Some Remarks on Partition Lattices

Roman König

Informatik 3

Universität Erlangen-Nürnberg

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Introduction

Every finite lattice L with operations $\wedge$ and $\vee$ has a set representation by the following construction.

L has two generating systems:

$$J(L) = \text{set of } \vee\text{−irreducibles is a } \vee\text{−generating system}$$

$$M(L) = \text{set of } \wedge\text{−irreducibles is a } \wedge\text{−generating system}$$

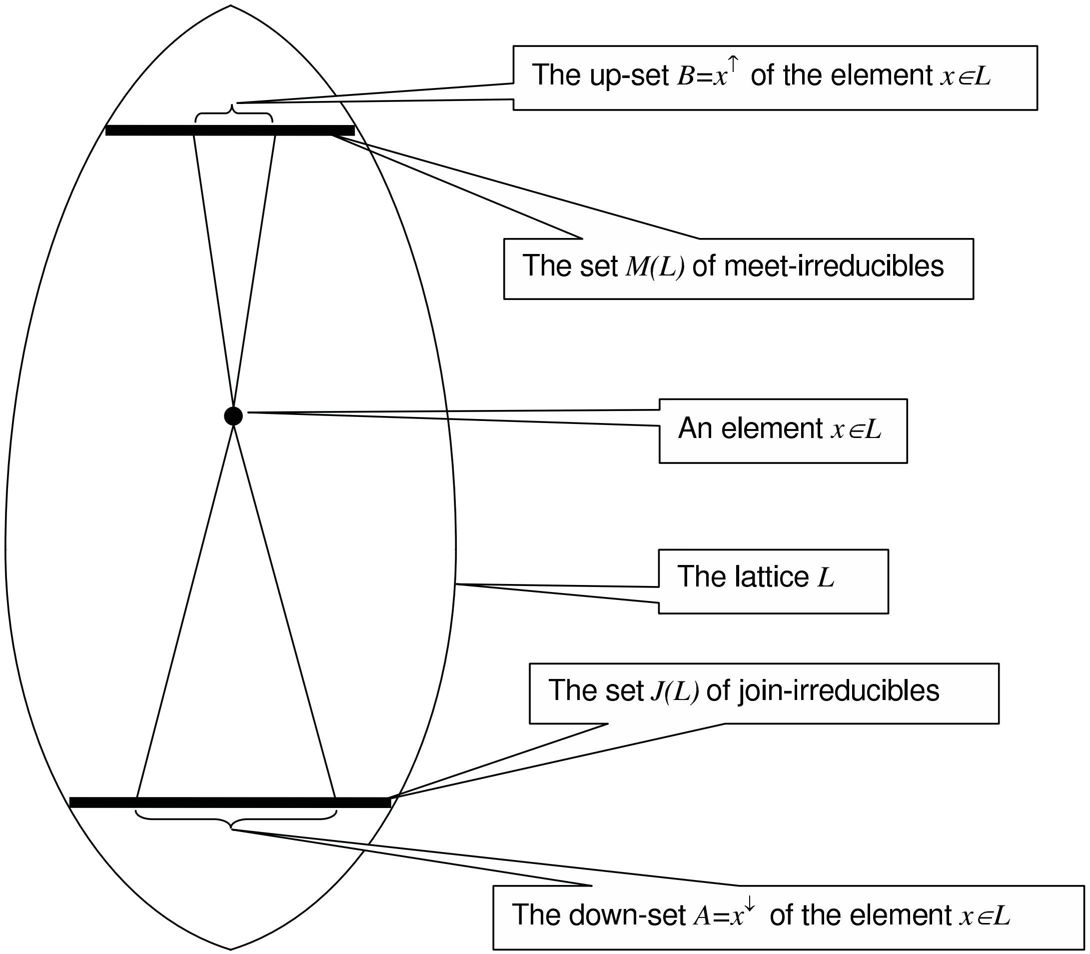
Every element $x \in L$ is represented as

$$x \simeq (A, B)$$

with $A \subseteq J(L), B \subseteq M(L)$ and $A = \{a \in J(L) \mid a \leq x\} = x^{\downarrow}$, the down-set of x, $B = \{b \in M(L) \mid x \leq b\} = x^{\uparrow}$, the up-set of x.

The general situation

Given a finite lattice L and an element $x \in L$, x is represented by the pair (A,B)



Two closure systems:

The system of all down-sets is a closure system on $J(L)$, the system of all up-sets is a closure system on $M(L)$.

Galois Connection:

$$A \mapsto A' := \{b \in M(L) \,|\, a \leq b \text{ for all } a \in A\} \text{ for } A \subseteq J(L)$$

$$B \mapsto B' := \{a \in J(L) \,|\, a \leq b \text{ for all } b \in B\} \text{ for } B \subseteq M(L)$$

Closure operators: $X \mapsto X''$ for $X \subseteq J(L)$, and $Y \mapsto Y''$ for $Y \subseteq M(L)$.

If $x \simeq (A,B)$, then A and B are closed sets and $A = x^{\downarrow}$, $B = x^{\uparrow}$

The order of L is now represented by set inclusion

$$(A,B) \leq (X,Y) \iff A \subseteq X \quad (\iff B \supseteq Y)$$

the lattice operations are

$$(A,B) \wedge (X,Y) = (A \cap X, (B \cup Y)'') \text{ and } (A,B) \vee (X,Y) = ((A \cup X)'', B \cap Y).$$

Application to partition lattices

Π_n: Lattice of set partitions of an $n-$element set $[1,n] = \{1,\ldots,n\}$ under refinement order. Π_n is a graded lattice with rank function $rk(\pi) = n - \#\pi$, where $\#\pi$ denotes the number of blocks of π.

Join irreducibles:
partitions with exactly $n-1$ blocks, (rank $=1$)
$\rightarrow$ $n-2$ singleton blocks and one $2-$block $\{k,l\}$ with $1 \leq k < l \leq n$
$\rightarrow$ are in bijection with $2-$subsets of n:
$J(\Pi_n) \leftrightarrow \binom{[1,n]}{2}$ (write $\binom{n}{2}$ for $\binom{[1,n]}{2}$)

Meet irreducibles:
partitions π with exactly 2 blocks, (rank $= n-2$)
$\rightarrow$ one of them does not contain the element n (= the proper class)
$\rightarrow$ are in bijection with non-empty subsets of $\{1,\ldots,n-1\}$:
$M(\Pi_n) \leftrightarrow 2^{n-1} - \{\emptyset\}$ ($\pi = \overline{531}, \overline{42} \in M(\Pi_5)$ has proper class $\{2,4\} \in 2^4$)

The relation $\leq$ in Π_n for irreducibles is

$$\{k,l\} \leq X \iff |\{k,l\} \cap X| = 0 \bmod 2$$

Galois connection:

$A \mapsto A' = \{X \in 2^{n-1} \mid |X \cap \{k,l\}| = 0 \mod 2$ for all $\{k,l\} \in A\}$ for $A \subseteq \binom{n}{2}$

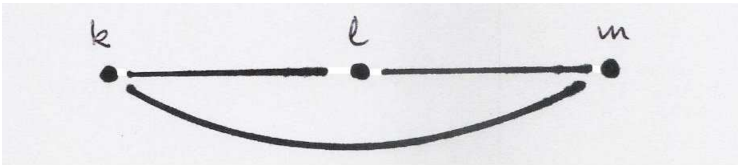
$B \mapsto B' = \{\{k,l\} \in \binom{n}{2} \mid |X \cap \{k,l\}| = 0 \mod 2$ for all $X \in B\}$ for $B \subseteq 2^{n-1}$

Closure operators:

1. For $A \subseteq \binom{n}{2}$ apply the rules

$\{\{k,l\},\{l,m\}\} \longrightarrow \{k,m\}$ if $k < l < m$
$\{\{k,m\},\{l,m\}\} \longrightarrow \{k,m\}$ if $k < l < m$
$\{\{k,l\},\{k,m\}\} \longrightarrow \{l,m\}$ if $k < l < m$

closure under transitivity



if two edges are present, the third must be present

2. For $B \subseteq 2^{n-1}$ apply the rules

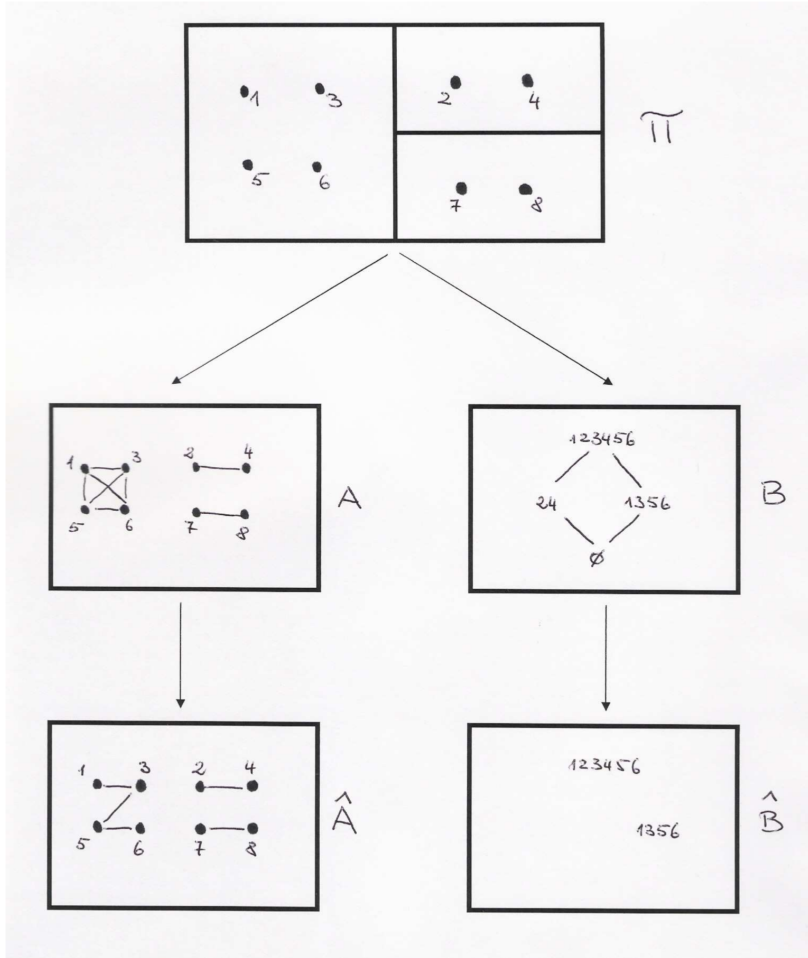
$$\{X,Y\} \longrightarrow X \cup Y$$
$$\{X,Y\} \longrightarrow X \setminus Y$$

Closed subsets of 2^{n-1} are boolean algebras contained (as sublattices) in 2^{n-1}.

In a partition $\pi \simeq (A,B)$ the set B is the boolean algebra defined by the proper classes of π (= classes not containing n).
A can be seen as the graph of the equivalence relation defined by π.

Example: $\pi = \overline{42}, \overline{6531}, \overline{87} \in \Pi_8$

The graph A has $\#\pi$ connected components (= blocks of π). A can be reduced (by transitivity) until we are left with a spanning forest $\hat{A}$, and then we have $\#\pi = n - |\hat{A}|$.
The boolean algebra B can be reduced (by boolean operations) until we are left with a minimal generating system $\hat{B}$. Every possible $\hat{B}$ has cardinality $|\hat{B}| = \dim B$.



$$\pi \simeq (A,B) = (\{\{1,3\},\{1,5\},\{1,6\},\{3,5\},\{3,6\},\{2,4\},\{5,6\},\{7,8\}\},$$
$$\{\emptyset,\{2,4\},\{1,3,5,6\},\{1,2,3,4,5,6\}\})$$

The improper class is $\overline{87} = \{1,\ldots,8\} \setminus \bigcup B$.

In our case:

$(\{\{1,3\},\{3,5\},\{5,6\},\{2,4\},\{7,8\}\},\{\{1,3,5,6\},\{1,2,3,4,5,6\}\})$ and $(\{\{1,3\},\{1,5\},\{1,6\},\{2,4\},\{7,8\}\},\{\{1,3,5,6\},\{2,4\}\})$ are possible $(\hat{A},\hat{B})$.

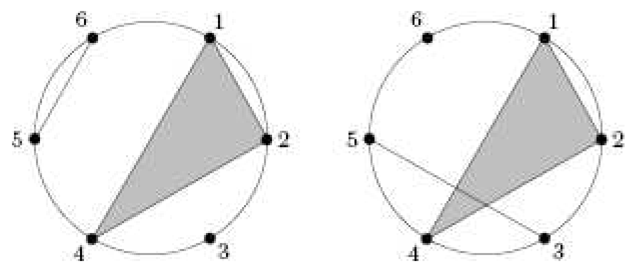
The atoms of B are the proper classes of the partition $\pi \simeq (A,B)$, hence $\#\pi = |\hat{B}|+1 = n-|\hat{A}|$ for every reduced representation $(\hat{A},\hat{B})$ of (A,B).

Consequence:

1. For every reduced representation $(\hat{A},\hat{B})$ we have $|\hat{A}|+|\hat{B}| = n-1$
2. $rk(A,B) = |\hat{A}|$
3. For every partition $\pi = (A,B)$ we have $rk(\pi)+\dim B = n-1$

Application to noncrossing partitions

Example: (from Armstrong [1])



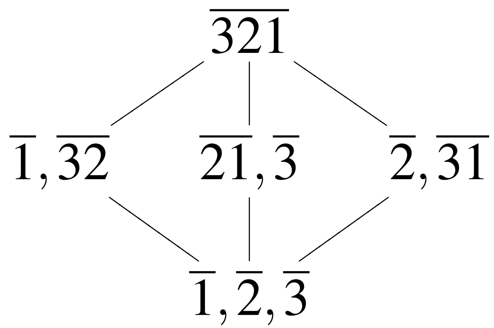
A noncrossing partition (nc-partition) and a (crossing) partition

A partition π of $\{1, \ldots, n\}$ is noncrossing if there is no crossing in the picture for π.

Let $NC(n)$ be the set of all nc-partitions on $\{1,\ldots,n\}$. The order of $NC(n)$ is inherited from the lattice Π_n. With this order $NC(n)$ is a lattice, but not a sublattice of the partition lattice:

For example, the join of the nc-partitions $\overline{2},\overline{31},\overline{4}$ and $\overline{1},\overline{3},\overline{42}$ in Π_4 is $\overline{31},\overline{42}$.

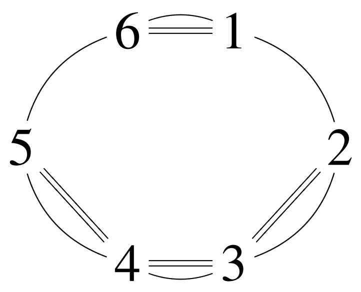
$NC(n)$ is not distributive since $NC(3) =$



To obtain a set-representation of $NC(n)$, we need

Join-irreducibles: Every join-irreducible partition is noncrossing, hence join-irreducible nc-partitions are in bijection with $\binom{n}{2}$ as before.

Meet-irreducibles: A meet-irreducible partition is noncrossing if and only if its proper class $X \subseteq \{1, \ldots, n-1\}$ is a nonempty interval:



The interval $[2,5] = \{2, \ldots, 5\}$ defines (together with its complement) the meet-irreducible nc-partition $\overline{5432}, \overline{61}$ of $\{1, \ldots, 6\}$.

Define $I_{n-1} := \{\text{intervals} \neq \emptyset \text{ of } \{1,\ldots,n-1\}\} = \{[i,j] \mid 1 \leq i \leq j \leq n-1\} \subseteq 2^{n-1}$ and observe:

$$|I_{n-1}| = \binom{n}{2}$$

Every nc-partition is represented by a pair (P,Q) with $P \subseteq \binom{n}{2}$, $Q \subseteq I_{n-1}$ such that

$$P = Q' \text{ and } Q = P' \cap I_{n-1}$$

and $Q'' = P'$. This means:

The boolean algebra P' is generated by the intervals contained in P'.

For example, the (crossing) partition $\overline{31}, \overline{42}, \bar{5}$ has $P = \{\{1,3\},\{2,4\}\}$ and defines (on $\{1,\ldots,4\}$) the boolean algebra $P' = \{\emptyset,\{1,3\},\{2,4\},[1,4]\}$ which is not generated by intervals.

The nc-partition $\overline{32}, \overline{41}, \bar{5}$ has $P = \{\{1,4\},\{2,3\}\}$ and generates the boolean algebra $P' = \{\emptyset,[2,3],\{1,4\},[1,4]\}$ which has the generating system
$Q = P' \cap I_4 = \{[2,3],[1,4]\} \subseteq I_4$

In other words:

- A partition $\pi \simeq (A,B) \in \Pi_n$ is an nc-partition iff the boolean algebra B has a generating system $Y \subseteq I_{n-1}$.

Remark: From the system Y a Dyck-word representing (A,B) can be uniquely constructed, thus showing that the number of nc-partitions on n points is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

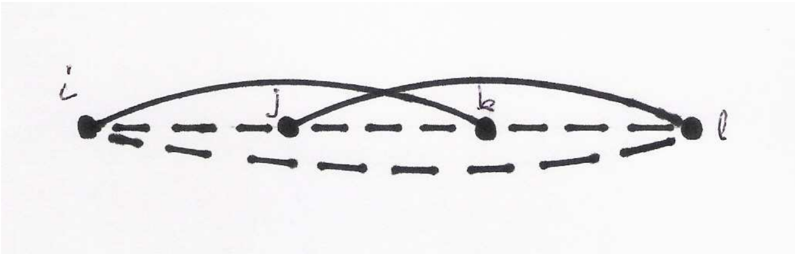
The closure operators for nc-partitions

Every $\pi \in NC(n)$ is represented by a pair (P,Q) with $P \subseteq \binom{n}{2}$ and $Q \subseteq I_{n-1}$ such that

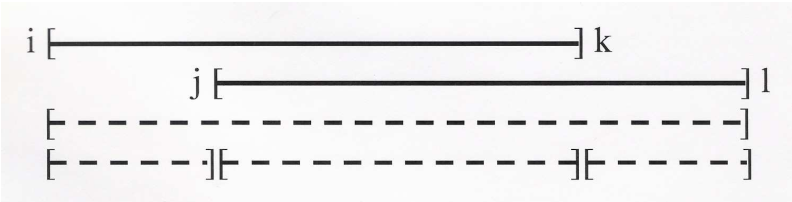
P is closed under transitivity and the "non-crossing rules"

$$\{\{i,k\},\{j,l\}\} \longrightarrow \{i,l\} \text{ if } i < j < k < l.$$

This results in four groups of rules for $2-$subsets



Q is closed under the rules describing the boolean operations $\cup, \cap, \setminus$ restricted to intervals:



This again results in four groups of interval rules.

Example: Given two nc-partitions on $\{1,\dots,8\}$

$$\pi = \overline{21}, \overline{3}, \overline{5}, \overline{6}, \overline{874} = (P, Q)$$
$$= (\{\{1,2\},\{4,7\},\{7,8\},\{4,8\}\},\{[1,2],[3],[1,3],[5],[6],[5,6]\})$$

$$\rho = \overline{43}, \overline{521}, \overline{876} = (R, S)$$
$$= (\{\{1,2\},\{1,5\},\{2,5\},\{3,4\},\{6,7\},\{7,8\},\{6,8\}\},\{[3,4],[1,5]\})$$

$\pi \wedge \rho = (U, V) \in (2^{\binom{8}{2}}, 2^{I_7})$ with $U = P \cap R = \{\{1,2\},\{7,8\}\}$.

$V =$ closure of $Q \cup S = \{[1,2],[3],[1,3],[5],[6],[5,6],[3,4],[1,5]\}$ under the interval implications:
$\rightsquigarrow V = \{[1,2],[3],[1,3],[5],[6],[5,6],[3,4],[1,5],[3,6],[3,5],[4],[1,4],[1,6],[4,5],[4,6]\}$

$\hat{U} = \{\{1,2\},\{7,8\}\}$ (here it is unique), $\hat{V}$ reduced for interval implications can be chosen as $\{[1,2],[3],[4],[5],[6]\}$ or as $\{[1,2],[3],[3,4],[3,5],[3,6]\}$ or ....

Note that $|\hat{U}| + |\hat{V}| = n - 1 = 8 - 1 = 7$.

For the nc-partition $\pi \vee \rho$ take intersection in the second component $Q \cap S = \{[1,2],[3],[1,3],[5],[6],[5,6]\} \cap \{[3,4],[1,5]\} = \emptyset$, and $\pi \vee \rho = \top$ (in the lattice $NC(8)$) follows.

$\pi = \overline{21}, \overline{3}, \overline{5}, \overline{6}, \overline{874}$ and $\rho = \overline{43}, \overline{521}, \overline{876}$ have join $\overline{521}, \overline{87643}$ in Π_8.

Kreweras-complement

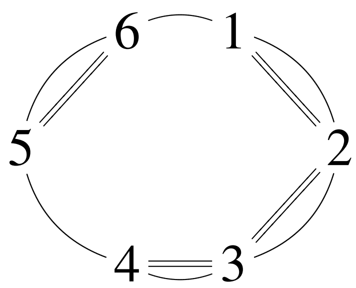
For every n define two functions

$$\phi_n : I_{n-1} \to \binom{n}{2} \text{ by } [i,j] \mapsto \begin{cases} \{i-1,j\} & \text{if } i>1 \\ \{j,n\} & \text{if } i=1 \end{cases}$$

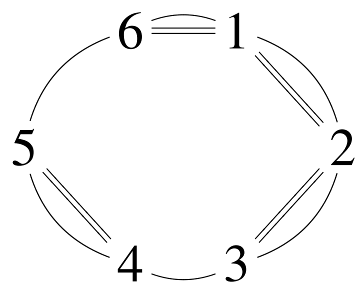
$$\psi_n : \binom{n}{2} \to I_{n-1} \text{ by } \{k,l\} \mapsto [k,l-1]$$

ϕ_n and ψ_n are bijections, but not mutually inverses.

Rather we have $$\phi_n \circ \psi_n(\{k,l\}) = \begin{cases} \{k-1, l-1\} & \text{if } k > 1 \\ \{l-1, n\} & \text{if } k = 1 \end{cases}$$
$$\psi_n \circ \phi_n([i,j]) = \begin{cases} [i-1, j-1] & \text{if } i > 1 \\ [j, n-1] & \text{if } i = 1 \end{cases}$$
are counterclockwise rotations of the circle of length n, compatible with nc-partitions. When applied to $2-$subsets $\{i,j\}$ or to intervals $[i,j]$ with $i \neq 1$ this is obvious. For the rest consider the interval $[1,4]$ on $\{1,\ldots,6\}$. It represents the meet-irreducible partition $\overline{4321}, \overline{65}$



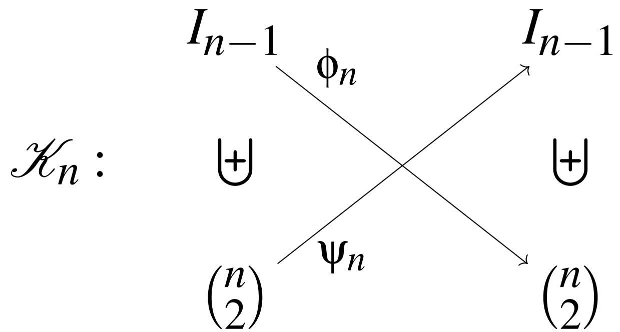
is transformed by $\psi_6 \circ \phi_6$ to $[4,5]$:



which is in fact the appropriate description of the rotated partition.

The same applies to $2-$subsets: $\{1,k\} \rightsquigarrow [1,k-1] \rightsquigarrow \{k-1,n\}$.

ϕ_n and ψ_n define a bijection $\mathscr{K}_n : I_{n-1} \uplus \binom{n}{2} \longrightarrow I_{n-1} \uplus \binom{n}{2}$ by the scheme



with the properties $\mathscr{K}_n^2 = \phi_n \circ \psi_n \cup \psi_n \circ \phi_n$ and $\mathscr{K}_n^{2n} = id$. It is clear that

– $(P,Q) \in NC(n) \iff (\mathscr{K}_n(Q), \mathscr{K}_n(P)) \in NC(n)$

– $(P,Q) \leq (R,S) \iff (\mathscr{K}_n(Q), \mathscr{K}_n(P)) \geq (\mathscr{K}_n(S), \mathscr{K}_n(R))$

– $(P,Q) \in NC(n) \implies \mathscr{K}_n(P) \cap Q = \emptyset$ and $P \cap \mathscr{K}_n(Q) = \emptyset$

Theorem: For every nc-partition $(P,Q) \in NC(n)$ the Kreweras-complement is the nc-partition $\mathcal{K}_n(P,Q) := (\mathcal{K}_n(Q), \mathcal{K}_n(P))$. This means

1. $(P,Q) \wedge (\mathcal{K}_n(Q), \mathcal{K}_n(P)) = \bot$
   $(P,Q) \vee (\mathcal{K}_n(Q), \mathcal{K}_n(P)) = \top$
2. The nc-partition $(\mathcal{K}_n(Q), \mathcal{K}_n(P))$ is the unique solution of the equation

$$perm(P,Q) \circ perm(\mathcal{K}_n(Q), \mathcal{K}_n(P)) = (1, \ldots, n)$$

$perm(P,Q)$ is the permutation that consists of the cycles defined by the blocks of the partition $\pi = (P,Q)$ (written in ascending order).

Example: $\pi = \overline{43}, \overline{521}, \overline{876}$ is noncrossing, $perm(\overline{43}, \overline{521}, \overline{876}) = (1,2,5)(3,4)(6,7,8)$

$\mathcal{K}_8(\pi) = \overline{1}, \overline{3}, \overline{42}, \overline{6}, \overline{7}, \overline{85}$ with $perm(\mathcal{K}_8(\pi)) = (2,4)(5,8)$ and

$(1,2,5)(3,4)(6,7,8) \circ (2,4)(5,8) = (1,2,3,4,5,6,7,8)$

$\mathscr{K}_n$ is not only a bijection $\mathscr{K}_n : I_{n-1} \uplus \binom{n}{2} \longrightarrow I_{n-1} \uplus \binom{n}{2}$.

$\mathscr{K}_n$ transforms the system of interval implications to the system of $2-$subset-implications and vice versa, hence $\mathscr{K}_n$ is a transformation of one closure operator to the other. For example, transitivity goes to

$$\mathscr{K}_n(\{\{k,l\},\{l,s\}\} \to \{k,s\}) = \{[k,l-1],[l,s-1]\} \to [k,s-1]$$

It follows that

1. If $(\hat{P},\hat{Q})$ is reduced, then $(\mathscr{K}_n(\hat{Q}),\mathscr{K}_n(\hat{P}))$ is reduced.
2. $rk(\mathscr{K}_n(\pi)) = n-1-rk(\pi)$
3. $|\hat{Q}| = \#\pi - 1$
4. $\#(\pi)+\#(\mathscr{K}_n(\pi)) = n - rk(\pi) + n - rk(\mathscr{K}_n(\pi)) = n+1$

Hence Kreweras' pictorial, not very transparent construction can be replaced by mere application of $\mathscr{K}_n$.

Further Consequences:

$\mathcal{K}_n$ is an anti-automorphism from $NC(n)$ onto itself interchanging level k and level $n-1-k$. (rank-inverting)

$\mathcal{K}_n^2$ is an isomorphism of $NC(n)$ with the property $type(perm(A,B)) = type(perm(\mathcal{K}_n^2(A), \mathcal{K}_n^2(B)))$.

$\mathcal{K}_n^{2n} = id \implies \mathcal{K}_n^n$ is an involution on $NC(n)$. If n is odd, then $\mathcal{K}_n^n$ is a rank-inverting involution on $NC(n)$: $rk(\mathcal{K}_n^n(\pi)) = n-1-rk(\pi)$.

G. Kreweras' construction has been modified by several authors for special purposes. For example, R. Simion defined a rank-inverting anti-isomorphism of $NC(n)$ (and V. Reiner for type B nc-partitions) for all n. These pictorial constructions can be described by the operator $\mathcal{K}_n$.

I imagine that people who are more experienced in Coxeter theory, root systems, Weyl groups ... than I am may ask questions that can be answered by extending this set representation approach.

References

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