Isometry classes of Generalized Associahedra

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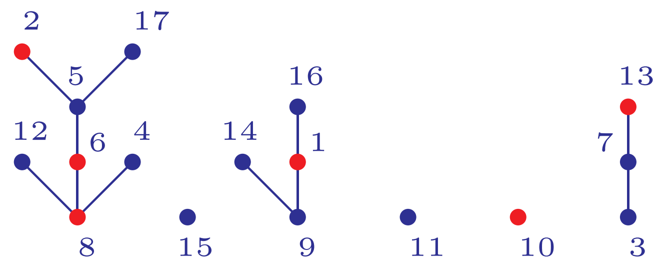
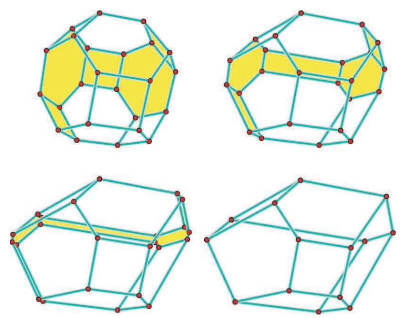
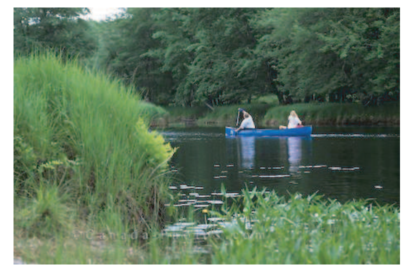
(joint work with **C. Hohlweg**, **C. Lange** and **H. Thomas**)

Fields Institute Workshop

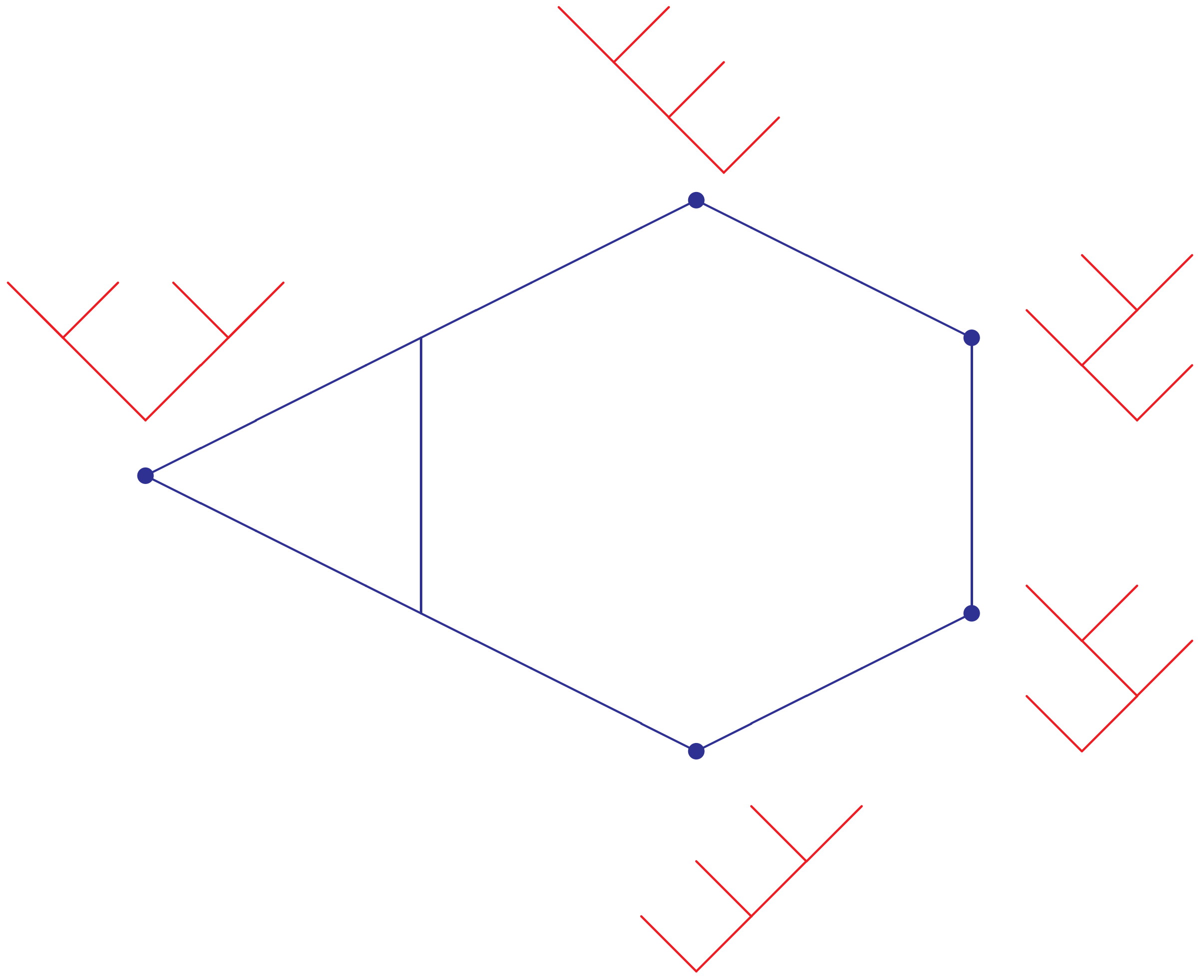
Pierre Leroux







Associahedron (Stasheff polytope)



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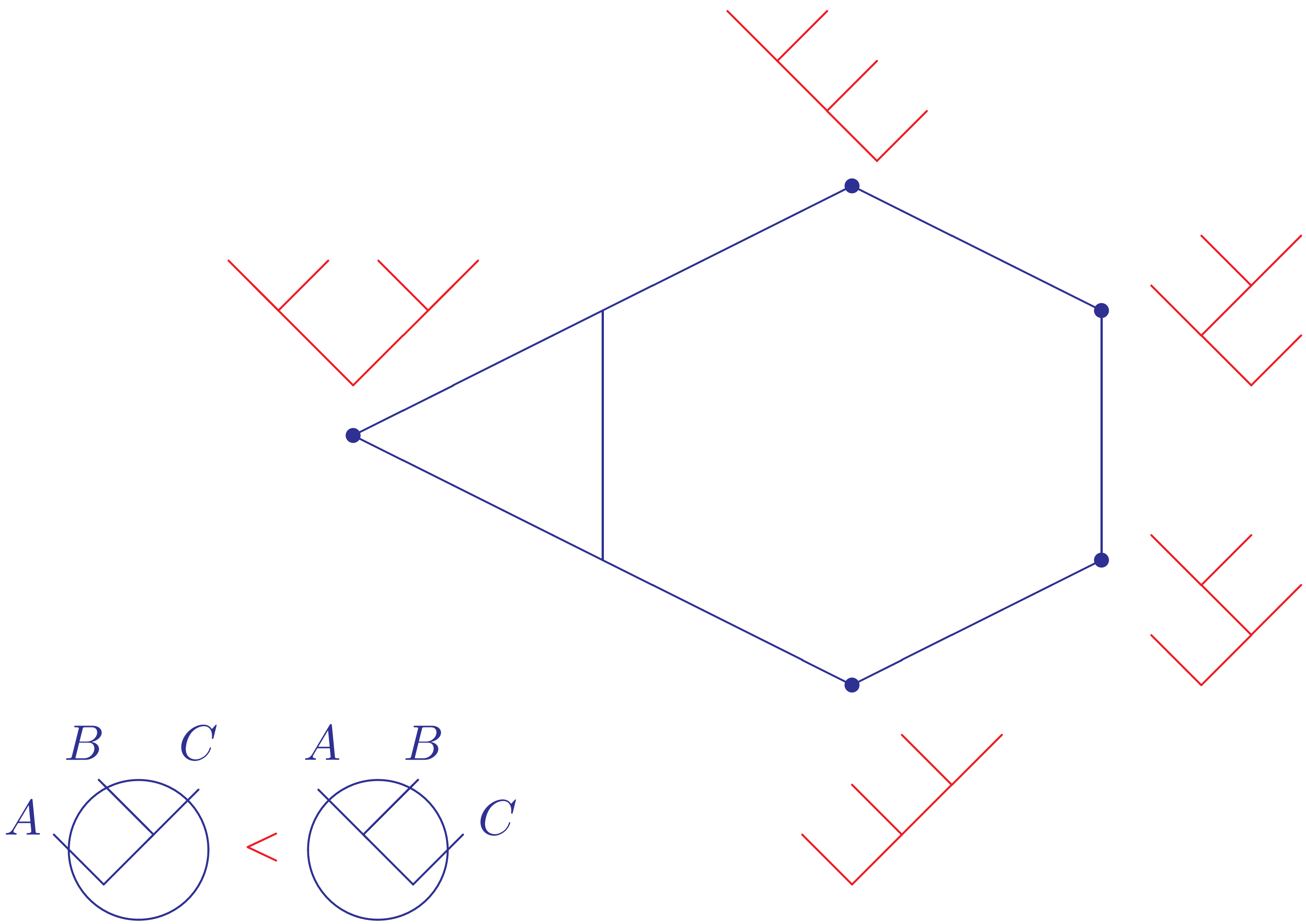

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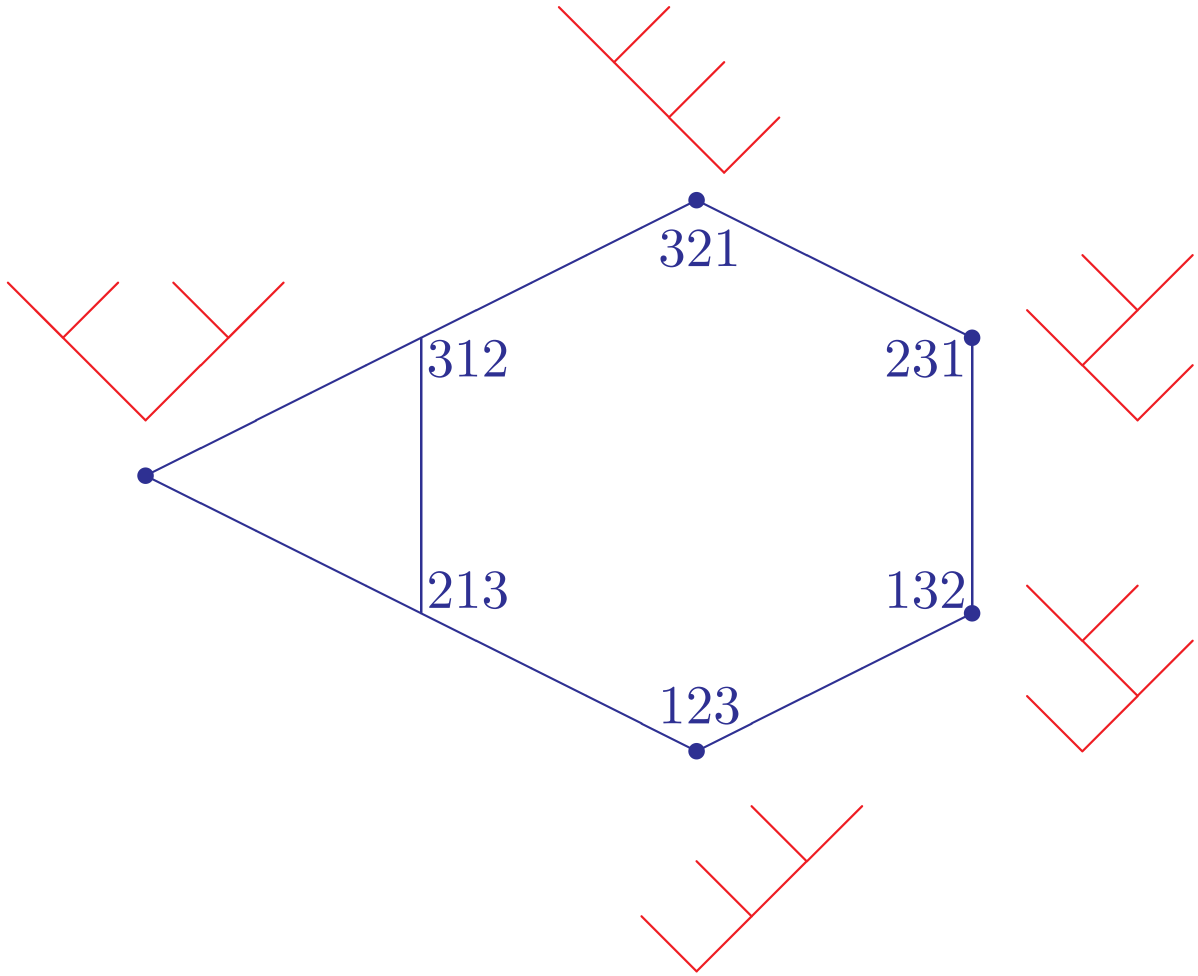

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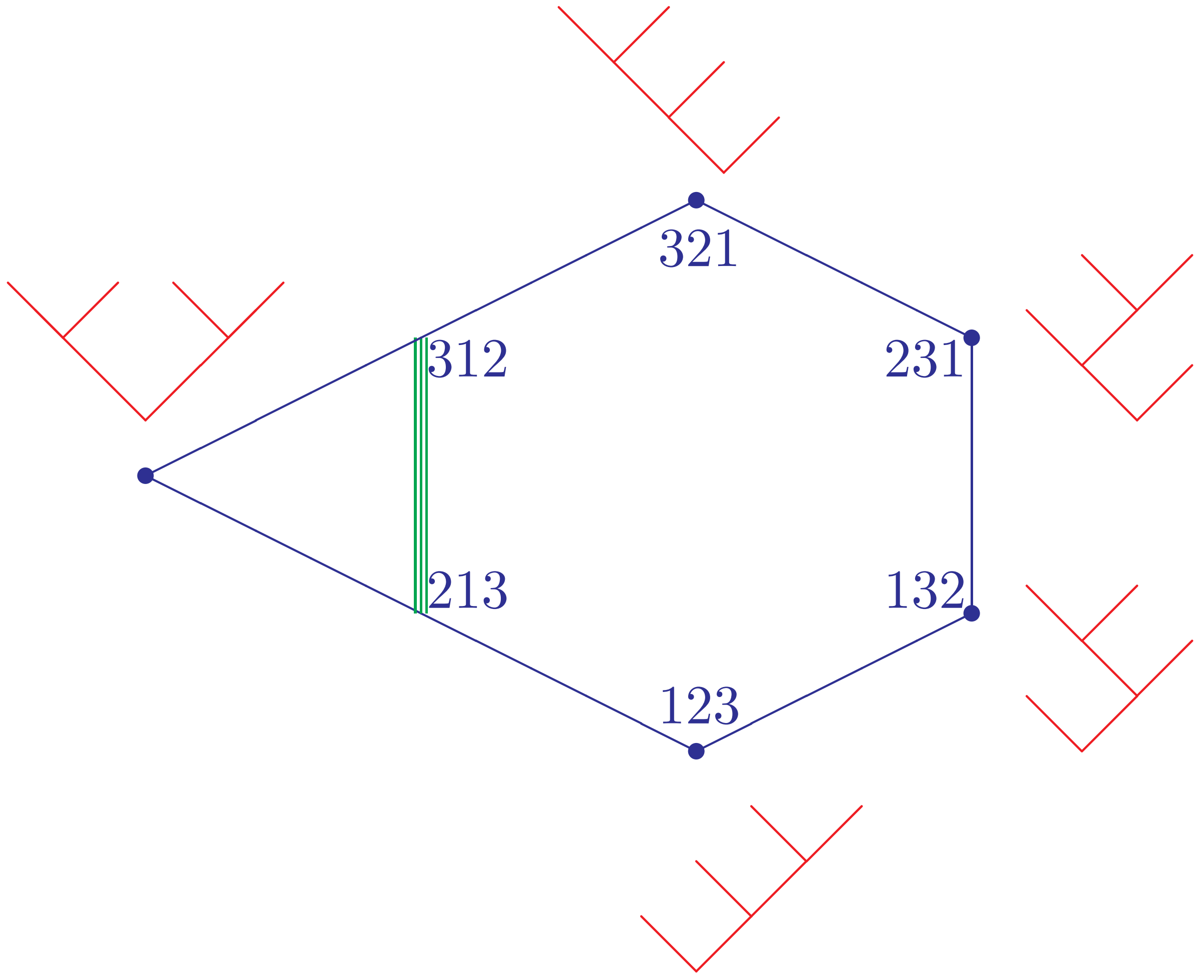
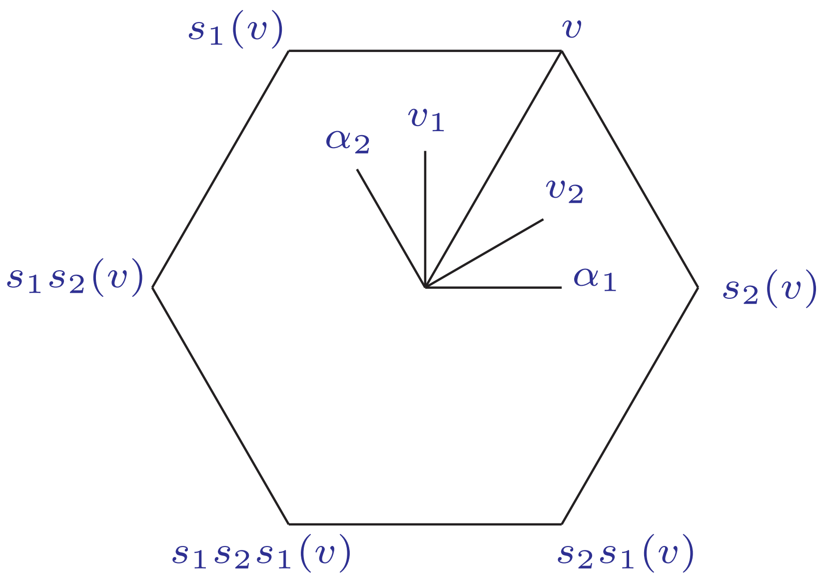
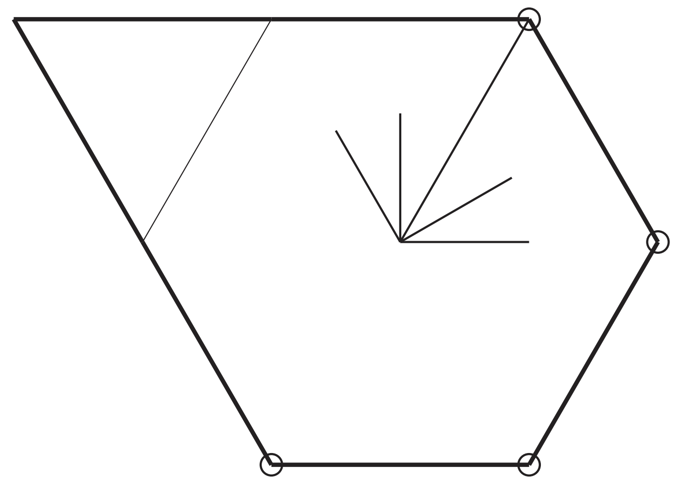


Loday's construction

Permutahedron ⟶ Associahedron





$\{\alpha_1, \alpha_2\}$ is a basis of the root system of type A_2

Generalized Associahedra

[Fomin, Zelevinski + Chapoton + Reading + HLT]

(W, S) a finite Coxeter system acting on $(V, \langle \cdot, \cdot \rangle)$.

Φ root system with simple roots $\Delta = \{\alpha_s \mid s \in S\}$.

$\Delta^* = \{v_s \mid s \in S\}$ be the dual simple roots of Δ.

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$$v = \sum_{s \in S} v_s$$

The permutahedron: Perm(W) = convex hull $\{w(v) \,|\, w \in W\}$.

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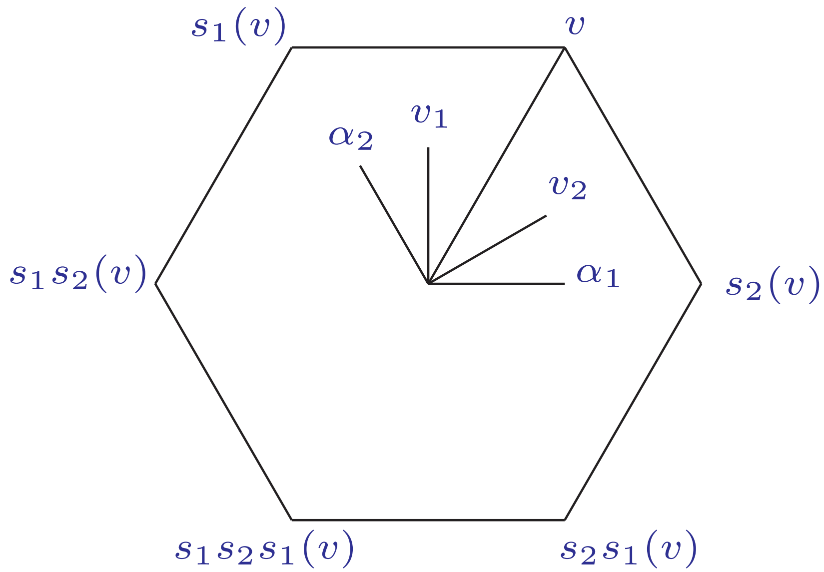

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Fix a coxeter element c of (W, S). $c = \overrightarrow{\prod_{s \in S}} s$ in some order.

example: for $W = A_3$ and $S = \{s_1, s_2, s_3\}$ we can choose

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Let $w_0 = c_{K_1} c_{K_2} \cdots c_{K_p}$ (unique) reduced factorization such that

$$K_1 \supseteq K_2 \supseteq \cdots \supseteq K_p \quad \text{and} \quad c_K = \overrightarrow{\prod_{s \in K}} s$$

example: for $W = A_3$ and $S = \{s_1, s_2, s_3\}$, if we choose

$$c = s_1 s_2 s_3 \quad \longrightarrow \quad w_0 = s_1 s_2 s_3 s_1 s_2 s_1 = c_{\{1,2,3\}} c_{\{1,2\}} c_{\{1\}}$$

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$T_c = \{u \in W : u$ is a prefix of $c_{K_1} c_{K_2} \cdots c_{K_p}$ up to commutations$\}$

Using only the allowed commutation $s_i s_j = s_j s_i$.

example: for $W = A_3$ and $S = \{s_1, s_2, s_3\}$, with $c = s_1 s_3 s_2$ we have $w_0 = s_1 s_3 s_2 \cdot s_1 s_3 s_2$ and

$$T_c = \{e, s_1, s_1 s_3, s_1 s_3 s_2, s_1 s_3 s_2 s_1, s_1 s_3 s_2 s_1 s_3, w_0, s_3, s_1 s_3 s_2 s_3\}$$

Generalized Associahedra

[Fomin, Zelevinski + Chapoton + Reading + HLT]

$\mathrm{Perm}(W)$ = convex hull $\{w(v) \mid w \in W\}$ where $v = \sum_{s \in S} v_s$.

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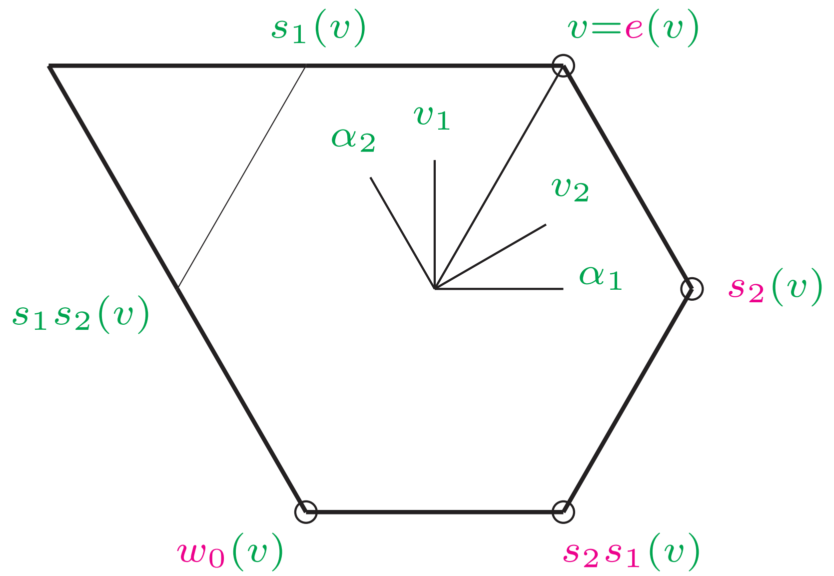
$$\mathrm{Ass}_c(W)$$

is the polytope defined by the hyperplanes of $\mathrm{Perm}(W)$ that contains elements $u(v)$ for $u \in T_c$.

Generalized Associahedra: A_2 and $c = s_2 s_1$

$w_0 = s_2 s_1 \cdot s_2$ and $T_c = \{e, s_2, s_2 s_1, w_0\}$



Generalized Associahedra: A_3 and $c = s_1 s_2 s_3$

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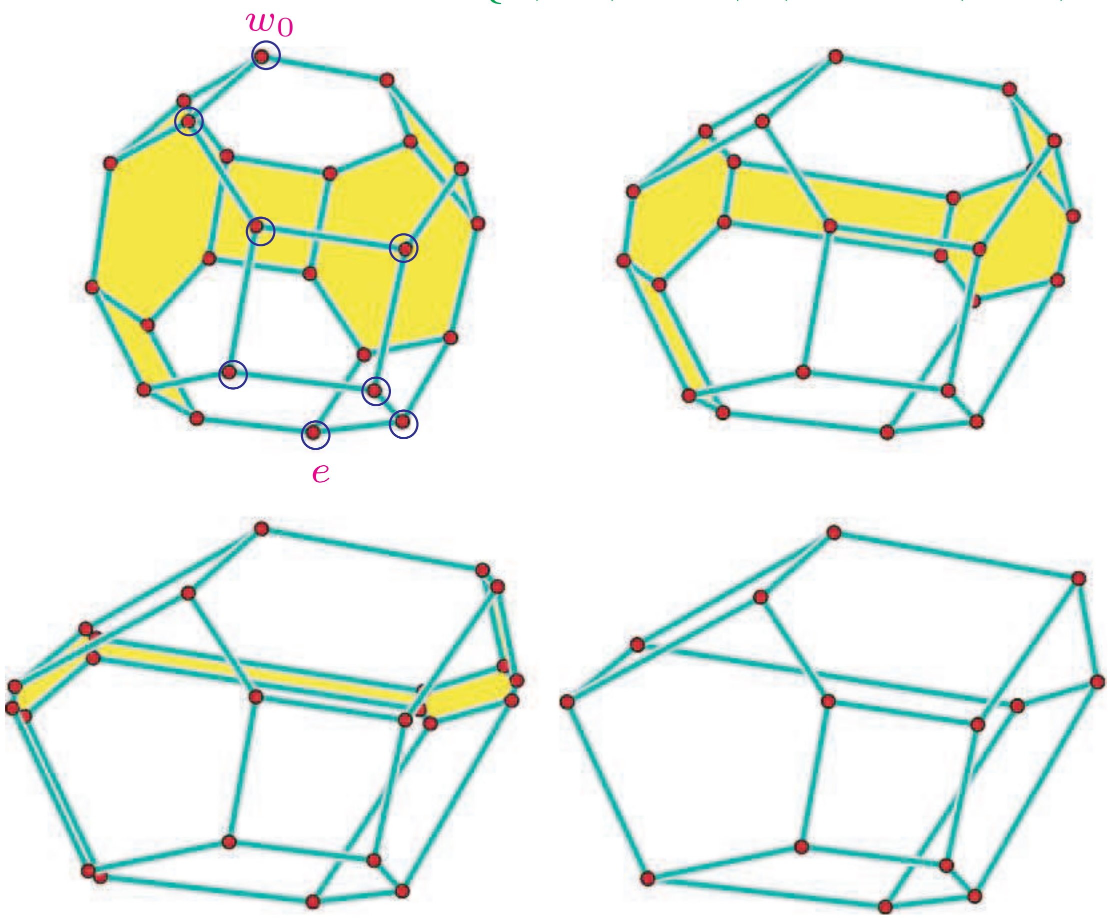

Generalized Associahedra: A_3 and $c = s_1 s_3 s_2$

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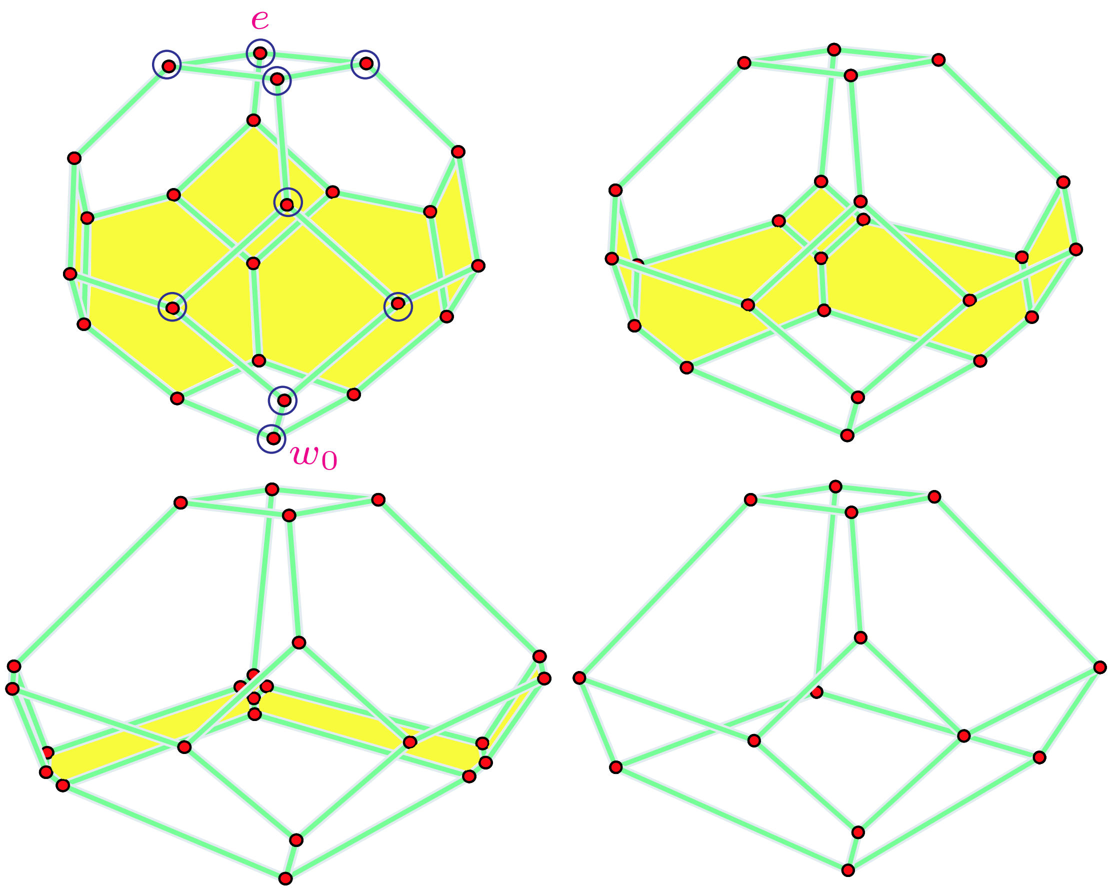

Some questions

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Theorem [BHLT] For (W, S) irreducible finite Coxeter system and c, c' Coxeter elements:

$$\mathrm{Ass}_c(W) \cong \mathrm{Ass}_{c'}(W) \qquad \Longleftrightarrow \qquad c' = \mu(c)^{\pm 1}$$

where μ is an automorphism of the Coxeter graph of W.

The Main Theorem

Theorem [BHLT] For (W, S) irreducible finite Coxeter system and c, c' Coxeter elements:

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Idea of proof

1. An isometry $\mathrm{Ass}_c(W) \to \mathrm{Ass}_{c'}(W)$ must fix the set $\{e, w_0\}$ and $\mathrm{Perm}(W)$.
2. Such isometry send coxeter elements c to $c' = \mu(c)^{\pm 1}$.
3. Conversely, there is such an isometry for any μ and the map $w \mapsto ww_0$ induces an isometry $\mathrm{Ass}_c(W) \to \mathrm{Ass}_{c^{-1}}(W)$.

For more details, see paper...[ArXive]