A bijection between non-crossing and non-nesting partitions of types A and B

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Overview

Classical non-crossing and non-nesting partitions

Motivation

Non-crossing partitions

Non-nesting partitions

A bijection between $NN(W)$ and $NC(W)$ in types A and B

Non-crossing and non-nesting set partitions

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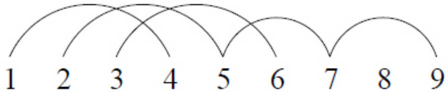
Example

$\mathcal{B} = \{\{1,4\},\{2,5,7,9\},\{3,6\},\{8\}\} \vdash [9]$:



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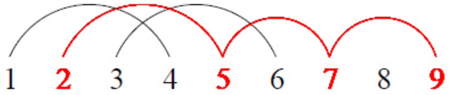

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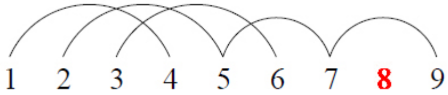

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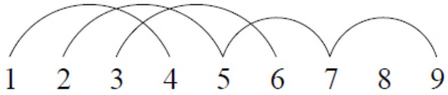

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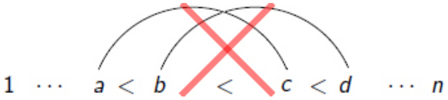

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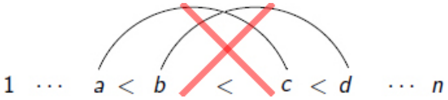

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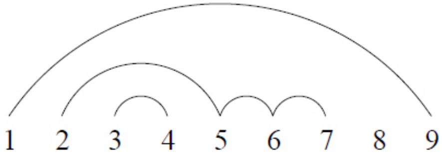

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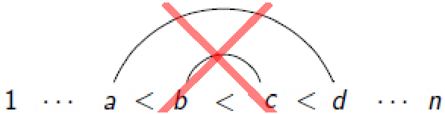

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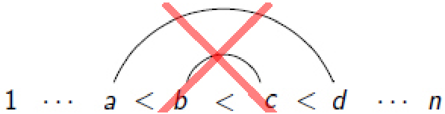

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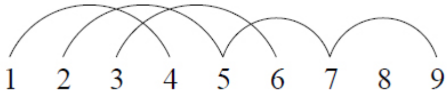

Non-crossing and non-nesting partitions

We will later see that non-crossing and non-nesting set partitions can be seen as the type A instances of more general constructions:

- **non-crossing partitions** $NC(W)$, attached to any *real reflection group* W (Reiner), and furthermore to any *well-generated complex reflection group* W (Bessis),

- **non-nesting partitions** $NN(W)$ attached to any *crystallographic real reflection group* (Postnikov).

General motivation

Both constructions seem to be related enumeratively in a very deep way, in particular

- both are counted by the *Catalan numbers*,
- both have a *positive part* which is counted by the *positive Catalan numbers*,
- both have a refinement which is counted by the *Narayana numbers*,

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Question / Open Problem

What is nature of the relationship between $NC(W)$ and $NN(W)$? Find a bijection between $NC(W)$ and $NN(W)$ that preserve "natural" statistics.

My motivation

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- its *bigraded Hilbert series* $\mathcal{H}(M; q, t)$ is a natural q, t-analogue of the Catalan numbers,
- the specialization $t = 1$ is conjectured to be counted by a certain statistic area on $NN(W)$,

$$\mathcal{H}(M; q, 1) = \sum_{I \in NN(W)} q^{\text{area}(I)}.$$

My personal motivation

Open Problem

Find a second statistic tstat on $NN(W)$ that describes those q,t-Catalan numbers combinatorially,

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Hope

A bijection between $NC(W)$ and $NN(W)$ for which some of those statistics can be "nicely" described in terms of $NC(W)$ could shade some light on this open problem.

Non-crossing set partitions and the symmetric group

When we order the elements in each block of a non-crossing set partition $\mathcal{B}$ increasingly, we can identify $\mathcal{B}$ with the permutation σ having cycles equal to the blocks of $\mathcal{B}$.

Example

$$\begin{array}{rcl} [n] \dashv \mathcal{B} & = & \{\{1,9\},\{2,5,6,7\},\{3,4\},\{8\}\} \\ \downarrow & & \\ \mathcal{S}_n \ni \sigma & = & (1,9)(2,5,6,7)(3,4) \\ & = & [9,5,4,3,6,7,2,8,1]. \end{array}$$

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- The image of this embedding is the set of all permutations which have
  - only increasing cycles,
  - no "crossing" cycles in the sense described above.

The absolute order on the symmetric group

Definition

For a permutation σ, let the **absolute length** $l_T(\sigma)$ be the minimal integer k such that σ can be written as the product of k *transpositions*,

$$l_T(\sigma) := \min\{k : \sigma = t_1 \cdots t_k, \text{ for transpositions } t_i\}.$$

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Theorem (Reiner)

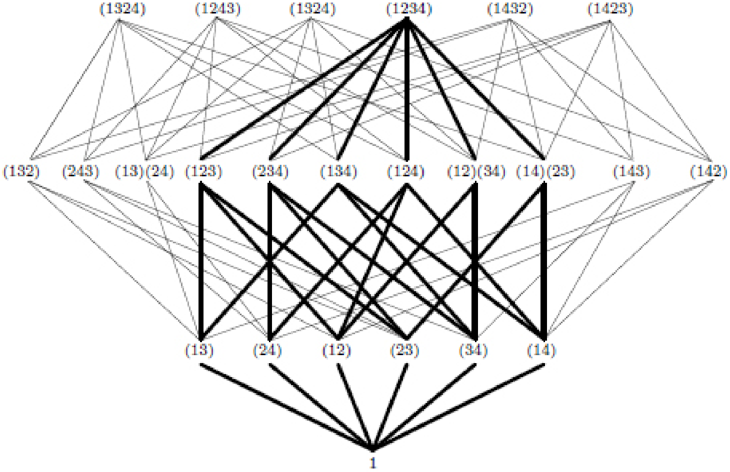
$\sigma \in \mathcal{S}_n$ is non-crossing if and only if

$$\sigma \leq_T (1, 2, \ldots, n) =: c \Leftrightarrow \sigma \in [1, c].$$

The non-crossing partition lattice of type A_3

Example

For $W = \mathcal{S}_4$, $NC(W) \subseteq \mathcal{S}_n$:



The absolute order for any real reflection group

Definition

Let W be a real reflection group (= *Coxeter group*) and let $\omega \in W$. Let its **absolute length** $l_T(\omega)$ be the minimal integer k such that ω can be written as the product of k *reflections*,

$$l_T(\omega) := \min\{k : \sigma = t_1 \cdots t_k, \text{ for reflections } t_i\}.$$

The **absolute order** on W is defined by

$$\omega \leq_T \omega' :\Leftrightarrow l_T(\omega') = l_T(\omega) + l_T(\omega^{-1}\omega').$$

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In the absolute order, the intervall $[1, c]$ does *not* depend on the specific choice of the Coxeter element c,

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Definition

Fix a real reflection group W and a Coxeter element c. The **non-crossing partition lattice** associated to W is defined as

$$NC(W) := [1, c].$$

- For any Coxeter element c, the intervall $[1, c]$ is a *lattice* with many nice properties.

Enumeration of $NC(W)$

Theorem (Reiner, Bessis-Reiner)

Let W be a real reflection group. Then non-crossing partitions are counted by the W-Catalan numbers,

$$\#NC(W) = \mathrm{Cat}(W) := \prod_{i=1}^{l} \frac{d_i + h}{d_i},$$

where

- *l is the number of simple reflections in W,*
- *h is the Coxeter number,*
- *$d_1, \ldots, d_l$ are the degrees of the fundamental invariants.*

Cat(W) for all irreducible real reflection groups

A_{n-1}	B_n	D_n
$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n}{n}$	$\binom{2n}{n} - \binom{2n-2}{n-1}$

$I_2(m)$	H_3	H_4	F_4	E_6	E_7	E_8
$m+2$	32	280	105	833	4160	25080

The root poset

Definition

Let W be a crystallographic reflection group with associated root system $\Phi \subseteq \mathbb{R}^l$ and let $\Delta \subseteq \Phi^+ \subseteq \Phi$ be a *simple system* and a *positive system* respectively.
Define a partial order on Φ^+ by the *covering relation*

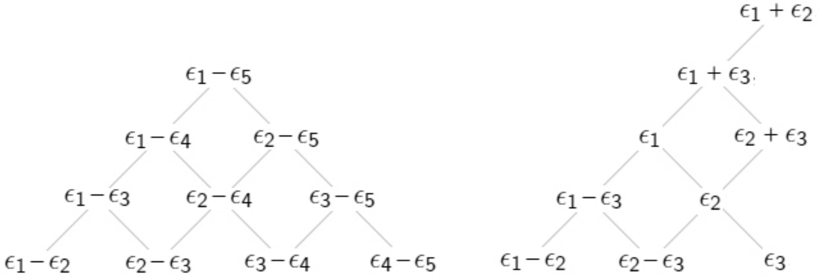
$$\alpha \prec \beta :\Leftrightarrow \beta - \alpha \in \Delta.$$

Equipped with this partial order, Φ^+ is the **root poset** associated to W.

The root poset

Example

The root posets of type A_4 and of type B_3:



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Non-nesting set partitions of $[n]$ are in bijection with *anti-chains* in the root poset of type A_{n-1}.

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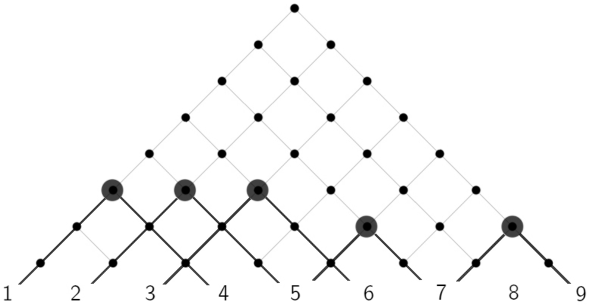

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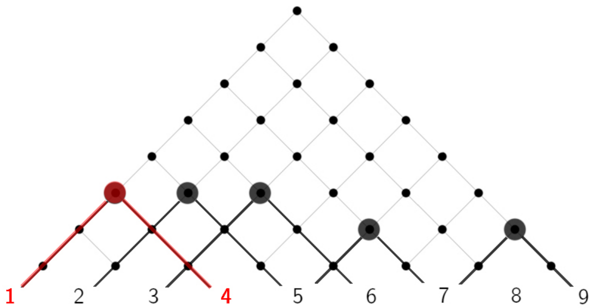

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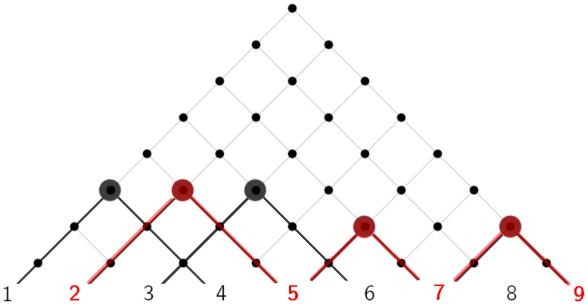

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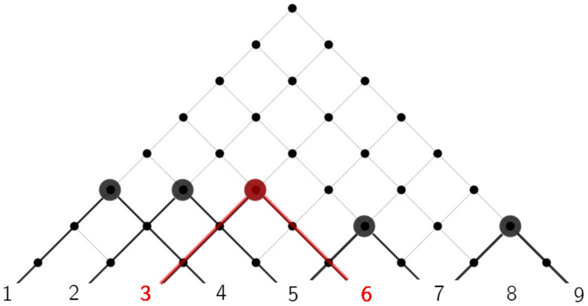

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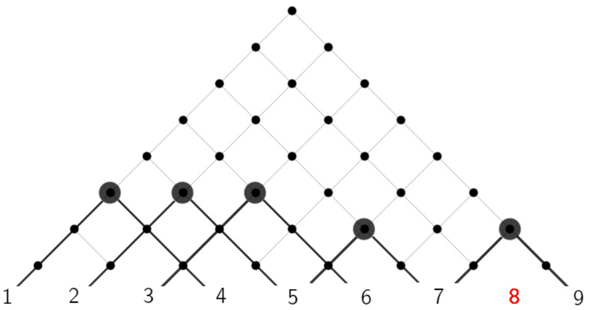

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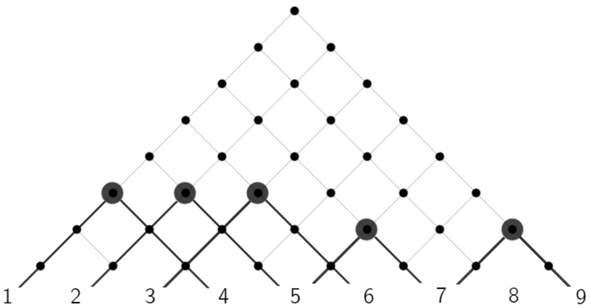

Non-nesting partitions

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Fix a crystallographic reflection group W with associated root poset Φ^+. An antichain in Φ^+ is called **non-nesting partition**,

$$NN(W) := \{\text{non-nesting partitions } A \subseteq \Phi^+\}.$$

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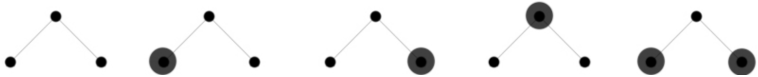
Example

The 5 antichains in the root poset of type A_2:



Enumeration of $NN(W)$

Theorem (Athanasiadis)

Let W be a crystallographic reflection group. Then

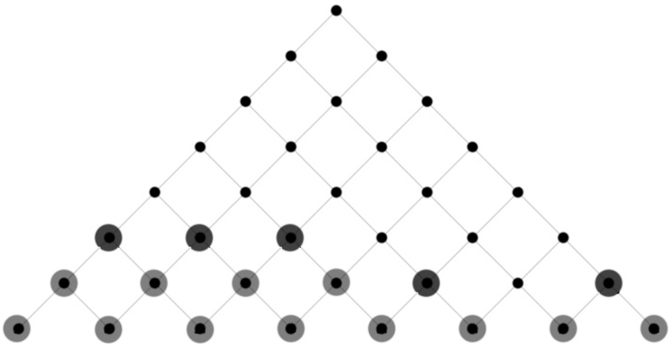
$$\#NN(W) = \mathrm{Cat}(W) = \#NC(W).$$

The area statistic on $NN(W)$

To an antichain $A \subseteq \Phi^+$, define the **area statistic** as area$(A) := \#I_A$, where

$$I_A := \{\alpha \in \Phi^+ : \alpha \leq \beta \text{ for some } \beta \in A\}.$$

Example (continued)



area$(A) = 17$

Back to the module from the beginning

Conjecture

$$\mathcal{H}(M;q,1) = \sum_{A \in NN(W)} q^{\mathrm{area}(A)}.$$

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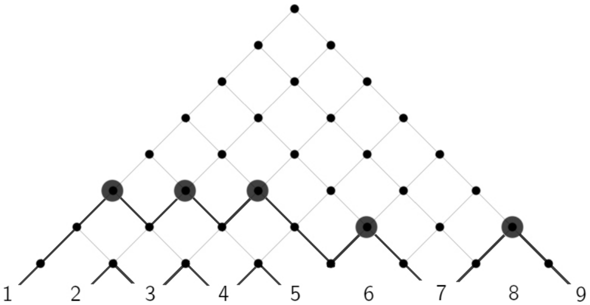
In type A the conjecture is known to be true and furthermore *MacMahon's Major index* maj on the associated Dyck path provides

$$\begin{aligned} q^N \mathcal{H}(M;q,q^{-1}) &= \sum_{A \in NN(W)} q^{\mathrm{maj}(A)} \\ &= \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \prod_{i=1}^{l} \frac{[d_i + h]_q}{[d_i]_q}. \end{aligned}$$

where N is the number of *positive roots* (Garsia & Haiman).

The bijection in type A

Define ϕ to be the map from $NN(\mathcal{S}_n)$ to $NC(\mathcal{S}_n) = [1, (1, \ldots, n)]$ by the rule shown in the following picture:



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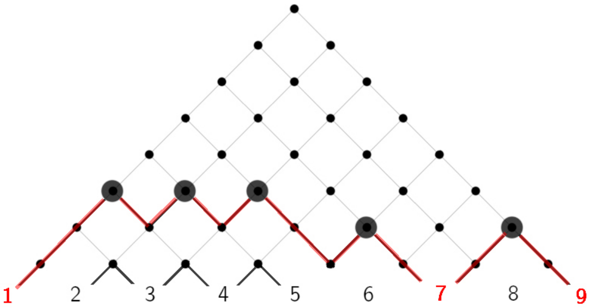

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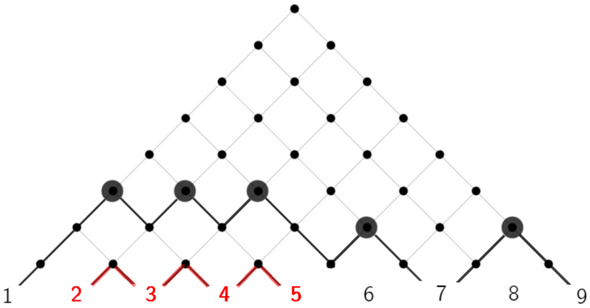

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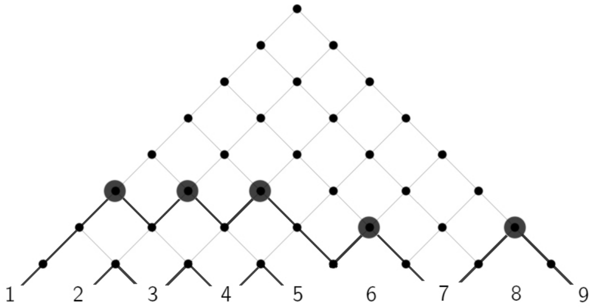

$$\downarrow$$

$$(1, 7, 9)(2, 3, 4, 5) \in NC(S_n)$$

Properties of the bijection

Theorem

The map ϕ is a bijection between $NN(\mathcal{S}_n)$ and $NC(\mathcal{S}_n)$ which sends

- area *to the (ordinary) length function* l_S *in the Coxeter group of type* A,
- maj *to* $2N - \mathrm{maj} - \mathrm{imaj}$, *where* maj *is the Major index of a permutation and* imaj *the Major index of the inverse permutation.*

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Theorem

The map ϕ *is a bijection between* $NN(\mathcal{S}_n)$ *and* $NC(\mathcal{S}_n)$ *which sends*

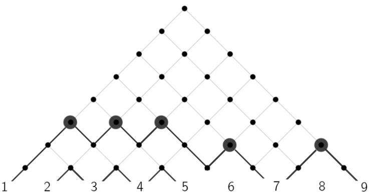
- area *to the (ordinary) length function* l_S *in the Coxeter group of type* A*,*
- maj *to* $2N - \mathrm{maj} - \mathrm{imaj}$*, where* maj *is the Major index of a permutation and* imaj *the Major index of the inverse permutation.*

Corollary

$$\begin{aligned} \mathcal{H}(M; q, 1) &= \sum_{\sigma \in NC(W)} q^{\mathrm{inv}(\sigma)}, \\ q^N \mathcal{H}(M; q, q^{-1}) &= \sum_{\sigma \in NC(W)} q^{\mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma)}. \end{aligned}$$

Properties of the bijection

Example (continued)



$$\mapsto$$

$$(1,7,9)(2,3,4,5) = [7,3,4,5,2,6,9,8,1] \in NC(S_n)$$

$$\mathrm{l}_S = \mathrm{inv} = \#\left\{ \begin{array}{c} (7,3),(7,4),(7,5),(7,2),(7,6),(7,1), \\ (3,2),(3,1),(4,2),(4,1),(5,2),(5,1), \\ (2,1),(6,1),(9,8),(9,1),(8,1) \end{array} \right\} = 17$$

The bijection in type B

We can also define an bijection between $NN(W)$ and $NC(W) = [1, c]$ where c is a certain Coxeter element with the following properties:

- it sends the statistic area *almost* to the (ordinary) length function l_S in the Coxeter group of type B,
- it sends the statistic maj *almost* to $2N - \text{maj} - \text{imaj}$, where maj is *almost* the f-Major index of a signed permutation (Adin, Roichman) and imaj is the f-Major index of the inverse signed permutation.

The bijection in type B

Corollary

$$q^N \mathcal{H}(M; q, q^{-1}) = \sum_{\sigma \in NC(W)} q^{\mathrm{maj}(\mathrm{rev}(\sigma)) + \mathrm{imaj}(\mathrm{rev}(\sigma))}$$

$$= \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} = \prod_{i=1}^{l} \frac{[d_i + h]_q}{[d_i]_q}.$$

Conjecture

$$\mathcal{H}(M; q, 1) = \sum_{\sigma \in NC(W)} q^{\mathrm{l}_S(\mathrm{rev}(\sigma))}.$$

A hope in type B and a remark on type D

Hope

In type B, this bijection equips $NN(W)$ with more structure and we hope that this could help to find a statistic tstat on $NN(W)$ to describe the whole Hilbert series of the W-module M in this type.

Remark

As the involution rev makes the situation much more difficult we were so far not able to find an analogous bijection in type D.