# Rook placements in Young diagrams

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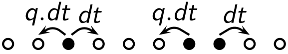
from the Erwin Schrödinger Institute, Vienna

# Introduction

Context: The PASEP, Partially Assymetric Self-Exclusion Process, is a 1D-model of particles in $n$ sites, hopping from each site to its neighbours.




This model is solved by a matrix ansatz (cf. Derrida &al). If:

$$DE - qED = D + E,$$

we can write $(D+E)^n$ in normal form:

$$(D+E)^n = \sum_{i,j\geq 0} c_{ij} E^i D^j,$$

Then the partition function is $Z = < (D+E)^n > = \sum c_{ij}$.

# Introduction

If we define:

$$\hat{D} = \tfrac{q-1}{q} D + \tfrac{1}{q}, \qquad \hat{E} = \tfrac{q-1}{q} E + \tfrac{1}{q}.$$

Then we have inversion formulas:

$$(1-q)^n (D+E)^n = \sum_{k=0}^{n} \tbinom{n}{k} 2^{n-k} (-1)^k q^k (\hat{D}+\hat{E})^k, \quad \text{and}$$

$$q^n (\hat{D}+\hat{E})^n = \sum_{k=0}^{n} \tbinom{n}{k} 2^{n-k} (-1)^k (1-q)^k (D+E)^k.$$

And the commutation relation is (cf. Uchiyama-Sasamoto, Evans) :

$$\hat{D}\hat{E} - q\hat{E}\hat{D} = \tfrac{1-q}{q^2}$$

# Introduction

The rewriting of $(D+E)^n$ in normal form is combinatorially described by alternative tableaux (cf. Viennot).

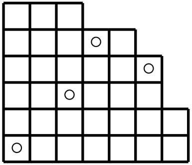
This explains the link between the PASEP and the combinatorics of permutations (cf. Corteel-Williams).

The rewriting of $(\hat{D}+\hat{E})^n$ in normal form is combinatorially described by rook placements in Young diagrams.

# Rewriting rules for $\hat{D}$ and $\hat{E}$

## Definition

A rook placement is a filling of the cells of a Young diagram with $\circ$, with at most one $\circ$ per line (resp. column).



We distinguish by a $\times$ the cells that are not directly below or to the left of a $\circ$ (cf. Garsia-Remmel).

Each $\circ$ has a weight $p$.
Each $\times$ has a weight $q$.

## Theorem

*Suppose more generally that $\hat{D}\hat{E} - q\hat{E}\hat{D} = p$, then $< (\hat{D} + \hat{E})^n >$ is the sum of weight of rook placements of half-perimeter $n$.*

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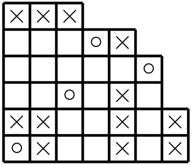

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# Rewriting rules for $\hat{D}$ and $\hat{E}$

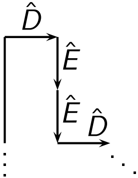
Since $(\hat{D}+\hat{E})^n$ expands into the sum of all words of length $n$ in $\hat{D}$ and $\hat{E}$, it is consequence of:

## Proposition

*Let $w$ be a word in $\hat{D}$ and $\hat{E}$. Then $< w >$ is the sum of weights of rook placements of shape $\lambda(w)$.*

$$w = \hat{D}\hat{E}\hat{E}\hat{D}\ldots$$

$\lambda(w)=$



# Rewriting rules: Sketch of proof

Operator point of view:

$$\hat{D}\hat{E}\hat{D}(\hat{D}\hat{E})\hat{D}\hat{E}\hat{E} = \hat{D}\hat{E}\hat{D}(q\hat{E}\hat{D})\hat{D}\hat{E}\hat{E} + \hat{D}\hat{E}\hat{D}(p)\hat{D}\hat{E}\hat{E}$$

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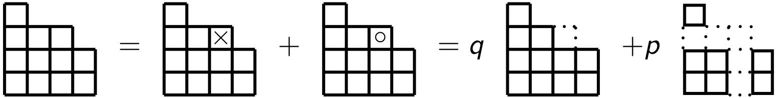

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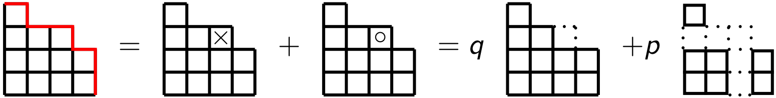

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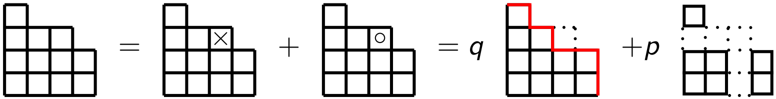

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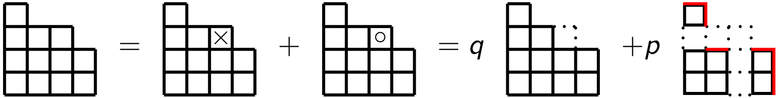

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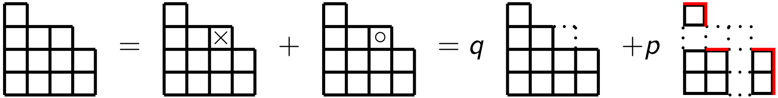



These are identical recurrence relations.

# Enumeration of rook placements: Examples

Let $T_{j,k,n}$ be the sum of weights of rook placements of half-perimeter $n$, with $k$ lines and $j$ lines without rook. We have:

## Proposition

$$T_{k,k,n} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

## Proposition

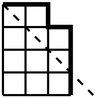
*When $p = 1$ and $q = 0$, $T_{0,k,n}$ is the number of (left factor of) Dyck paths of $n$ steps ending at height $n - 2k$. Hence:*

$$T_{0,k,n} = \binom{n}{k} - \binom{n}{k-1}.$$

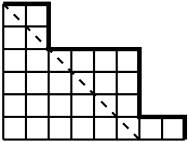
This is a consequence of:

## Proposition

*For any $\lambda$ there is at most one rook placement of shape $\lambda$ with no $\times$ and one rook per line, with equality in the case where the NE boudary of $\lambda$ is a Dyck path.*



If the path goes below the diagonal, it is impossible to place one rook per line.



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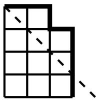

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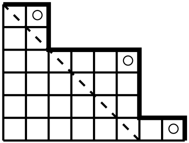

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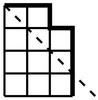

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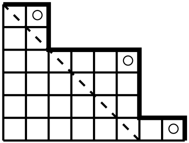

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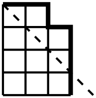

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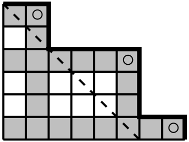

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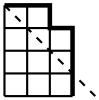

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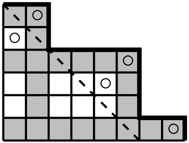

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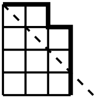

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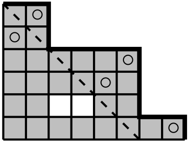

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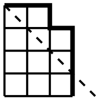

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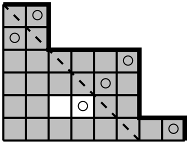

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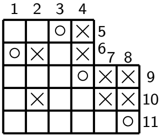
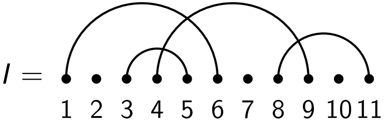
# Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):



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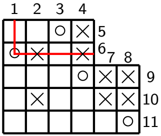


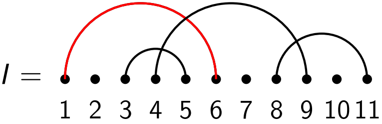

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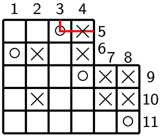


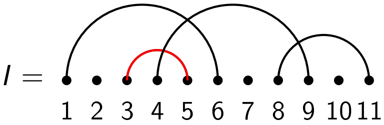

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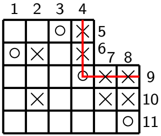


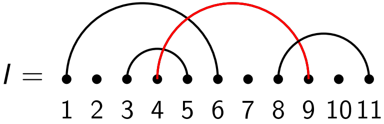

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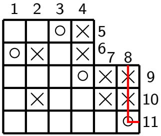


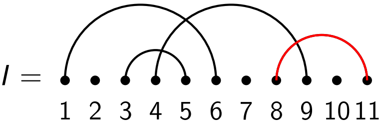

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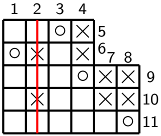


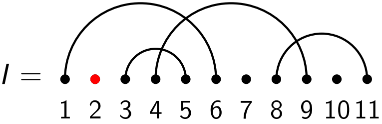

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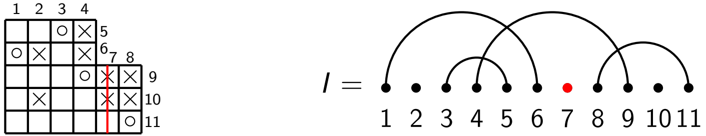

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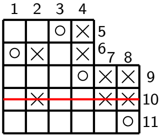
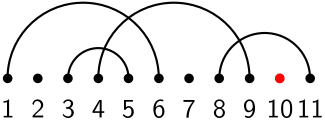



$I =$ 

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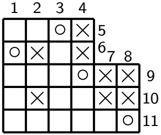
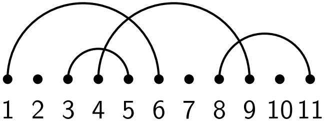



$I =$ 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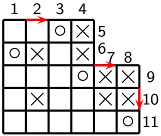


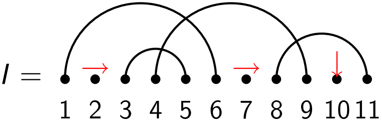


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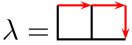
To keep track of empty lines or columns, we also define:



We have a bijection between rook placements of half-perimeter $n$, and couples $(I, \lambda)$ where:

- $I$ is an involution on $\{1, \ldots, n\}$,
- $\lambda$ is a Young diagram of half-perimeter $\#\mathrm{Fix}(I)$.

## Proposition

*With respect to this decomposition $R \mapsto (I, \lambda)$, the parameter "number of crosses" is additive:*

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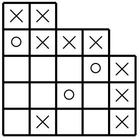
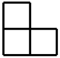
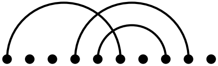
It is possible to describe $\mu$ precisely:

$$\mu(I) = \#\mathrm{crossings}(I) + \sum_{x \in \mathrm{Fix}(I)} height(x)$$

$R =$ 

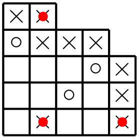
$\lambda =$ 

$I =$ 

- $|\lambda|$ counts the number of $\times$ with no rook in the same line, no rook in the same column.
- #crossings counts the number of $\times$ with one rook in the same line, one rook in the same column.
- $\sum$height$(x)$ counts all remaining $\times$.

$$|\lambda| = 3, \qquad \#\text{crossings} = 2,$$

$$\sum \text{height}(x) = 1 + 1 + 2 + 0 = 4$$

$R =$ 

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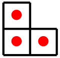
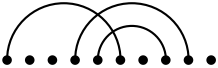
$\lambda =$ 

I= 

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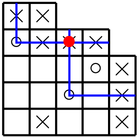
$R =$ 

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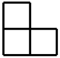
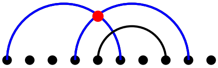
$\lambda=$ 

I= 

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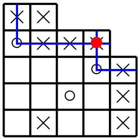
$R =$ 

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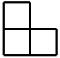
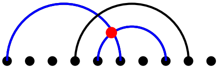
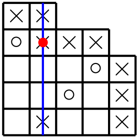
$\lambda =$ 

I= 

$|\lambda| = 3, \qquad \#\text{crossings} = 2,$

$$\sum \text{height}(x) = 1 + 1 + 2 + 0 = 4$$

$R =$



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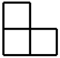

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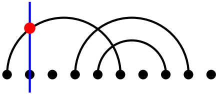

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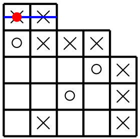
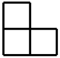
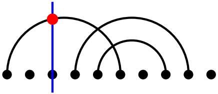
$$|\lambda| = 3, \qquad \#\text{crossings} = 2,$$

$$\sum \text{height}(x) = {\color{red}1} + 1 + 2 + 0 = 4$$

$R =$ 

$\lambda =$ 

$I =$ 

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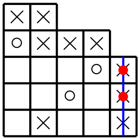
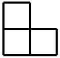
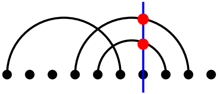
$R =$ 

$\lambda =$ 

$I =$ 

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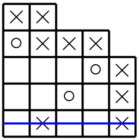
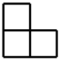
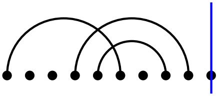
$R =$ 

$\lambda =$ 

I= 

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Consequence : Remember that $T_{j,k,n}$ is the sum of weights of rook placements of half-perimeter $n$, with $k$ lines, $j$ lines without rook.

Then we have a factorization:

$$T_{j,k,n} = \begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q T_{0,k-j,n}.$$

$$\nearrow \qquad \uparrow \qquad \nwarrow$$

$$\sum_R w(R) \qquad \sum_\lambda q^{|\lambda|} \qquad p^{k-j} \sum_I q^{\mu(I)}$$

Besides this factorization property, we have a recurrence relation:

$$T_{0,k,n} = T_{0,k,n-1} + pT_{1,k,n-1}.$$

Case 1:
The first column contains no rook.

Case 2:
The first column contains a rook.

Hence:

$$T_{0,k,n} = T_{0,k,n-1} + p[n+1-2k]_q T_{0,k-1,n-1}.$$

## Proposition

*This recurrence is solved by:*

$$T_{0,k,n} = \left(\frac{p}{1-q}\right)^k \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \left( \binom{n}{k-i} - \binom{n}{k-i-1} \right).$$

It remains to compute:

$$< (\hat{D}+\hat{E})^n >= \sum_{j,k} T_{j,k,n} = \sum_{j,k} \begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q T_{0,k-j,n}.$$

In the PASEP case, ie. $p = \frac{1-q}{q^2}$, we can simplify this sum with q-binomial identities. We obtain:

## Proposition

$$< (\hat{D} + \hat{E})^n >= \frac{2F(n) - F(n+1)}{q^n(1-q)},$$

*where*

$$F(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{n}{k} - \binom{n}{k-1} \right) \sum_{j=0}^{n-2k} q^{j(n+1-2k-j)}.$$

Remember that $(\hat{D}+\hat{E})^n$ and $(D+E)^n$ are linked by inversion formulas. We get a new proof of:

## Theorem

$$< (D+E)^{n-1} >= \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left( \binom{2n}{n-k} - \binom{2n}{n-k-1} \right) \times \left( \sum_{j=0}^{k} q^{j(k+1-j)} - \sum_{j=0}^{k-1} q^{j(k-j)} \right).$$

(Conjecture of Corteel-Rubey, March 2008. Proof T. Prellberg, May 2008. Alternative proof, J-V, August 2008)

# Conclusion

$< (D + E)^n >$ is the one-parameter function partition of the PASEP, but also:

- The $q$-enumeration of permutations wrt the number of 13-2 patterns (or equivalently, the number of crossings)
- The $q$-enumeration of permutation tableaux wrt the number of non-topmost 1's.
- The momentum of simple q-Laguerre polynomials.

These results also give an expression for the 3-parameter partition function of the PASEP, although it seems there is no nice simplification.

A generalization to $(\alpha D + E)^n$ and $(\alpha \hat{D} + \hat{E})^n$ would give the momentum of (non-simple) q-Laguerre polynomials.