck path which corresponds to λ via the AKOP-bijection. Let $\alpha_{i,j} \in \Phi_{min}$. Then the cell $(i, \ell + 1 - j) = (i, \lambda_i)$ of $\alpha_{i,j}$ in F is a south-east corner of the diagram and two cases are possible: there exists a rectangle R_p such that $(i, \lambda_i) \in R_p$ or (i, λ_i) is not in any rectangle. If the latter case occurs, then $(i, \ell + 1 - j)$ is above a rectangle R_p. For example, if $\lambda = (5, 3, 1, 1, 1, 0, 0)$, we have that $\alpha_{2,5}, \alpha_{5,7}$ are in the first case and

$\alpha_{1,3}$ is in the second case.



If $\alpha_{i,j}$ is in the rectangle R_p, then the cell $(i, \lambda_i) = (i, \ell - j + 1)$ which corresponds to $\alpha_{i,j}$ in F satisfies

$$\ell - i_{p+1} + 2 < i < \ell - i_p + 2, \tag{6}$$

$$i_{p-1} < \lambda_i \leqslant i_p, \tag{7}$$

and so we have

$$\ell - i_p + 1 \leqslant j < \ell - i_{p-1} + 1. \tag{8}$$

If $\alpha_{i,j}$ is above the rectangle R_p, then the cell $(i, \ell - j + 1)$ which corresponds to $\alpha_{i,j}$ in F satisfies

$$(i, \ell - j + 1) = (\ell - i_{p+1} + 2, i_p). \tag{9}$$

Define the map r from Φ_{min} to $\{1, \ldots, k\}$ which associates to $\alpha_{i,j}$ the integer $r(\alpha_{i,j}) = p$ such that $\alpha_{i,j}$ is in or immediately above the rectangle R_p.

Let $\alpha_{i,j} \in \Phi_{min}$ and $p = r(\alpha_{i,j})$. Since the cell $(i, \ell - j + 1)$ which contains $\alpha_{i,j}$ in T_ℓ is a south-east corner, there is a horizontal line under this cell. If $c = (i, \ell - j + 1)$ is in the rectangle R_p, then it is at the row $q = i - (\ell - i_{p+1} + 2)$ of R_p and the line under c correspond to the part $l^{a_{p,q}}$ in M_p. Furthermore $(p, q) \in \mathcal{A}_p$.

If c is immediately above the rectangle R_p, then the line under c corresponds to $l^{a_{p,0}}$ in M_p and $(p, 0) \in \mathcal{A}_p$. Since in this case, by (9) we have $(i, \ell - j + 1) = (\ell - i_{p+1} + 2, i_p)$, we obtain that $i - (\ell - i_{p+1} + 2) = 0$. We can define in any case the map s from Φ_{min} to $\mathbb{N}$ by

$$s(\alpha_{i,j}) = i - (\ell - i_{r(\alpha_{i,j})+1} + 2). \tag{10}$$

Furthermore, in both cases, the line under the cell which contains $\alpha_{i,j}$ is the part $l^{a_{r(\alpha_{i,j}), s(\alpha_{i,j})}}$ in $M_{r(\alpha_{i,j})}$ and $(r(\alpha_{i,j}), s(\alpha_{i,j})) \in \mathcal{A}_{r(\alpha_{i,j})}$.

Conversely, let $(p,q) \in \mathcal{A}_p$. Then, there is a horizontal line under the row $i = q - \ell - i_{p+1} + 2$ of F which is under a south-east corner of F. This south-east corner is a cell (i, λ_i) which corresponds to a root $\alpha_{i,j}$, where $\ell - j + 1 = \lambda_i$. So we have a bijection

$$\begin{array}{rcccl} \Psi & : & \Phi_{min} & \to & \mathcal{A} \\ & & \alpha_{i,j} & \mapsto & (r(\alpha_{i,j}), s(\alpha_{i,j})). \end{array}$$

Lemma 5.2. *We have* $\Psi(U) = \mathcal{U}$ *and* $\Psi(L) = \mathcal{L}$.

Proof. Since $L = \Phi_{min} \setminus U$ and $\mathcal{L} = \mathcal{A} \setminus \mathcal{U}$, it suffices to prove that $\Psi(L) = \mathcal{L}$.

Let $\alpha_{i,j} \in L$. Set $p = r(\alpha_{i,j})$, $q = s(\alpha_{i,j})$ and let $c = (i, \lambda_i)$ be the cell which corresponds to $\alpha_{i,j}$ in F.

First assume that $i = j$. Then, we have $c = (i, \ell - i + 1)$. If $c \in R_p$, then by (6) and (8), we have

$$i = \ell - i_p + 1,$$

so by (10), we have that $q = i_{p+1} - i_p - 1$ so by (3), $a_{p,q} \in \mathcal{L}_p$.

If c is above R_p, then by (9), we have $c = (i, \ell - i + 1) = (\ell - i_{p+1} + 2, i_p)$, so $q = 0$ and $i_{p+1} - i_p = 1$, hence by (3) we also have $a_{p,q} \in \mathcal{L}_p$.

Now assume that $i \neq j$ and there exists a root of shape $\alpha_{m,i} \in \Phi_{min}$. Set $t = r(\alpha_{m,i})$. Let $(m, \lambda_m) = (m, \ell - i + 1)$ be the cell which corresponds to $\alpha_{m,i}$ in λ. If $c \in R_p$, then by (6), we have

$$i_p \leqslant \lambda_m \leqslant i_{p+1} - 2.$$

So either $(m, \lambda_m) \in R_{p+1}$ or $(m, \lambda_m) = (\ell - i_{p+1} + 2, i_p)$.

If $(m, \lambda_m) \in R_{p+1}$, then between the columns i_{p+1} and $\lambda_m = \ell - i + 1$, we have $i_{p+1} - (\ell - i + 1)$ columns, so there exists n such that $\sum_{u=0}^{n} a_{p+1,u} = i_{p+1} - (\ell - i + 1)$. Furthermore, by (10), we have $q = i - (\ell - i_{p+1} + 2)$, hence $a_{p,q} \in \mathcal{L}_p$.

If $(m, \lambda_m) = (\ell - i_{p+1} + 2, i_p)$, then $i = \ell - i_p + 1$ and by (10), we have that

$$q = (\ell - i_p + 1) - (\ell - i_{p+1} + 2) = i_{p+1} - i_p - 1.$$

Hence, by (3), we have $a_{p,q} \in \mathcal{L}_p$.

Conversely, let $a_{p,q} \in \mathcal{L}_p$, then there exists $0 \leqslant t \leqslant h_{p+1}$ such that $q + 1 = \sum_{f=0}^{t} a_{p+1,f}$. There also exists $\alpha_{i,j} \in \Phi_{min}$ such that $r(\alpha_{i,j}) = p$ and $s(\alpha_{i,j}) = q$. By (10), we have that

$$q = i - (\ell - i_{p+1} + 2).$$

Observe that for all $0 \leqslant j \leqslant h_{p+1}$, there exists a south-east corner (n_j, λ_{n_j}) in or above the rectangle R_{p+1} such that

$$\lambda_{n_j} = i_{p+1} - \sum_{f=0}^{j} a_{p+1,f}.$$

So there exists a south-east corner (n_j, λ_{n_j}) such that

$$\lambda_{n_j} = i_{p+1} - (q+1) = \ell - i + 1.$$

The element of Φ_{min} which corresponds to the cell (n_j, λ_{n_j}) is $\alpha_{n_j,i}$, so we have $\alpha_{i,j} \in L$. □

It follows by Proposition 4.1 and Equation (5) that we have the following theorem.

Theorem 5.3. *There is a bijection between the elements* $\Phi \in \mathcal{F}_\emptyset$ *such that* $\sharp I_\Phi = r$ *and the Dyck paths of length* $2\ell + 2$ *having* r *occurrences of "udu".*

Since the number of Dyck paths having a fixed number of occurrences of *udu* is calculated in Theorem 2.1 of [Sun], we have the following corollary.

Corollary 5.4. *The number of elements of* $\Phi \in \mathcal{F}_\emptyset$ *such that* $\sharp I_\Phi = r$ *is*

$$\binom{\ell}{r} \sum_{k=0}^{[\ell - r/2]} \binom{\ell - r}{2k} \mathcal{C}_k$$

where $\mathcal{C}_k$ *denotes the* k*-th Catalan number.*

Example 5.5. *Let* N_r^ℓ *be the number of elements* $\Phi \in \mathcal{F}_\emptyset$ *such that* $\sharp I_\Phi = r$*. We have by Corollary 5.4:*

r	N_r^1	N_r^2	N_r^3	N_r^4	N_r^5
0	1	2	4	9	21
1	1	2	6	16	45
2		1	3	12	40
3			1	4	20
4				1	5
5					1

6. Duality

We shall construct a duality between the elements of $\mathcal{F}_\emptyset$ such that $\sharp\Phi_{min} = p$ and those such that $\sharp\Phi_{min} = \ell - p$.

Proposition 6.1. *Let $\Phi \in \mathcal{F}_\emptyset$. Let N be the number of peaks in $(D \circ \sigma)(\Phi)$, then we have*

$$\sharp\Phi_{min} = \ell - (N - 1).$$

Proof. Let $\lambda = \sigma(\Phi)$ be the corresponding ℓ-partition. Recall that the construction of $D(\lambda)$ is iterative. At each step, when we add $a_{p,q}$ peaks to a highest peak, for $(p, q) \in \mathcal{A}_p$, we also "destroy" the initial highest peak. So, we add only $a_{p,q} - 1$ peaks. At the end of the construction we have

$$\ell - \lambda_1 + 1 + \sum_{p=1}^{k} \sum_{(p,q)\in\mathcal{A}_p} (a_{p,q} - 1)$$

peaks. Since $\sum_{p=1}^{k} \sum_{(p,q)\in\mathcal{A}_p} a_{p,q} = \sum_{(p,q)\in\mathcal{A}} a_{p,q} = \lambda_1$ and $\mathcal{A}$ is in bijection with Φ_{min} by Section 5, we obtain the result. □

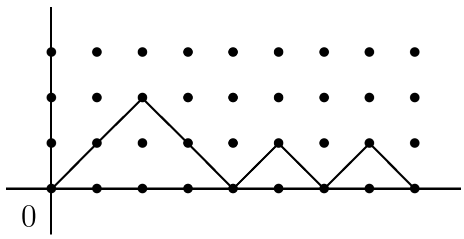
Proposition 6.2. *Let $\Phi \in \mathcal{F}_\emptyset$ and p be the number of peaks in $(P \circ \sigma)(\Phi)$, then we have*

$$\sharp\Phi_{min} = p - 1.$$

Proof. The result is clear by the construction of $(P \circ \sigma)(\Phi)$ defined in Section 2. □

Theorem 6.3. *The map $\sigma^{-1} \circ P^{-1} \circ D \circ \sigma$ induces a bijection from $\mathcal{F}_\emptyset$ to $\mathcal{F}_\emptyset$ which sends $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp\Phi_{min} = p$ to $\Psi \in \mathcal{F}_\emptyset$ such that $\sharp\Psi_{min} = \ell - p$.*

For example, in $sl_4(\mathbb{C})$, the element $\Phi = \{\theta\} \in \mathcal{F}_\emptyset$ corresponds to the partition $\lambda = (1, 0, 0)$, and the Dyck path D_λ is:



Then, $P^{-1}(D_\lambda) = (3, 2, 0)$ which is the partition which corresponds to Ψ such that $\Psi_{min} = \{\alpha_1, \alpha_2\}$.

Remark 6.4. *It was proved in* [Pa] *that when $\mathfrak{g}$ is a simple Lie algebra of type A or C, the number of elements $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp\Phi_{min} = p$ is the same as the number of elements $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp\Phi_{min} = \ell - p$. But the duality of* [Pa] *is not the same as the one defined above. For example, in $sl_4(\mathbb{C})$, if we consider $\Phi = \{\theta\}$ like above, the dual ideal defined by* [Pa] *is Ψ where $\Psi_{min} = \{\alpha_1 + \alpha_2, \alpha_3\}$.*

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