Combinatorial aspects of supercharacter theories of the unitriangular groups

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UNITRIANGULAR GROUP $= 1 +$ NILTRIANGULAR ALGEBRA

$$\begin{bmatrix} 1 & \star & \star & \cdots & \star \\ & 1 & \star & \cdots & \star \\ & & 1 & \cdots & \star \\ & & & \ddots & \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \end{bmatrix} + \begin{bmatrix} 0 & \star & \star & \cdots & \star \\ & 0 & \star & \cdots & \star \\ & & 0 & \cdots & \star \\ & & & \ddots & \end{bmatrix}$$

- *(number of) conjugacy classes?*
- *(number of) irreducible characters?*

(These are “wild” problems.)

Supercharacters and superclasses

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Roughly, a “SUPERCHARACTER THEORY” replaces

IRREDUCIBLE CHARACTERS	$\longrightarrow$	“SUPERCHARACTERS”
CONJUGACY CLASSES	$\longrightarrow$	“SUPERCLASSES”
CHARACTER TABLE	$\longrightarrow$	“SUPERCHARACTER TABLE”

A SUPERCHARACTER THEORY of a finite group G consists of

- a *partition* $\mathcal{K}$ *of* G
- a *set* $\mathcal{X}$ *of (complex) characters of* G

satisfying the following three *axioms*:

1. $|\mathcal{K}| = |\mathcal{X}|$;
2. *every irreducible character of* G *is a constituent of a unique* $\xi \in \mathcal{X}$;
3. *the characters in* $\mathcal{X}$ *are constant on the members of* $\mathcal{K}$.

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- The elements of $\mathcal{K}$ are called the *superclasses* of G,
- the elements of $\mathcal{X}$ are called the *supercharacters* of G.

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For some groups these are the only possibilities, there are many groups for which nontrivial supercharacter theories exist.

In many cases it may be possible to obtain useful information using some particular supercharacter theory. For instance, E. Arias-Castro, P. Diaconis and R. Stanley showed that a special supercharacter theory can be applied to study a *random walk on uppertriangular matrices over finite fields* using techniques that traditionally required the knowledge of the full character theory.

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EXAMPLE (UNITRIANGULAR GROUP).

- $A = \mathfrak{b}_n(q) = \{$*uppertriangular* $n \times n$ *matrices over* $\mathbb{F}_q\}$;
- $J = \mathfrak{u}_n(q) = \{$*nilpotent uppertriangular matrices over* $\mathbb{F}_q\}$;
- $G = 1 + J = U_n(q) = \{$*unipotent uppertriangular matrices over* $\mathbb{F}_q\}$.

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- $\mathcal{L}(f) = \{a \in J : af = 0\}$,
- $L(f) = 1 + \mathcal{L}(f)$,
- $\vartheta_f : L(f) \to \mathbb{C}$ the map defined by

$$\vartheta_f(1 + a) = \vartheta(a) \qquad (a \in \mathcal{L}(f))$$

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- $L(f)$ is an *algebra subgroup* of $G = 1 + J$,
- ϑ_f is a *linear character* of $L(f)$.

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Theorem. *For any $f \in J^*$,*

- $\xi_f(1) = |Gf| = |Jf|$;
- $\langle \xi_f, \xi_g \rangle \neq 0 \iff g \in GfG \iff \xi_f = \xi_g$;
- $\langle \xi_f, \xi_f \rangle = |Gf \cap fG| = |Jf \cap fJ|$.

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Remark. We obtain a similar result by replacing the *left* by the *right representation* of A on J^*. In particular, $\xi_f(1) = |Jf| = |fJ|$.

It is clear that $\{GfG \colon f \in J^*\}$ is a partition of J^*. In fact, we have the following.

Theorem. *The regular character ρ_G decomposes as the (orthogonal) sum*

$$\rho_G = \sum_{GfG \subseteq J^*} m_f \, \xi_f$$

where $m_f = \frac{|Gf \cap fG|}{|Gf|} = \frac{|Gf|}{|GfG|} \quad (f \in J^*)$.

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In particular, we deduce that

Theorem. *Every irreducible character of $G = 1 + J$ is a constituent of a unique supercharacter.*

Theorem. *For all* $f \in J^*$ *and all* $a \in J$, *we have*

$$\xi_f(1+a) = \frac{\xi_f(1)}{|GfG|} \sum_{g \in GfG} \vartheta_g(a) = \frac{\xi_f(1)}{|GaG|} \sum_{b \in GaG} \vartheta_f(b).$$

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In fact, a *theorem of Brauer* implies that $|G\backslash J/G| = |G\backslash J^*/G|$.

Thus, we may define the SUPERCHARACTER TABLE of G to be the square (complex) matrix with entries given by the supercharacter values on the superclasses.

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In particular,

- The restriction of a supercharacter to an algebra subgroup of G is a $\mathbb{Z}$-linear combination of supercharacters.
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REMARK. In general, induction of supercharacters is not a superclass function. However, it is possible to define a SUPERINDUCTION of superclass functions (via Frobenius reciprocity).

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Let

- $e_{i,j}$ the elementary matrix with 1 in position (i,j) and 0's elsewhere.
- $\{e_{i,j} : 1 \leq i < j \leq n\}$ the canonical basis of $\mathfrak{u}_n(q)$.
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The *supercharacters* and *superclasses* of $U_n(q)$ are in one-to-one correspondence with pairs (D, ϕ) where D is a BASIC SUBSET of

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A subset $D \subseteq \Phi(n)$ is said to be BASIC if D has *at most one* entry from each row and each column.

- The supercharacter $\xi_{D,\phi}$ corresponding to the basic pair (D,ϕ) is defined by the linear function

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- The superclass $K_{D,\phi}$ corresponding to the basic pair (D, ϕ) is defined by the matrix

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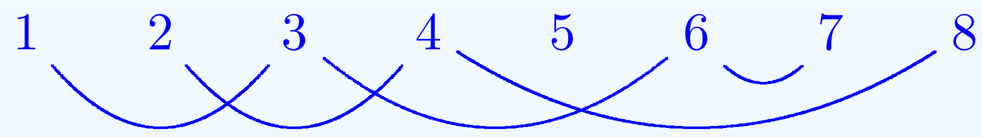
which we represent by “arcs”:



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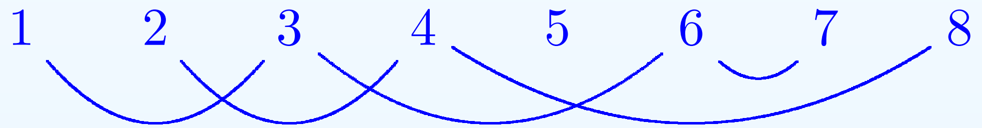

Theorem. *Let $D \subseteq \Phi(n)$ be a basic subset and let $\phi\colon D \to \mathbb{F}_q^\times$ be any map. Then,*

$$\langle \xi_{D,\phi}, \xi_{D,\phi} \rangle = q^{c(D)}$$

where $c(D)$ is the number of crossings of the set partition corresponding to D.

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Example (ELEMENTARY CHARACTERS)**.** A *labelled arc* $i \underset{\alpha}{\smile} j$ (with label $\alpha \in \mathbb{F}_q^{\times}$) corresponds uniquely to an *irreducible supercharacter* $\xi_{i,j}(\alpha)$ defined by $\alpha e_{i,j}^{*} \in \mathfrak{u}_n(q)^{*}$.

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Theorem. *Given a basic subset* $D \subseteq \Phi(n)$ *and a map* $\phi\colon D \to \mathbb{F}_q^{\times}$*, the supercharacter* $\xi_{D,\phi}$ *is the product*

$$\xi_{D,\phi} = \prod_{(i,j)\in D} \xi_{i,j}(\alpha_{i,j})$$

where $\alpha_{i,j} = \phi(i,j)$*.*

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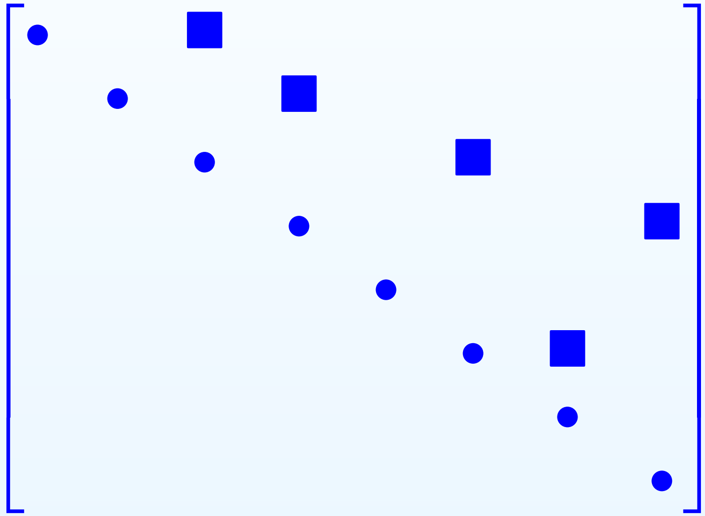

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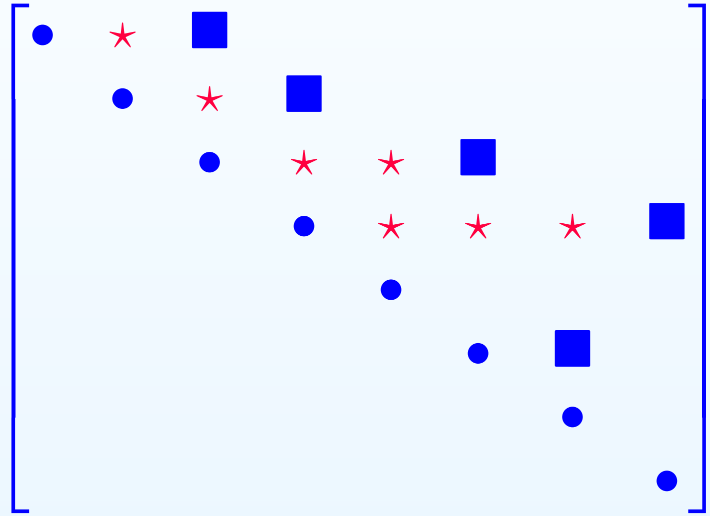

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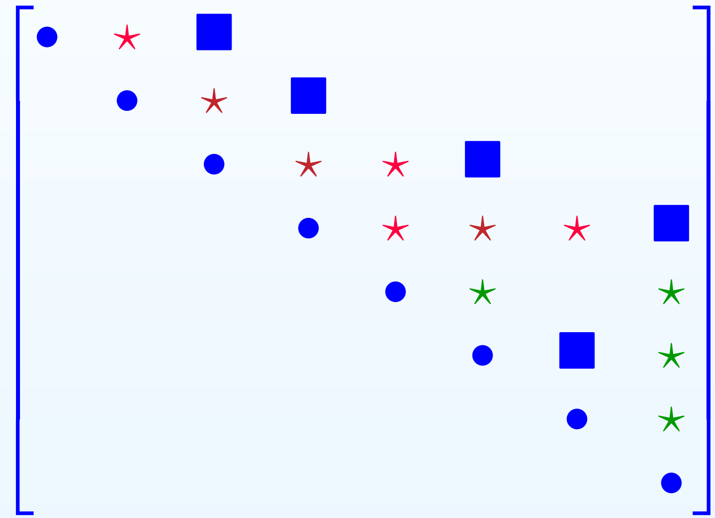

$$\langle \xi_{D,\phi}, \xi_{D,\phi} \rangle = |Je^*_{D,\phi} \cap e^*_{D,\phi}J| = q^3$$

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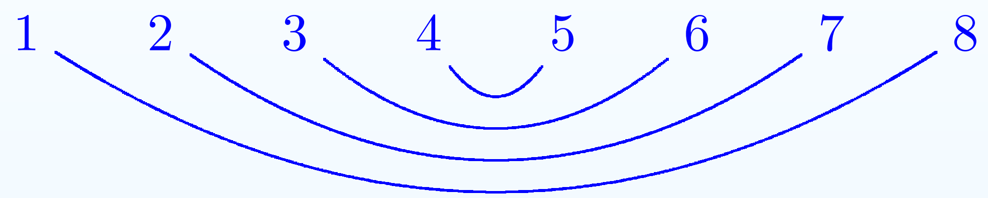

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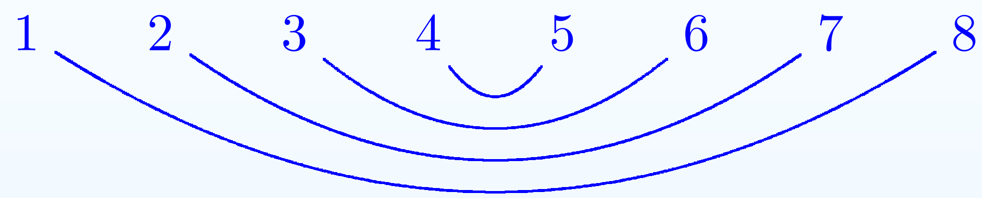

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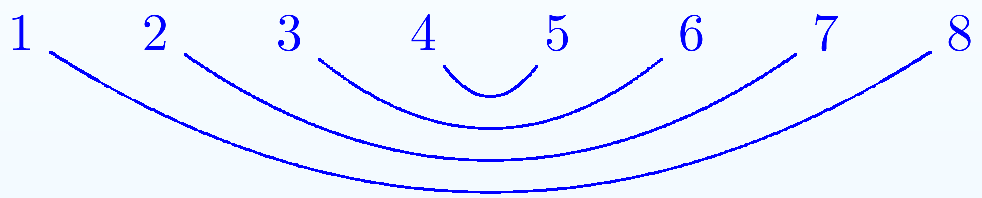

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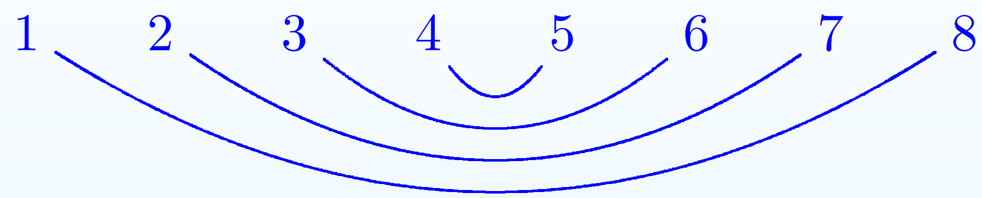

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EXERCISE. Evaluate $\xi_{D,\phi}$ at the superclass $K_{D,\phi}$, and obtain relation with the *nesting number* of the set partition corresponding to D.

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Conversely,

Theorem. *Let $f_1, \ldots, f_t \in J^*$, and suppose that $\xi_{f_1} \cdots \xi_{f_t} = \xi_f$ for some $f \in J^*$. Then, $xfy = f_1 + \cdots + f_t$ for some $x, y \in G$, and $Jf \cong Jfy = Jf_1 \oplus \cdots \oplus Jf_t$.*

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$$\begin{aligned} \xi_f = \xi_{f_1} \cdots \xi_{f_t} \quad &\Longleftrightarrow \quad Jf = Jf_1 \oplus \cdots \oplus Jf_t \\ &\Longleftrightarrow \quad fJ = f_1J \oplus \cdots \oplus f_tJ. \end{aligned}$$

“Indecomposable” supercharacters

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A special situation occurs when we consider the decomposition $Af = M_1 \oplus \cdots \oplus M_t$ of the left A-module Af into *indecomposable submodules*. Then, $M_i = Af_i$ for some $f_i \in J^*$ with $f = f_1 + \cdots + f_t$, and thus

$$Af = Af_1 \oplus \cdots \oplus Af_t.$$

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We say that a supercharacter ξ_f is *“indecomposable”* if the A-module Af is indecomposable.

Open question. Is it true that every indecomposable supercharacter is irreducible?

"Indecomposable" supercharacters

A special situation occurs when we consider the decomposition $Af = M_1 \oplus \cdots \oplus M_t$ of the left A-module Af into *indecomposable submodules*. Then, $M_i = Af_i$ for some $f_i \in J^*$ with $f = f_1 + \cdots + f_t$, and thus

$$Af = Af_1 \oplus \cdots \oplus Af_t.$$

It follows that

$$Jf = Jf_1 \oplus \cdots \oplus Jf_t \qquad \text{and} \qquad \xi_f = \xi_{f_1} \cdots \xi_{f_t}.$$

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Open question. Is it true that every indecomposable supercharacter is irreducible?

Theorem. *Let $f \in J^*$, and suppose that Af is indecomposable. Then,*

$$\mathrm{End}_A(Af) \cong \mathbb{F}_q \quad \Longrightarrow \quad \xi_f \text{ is irreducible.}$$

The unitriangular group II

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- For $f \in J^*$, there exists a unique basic subset $D \subseteq \Phi(n)$ such that

$$Af \cong \sum_{(i,j)\in D} E_{i,j} = Ae^*_D, \qquad e^*_D = \sum_{(i,j)\in D} e_{i,j}.$$

Therefore, we conclude the proof of the following.

Theorem. *Let ξ be a supercharacter of $U_n(q)$. Then, there exists a basic subset $D \subseteq \Phi(n)$ and a map $\phi\colon D \to \mathbb{F}_q^\times$ such that*

$$\xi = \xi_{D,\phi} = \prod_{(i,j)\in D} \xi_{i,j}(\alpha_{i,j})$$

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(Supercharacters of $U_n(q)$ were originally defined by the formula above and called *basic characters*. The name *supercharacters* was suggested by Roger W. Carter.)

Pattern groups

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Let $(I, \preceq)$ be a finite poset, and assume that

- $I = \{1, 2, \ldots, n\}$;
- $i \leq j \quad \Longrightarrow \quad i \preceq j$.

Let $A = A_I$ be the incidence algebra of I over $\mathbb{F}_q$ (naturally identified with a subalgebra of $\mathfrak{b}_n(q)$).

Moreover, let

$$J = J(A_I) \quad \text{and} \quad G_I = 1 + J.$$

$G = G_I$ is called the PATTERN GROUP (over $\mathbb{F}_q$) of the poset $(I, \preceq)$.

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EXAMPLE. $U_n(q)$ is the pattern group associated with the *chain*

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n$$

The incidence algebra is $\mathfrak{b}_n(q)$.

Example.

$$\begin{bmatrix} 1 & & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & 1 & & & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & & 1 & & & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & & & 1 & & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & & & & 1 & & \blacksquare & \blacksquare & \blacksquare \\ & & & & & 1 & & \blacksquare & \blacksquare \\ & & & & & & 1 & \blacksquare & \blacksquare \\ & & & & & & & 1 & \blacksquare \\ & & & & & & & & 1 \end{bmatrix}$$

A pattern group G_I

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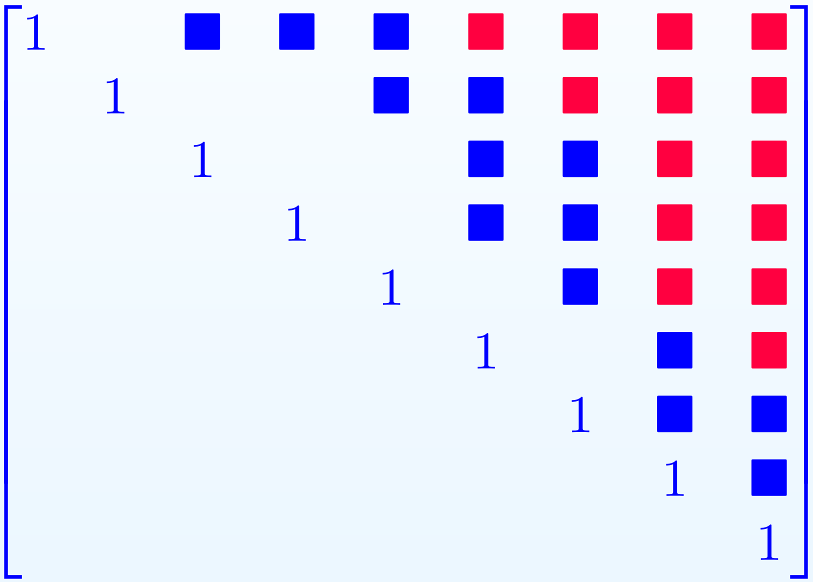
$$\begin{bmatrix} 1 & & & & & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & 1 & & & & & \blacksquare & \blacksquare & \blacksquare \\ & & 1 & & & & & \blacksquare & \blacksquare \\ & & & 1 & & & & \blacksquare & \blacksquare \\ & & & & 1 & & & \blacksquare & \blacksquare \\ & & & & & 1 & & & \blacksquare \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}$$

A pattern group G_J

Example.



G_J is a normal subgroup of G_I

Example.

$$\begin{bmatrix}
1 & & \blacksquare & \blacksquare & \blacksquare & & & & \\
 & 1 & & & \blacksquare & \blacksquare & & & \\
 & & 1 & & & \blacksquare & \blacksquare & & \\
 & & & 1 & & \blacksquare & \blacksquare & & \\
 & & & & 1 & & \blacksquare & & \\
 & & & & & 1 & & \blacksquare & \\
 & & & & & & 1 & \blacksquare & \blacksquare \\
 & & & & & & & 1 & \blacksquare \\
 & & & & & & & & 1
\end{bmatrix}$$

The algebra group G_I/G_J

We expect to use (combinatorial) methods from the *theory of quiver representations* (more precisely, *linear representations of posets*) to obtain results about supercharacters of finite pattern groups... (work in progress)...