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Noncommutative hypergeometric and basic hypergeometric differential equations over an abstract unital Banach algebra

joint work with Michael Schlosser

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Known results

$$ {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(A)_n\,(B)_n}{(C)_n} \frac{z^n}{n!} $$

where

$$ (\alpha)_0 = 1, \qquad (\alpha)_n = \prod_{j=0}^{n-1} (\alpha + j) $$

Proposition 1 (Euler)**.**

$$ F(z) = {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix} ; z\right] F_0 $$

is the unique solution analytic at $z = 0$ *of the hypergeometric differential equation*

$$ z(1-z)F''(z) + (C - z(1 + A + B))F'(z) \\ - ABF(z) = 0, $$

where $F(0) = F_0$.

Notation

Let R be a unital Banach algebra with norm $\|\cdot\|$, and identity I and zero element O.

Noncommutative product ($\forall m \geq l-1 \in \mathbb{N}$)

$$\prod_{j=l}^{m} A_j = \begin{cases} I & m = l-1 \\ A_l A_{l+1} \cdots A_m & m \geq l. \end{cases}$$

Noncommutative shifted factorial of type I $(\forall k, r \in \mathbb{N})$

$$\left\lceil \begin{matrix} A_1, A_2, \ldots, A_r \\ C_1, C_2, \ldots, C_r \end{matrix} ; Z \right\rceil_k :=$$

$$\prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (C_i + (k-j)I)^{-1} (A_i + (k-j)I) \right) Z \right]$$

noncommutative shifted factorial of type II $(\forall k, r \in \mathbb{N})$

$$\left\lfloor \begin{matrix} A_1, A_2, \ldots, A_r \\ C_1, C_2, \ldots, C_r \end{matrix} ; Z \right\rfloor_k :=$$

$$\prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (C_i + (j-1)I)^{-1} (A_i + (j-1)I) \right) Z \right].$$

Noncommutative hypergeometric series of type I

$$
{}_{r+1}F_r \left\lceil \begin{matrix} A_1, A_2, \ldots, A_{r+1} \\ C_1, C_2, \ldots, C_r \end{matrix} ; Z \right\rfloor :=
$$

$$
\sum_{k \geq 0} \left\lceil \begin{matrix} A_1, A_2, \ldots, A_{r+1} \\ C_1, C_2, \ldots, C_r, I \end{matrix} ; Z \right\rfloor_k
$$

Noncommutative hypergeometric series of type II

$$
{}_{r+1}F_r \left\lfloor \begin{matrix} A_1, A_2, \ldots, A_{r+1} \\ C_1, C_2, \ldots, C_r \end{matrix} ; Z \right\rceil :=
$$

$$
\sum_{k \geq 0} \left\lfloor \begin{matrix} A_1, A_2, \ldots, A_{r+1} \\ C_1, C_2, \ldots, C_r, I \end{matrix} ; Z \right\rceil_k
$$

The series terminates if one of the upper parameters A_i is of the form $-nI$. If the series is nonterminating, then the series converges in R if $\|Z\| < 1$.

Given an identity $E \in R$, we get a new one $^{\sim}E$ by simply reversing all the products (e.g. if $R = M_{n \times n}(\mathbb{K})$ it is the transposition of matrices).

For instance,

$$^{\sim}\left[\begin{matrix} A_1, A_2, \ldots, A_r \\ C_1, C_2, \ldots, C_r \end{matrix}; Z\right]_k = \prod_{j=1}^{k} \left(Z \prod_{i=1}^{r} (A_i + (j-1)I)(C_i + (j-1)I)^{-1} \right).$$

From now on, let always $Z \in \{X \in R : XY = YX,\ \forall Y \in R\}$ with $\|Z\| < 1$.

Type I noncommutative hypergeometric equations

Theorem 2.

$$F(Z) = {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] F_0$$

is the unique solution analytic at $Z = O$ of the noncommutative hypergeometric equation

$$Z(I - Z)F''(Z) + (C - Z(I + A + B))F'(Z)$$
$$- ABF(Z) = O,$$

where $F(O) = F_0$.

Type II noncommutative hypergeometric equations

Theorem 3. *Let* $C(C-A-B)+AB$ *be invertible.*
Then

$$F(Z) = F_0\ {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right]$$

is the unique solution analytic at $Z=O$ *of the noncommutative hypergeometric equation*

$$Z(I-Z)F''(Z) + ZF'(Z)(C-I-A-B)$$
$$+\big((I-Z)F'(Z) - F(Z)C^{-1}AB\big)\cdot$$
$$\big(C(C-A-B)+AB\big)^{-1}C\big(C(C-A-B)+AB\big)$$
$$= O,$$

where $F(O) = F_0$.

Known results over basic hypergeometric series

Let $0 < |q| < 1$.

Define

$$(a,q)_\infty := \prod_{j\geq 0}\left(1-aq^j\right), \qquad (a,q)_k = \frac{(a,q)_\infty}{(aq^k,q)_\infty}.$$

Therefore

$$(a,q)_k = \prod_{j=0}^{k-1}\left(1-aq^j\right) \qquad \text{if } k \in \mathbb{N}.$$

$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] := \sum_{k\geq 0} \frac{(a,q)_k\,(b,q)_k}{(c,q)_k\,(q,q)_k} z^k$$

satisfies a differential equation of the second order.

Let Q be a parameter which commutes with any of the other parameters appearing in the series, e.g. $Q = qI$.

Noncommutative Q-shifted factorial of type I $(\forall k, r \in \mathbb{N})$

$$\left\lceil \begin{matrix} A_1, A_2, \ldots, A_r \\ C_1, C_2, \ldots, C_r \end{matrix} ; Q, Z \right\rfloor_k :=$$

$$\prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (I - C_i Q^{k-j})^{-1} (I - A_i Q^{k-j}) \right) Z \right]$$

Noncommutative Q-shifted factorial of type II

$$\left\lfloor \begin{matrix} A_1, A_2, \ldots, A_r \\ C_1, C_2, \ldots, C_r \end{matrix} ; Q, Z \right\rceil_k :=$$

$$\prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (I - C_i Q^{j-1})^{-1} (I - A_i Q^{j-1}) \right) Z \right]$$

Noncommutative basic (or Q) hypergeometric series of type I

$$
{}_{r+1}\phi_r \left\lceil \begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r \end{matrix} ; Q, Z \right\rfloor :=
$$

$$
\sum_{k \geq 0} \left\lceil \begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r, Q \end{matrix} ; Q, Z \right\rfloor_k
$$

Noncommutative basic (or Q) hypergeometric series of type II

$$
{}_{r+1}\phi_r \left\lfloor \begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r \end{matrix} ; Q, Z \right\rceil :=
$$

$$
\sum_{k \geq 0} \left\lfloor \begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r, Q \end{matrix} ; Q, Z \right\rceil_k
$$

The series terminates if one of the upper parameters A_i is of the form Q^{-n}. If the series does not terminate, then it converges if $\|Z\| < 1$.

Type I and type II noncommutative basic hypergeometric equations

Q-difference operator $\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}$

$$\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}F(Z) = (I-Q)^{-1}Z^{-1}(F(Z)-F(QZ)).$$

Remarks:

$$\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}Z^k = (I-Q)^{-1}(I-Q^k)Z^{k-1}$$

$$\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}\left(ZF\left(Z\right)\right) = F\left(Z\right) + QZ\frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}F\left(Z\right)$$

$$\lim_{Q\to I}\ \frac{\mathrm{d}_Q}{\mathrm{d}_Q Z} = \frac{\mathrm{d}}{\mathrm{d}\,Z}$$

the standard differentiation operator

Theorem 4.

$$F(Z) = {}_2\phi_1\left[\begin{matrix} A, B \\ C \end{matrix}; Q, Z\right] F_0$$

is the unique solution analytic at $Z = O$ of the noncommutative basic hypergeometric equation

$$Z(C - ABQZ)\frac{\mathrm{d}_Q^2}{\mathrm{d}_Q Z^2}F(Z) + (I - Q)^{-1}$$
$$\cdot \big[(I - C) + (I - A)(I - B)Z - (I - ABQ)Z\big]$$
$$\cdot \frac{\mathrm{d}_Q}{\mathrm{d}_Q Z}F(Z)$$
$$- (I - Q)^{-2}(I - A)(I - B)F(Z) = O,$$

where $F(O) = F_0$.

Let $\widetilde{\frac{\mathrm{d}_Q}{\mathrm{d}_Q\,Z}}$ be the Q-difference operator acting from the *right* on functions over R, i.e.

$$F(Z)\frac{\widetilde{\mathrm{d}_Q}}{\mathrm{d}_Q\,Z} = {}^{\sim}\!\left(\frac{\mathrm{d}_Q}{\mathrm{d}_Q\,Z}\left({}^{\sim}F(Z)\right)\right)$$

Let C and $(I - C^{-1}A - C^{-1}(I - C^{-1}A)B)$ be invertible. Then

$$F(Z) = {}_2\phi_1\left[\begin{matrix}A, B\\ C\end{matrix}; Q, Z\right] F_0$$

is the unique solution analytic at $Z = O$ of the noncommutative basic hypergeometric equation

$$
\begin{aligned}
&F(Z)\frac{\widetilde{\mathrm{d}^2_Q}}{\mathrm{d}_Q\, Z^2} Z(I - C^{-1}ABQZ)\cdot \\
&(I - C^{-1}A - C^{-1}(I - C^{-1}A)B)^{-1}C\cdot \\
&(I - C^{-1}A - C^{-1}(I - C^{-1}A)B) \\
&+ F(Z)\frac{\widetilde{\mathrm{d}_Q}}{\mathrm{d}_Q\, Z}(I - Q)^{-1}\cdot \\
&(I - C^{-1}A - C^{-1}(I - C^{-1}A)B)^{-1}(I - C)\cdot \\
&(I - C^{-1}A - C^{-1}(I - C^{-1}A)B) \\
&+ F(Z)\frac{\widetilde{\mathrm{d}_Q}}{\mathrm{d}_Q\, Z} Z(I - Q)^{-1}\Big[C - A - B + (C^{-1}AB + C^{-1}ABQ - I)\cdot \\
&(I - C^{-1}A - C^{-1}(I - C^{-1}A)B)^{-1}C\cdot \\
&(I - C^{-1}A - C^{-1}(I - C^{-1}A)B)\Big] \\
&+ F(Z)(I - Q)^{-2}\Big[A + B - C - I - (C^{-1}AB - I)\cdot \\
&(I - C^{-1}A - C^{-1}(I - C^{-1}A)B)^{-1}C\cdot \\
&(I - C^{-1}A - C^{-1}(I - C^{-1}A)B)\Big] \\
&= O,
\end{aligned}
$$

where $F(O) = F_0$.