Inverse relations between powers of Bochner's operator and differential operators of even order

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September, 2008
SLC'61, Curia - Portugal

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Some of the notation

- $\mathbb{N}$ set of all nonnegative integers
- $\mathbb{N}^*$ set of all positive integers
- $\mathbb{C}$ set of all complex numbers
- $\mathcal{P}$ vector space of polynomials with coefficients in $\mathbb{C}$
- $\mathcal{P}'$ dual space of $\mathcal{P}$
- $(p)^{(n)}$ n-th derivative of $p \in \mathcal{P}$, $n \in \mathbb{N}$
- $p^{[n]}$ n-th normalized derivative, so that $p^{[n]}$ is monic
- $\langle u, p \rangle$ action of $u \in \mathcal{P}'$ on $p \in \mathcal{P}$
- MPS Monic Polynomial Sequence $\{P_n\}_{n\geqslant 0}$ such that $P_n(x) = x^n + p_{n-1}(x)$, with $\deg p_{n-1} = n-1$

Preliminary results

Derivative and product by a polynomial of a form

$$\langle u', f \rangle := -\langle u, f' \rangle \quad , \quad \langle gu, f \rangle := \langle u, gf \rangle, \quad f \in \mathcal{P},$$

The **dual sequence** $\{u_n\}_{n\geqslant 0}$ of a monic polynomial sequence (**MPS**) $\{P_n\}_{n\geqslant 0}$ is defined by $\langle u_n, P_k \rangle = \delta_{n,k}, \quad n, k \geqslant 0.$

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Orthogonal polynomial sequences (OPS)

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A MPS $\{P_n\}_{n\geqslant 0}$ is said to be a MOPS with respect to $u \in \mathcal{P}'$ if

$$\langle u, P_n P_m \rangle = K_n \delta_{n,m}, \quad n, m \geqslant 0$$

$$K_n \neq 0, \quad n \geqslant 0 .$$

In this case, u is called a **regular form** and it is proportional to u_0

Some characterisation properties:

Consider $\{P_n\}_{n\geqslant 0}$ to be a MPS. The statements are equivalent:

(a) $\{P_n\}_{n\geqslant 0}$ is a MOPS with respect to u_0

(b) $\begin{cases} P_1(x) = x - \beta_0; & P_1(x) = 1 \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x) \end{cases}$

with $\beta_n = \dfrac{\langle u_0, xP_n^2 \rangle}{\langle u_0, P_n^2 \rangle}$ and $\gamma_{n+1} = \dfrac{\langle u_0, P_{n+1}^2 \rangle}{\langle u_0, P_n^2 \rangle} \neq 0,\ n \in \mathbb{N}.$

(c) $u_n = \left(\langle u_0, P_n^2 \rangle\right)^{-1} P_n\, u_0, \quad n \in \mathbb{N},$

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Classical polynomial sequence

Let $k \in \mathbb{N}^*$ and $\{P_n\}_{n\geqslant 0}$ be a MPS. The sequence $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ with $P_n^{[k]}(x) := \frac{1}{n+1}\left(P_{n+1}^{[k-1]}(x)\right)'$, $n \in \mathbb{N}$, (and $P_n^{[0]} := P_n$, $n \geqslant 0$), is also a MPS.

Definition

A MOPS $\{P_n\}_{n\in\mathbb{N}}$ is said to be classical when $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is also orthogonal (Hahn's property, [Hahn(1935)]) The associated regular form u_0 is called classical form (*Hermite, Laguerre, Bessel and Jacobi*).

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(c) $\exists \Phi\ \Psi \in \mathcal{P}$ such that the associated regular form u_0 satisfies

$$D(\Phi u_0) + \Psi u_0 = 0\ ,$$

where $\deg \Phi \leqslant 2$ (Φ monic) and $\deg(\Psi) = 1$

(d) There exist two polynomials Φ (monic with $\deg \Phi \leqslant 2$) and Ψ (with $\deg \Psi = 1$) and a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ with $\chi_0 = 0$ and $\chi_{n+1} \neq 0$, $n \in \mathbb{N}$, such that

$$\mathcal{F}\big(\mathbf{P_n(x)}\big) = \chi_\mathbf{n}\mathbf{P_n},\quad n \geqslant 0,\ \text{[Bochner (1929)]}$$

where

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Construction of a generalisation on the Bochner differential equation fulfilled by classical polynomials

Let $k \in \mathbb{N}^*$ and $\{P_n\}_{n\geqslant 0}$ be a MPS.

If $\{u_n\}_{n\in\mathbb{N}}$ and $\{u_n^{[k]}\}_{n\in\mathbb{N}}$ represent the dual sequences of $\{P_n\}_{n\geqslant 0}$ and $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ (*resp.*), then it holds

$$D^k\left(u_n^{[k]}\right) = (-1)^k \prod_{\mu=1}^{k} (n+\mu)\; u_{n+k},\; n \in \mathbb{N}$$

Suppose $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ are two MOPS.

Therefore, the elements of the corresponding dual sequences are related by

$$\boxed{\left(P_n^{[k]}\; u_0^{[k]}\right)^{(k)} = \lambda_n^k\; P_{n+k} u_0\;, \quad n \in \mathbb{N},}$$

with

$$\lambda_n^k = (-1)^k \frac{\left\langle u_0^{[k]}, \left(P_n^{[k]}\right)^2 \right\rangle}{\left\langle u_0, P_{n+k}^2 \right\rangle} \prod_{\mu=1}^{k} (n+\mu)\;, \quad n \in \mathbb{N}.$$

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construction of a generalisation on the Bochner (cont.)

Using Leibniz relation for derivation, it follows

$$\sum_{\nu=0}^{k} \binom{k}{\nu} \left(P_n^{[k]}\right)^{(\nu)} \left(\mathbf{u}_0^{[k]}\right)^{(k-\nu)} = \lambda_n^k \, P_{n+k} \, \mathbf{u}_0 , \quad n \in \mathbb{N},$$

Inasmuch as $\{P_n^{[j]}\}_{n\in\mathbb{N}}$, $0 \leqslant j \leqslant k$, is also classical
we derive

$$\left(\mathbf{u}_0^{[k]}\right)^{(k-\nu)} = \omega_{k,\nu} \, \lambda_0^k \, \Phi^\nu \, P_{k-\nu}^{[\nu]} \, \mathbf{u}_0 \, , \quad 0 \leqslant \nu \leqslant k$$

with $$\omega_{k,\nu} = \begin{cases} \left(-\Psi'(0)\right)^{-\nu} & \text{if} \quad 0 \leqslant \deg \Phi \leqslant 1 \, , \\ \dfrac{1}{\left(k-1-\Psi'(0)\right)_\nu} & \text{if} \quad \deg \Phi = 2 \, , \end{cases}$$

thereby...

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Generalisation of Bochner's differential equation

Theorem

Let $\{P_n\}_{n\in\mathbb{N}}$ be a MOPS. Suppose there is an integer $k \geqslant 1$ such that $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is a MOPS. Then any polynomial P_{n+k} fulfils the following differential equation of order $2k$:

$$\boxed{\sum_{\nu=0}^{\mathbf{k}} \boldsymbol{\Lambda}_{\nu}\,(\mathbf{k};\mathbf{x})\;\; \mathbf{D}^{\mathbf{k}+\nu}\mathbf{P}_{\mathbf{n+k}}\,(\mathbf{x}) = \boldsymbol{\Xi}_{\mathbf{n}}\,(\mathbf{k})\,\mathbf{P}_{\mathbf{n+k}}\,(\mathbf{x})\;,\;\; n\in\mathbb{N},}$$

where

$$\Lambda_\nu(k;x) = \frac{\lambda_0^k\;\;\omega_{k,\nu}}{\nu!}\;\;\Phi^\nu(x)\;\;\Big(P_k(x)\Big)^{(\nu)},\quad 0\leqslant\nu\leqslant k,$$

$$\Xi_n\,(k) = \lambda_n^k\;\{n+k\}_{(\mathbf{k})}\;,\qquad n\in\mathbb{N};$$

$$\lambda_n^k = (-1)^k\,\frac{\left\langle\; v_0,\, {Q_n}^2\;\right\rangle}{\left\langle\; u_0,\, P_{n+k}^2\;\right\rangle}\,(n+1)_k\;,\qquad n\in\mathbb{N};$$

with D representing the differential operator and $\{x\}_{(\mathbf{k})} := x(x-1)\ldots(x-k+1)$, $k\in\mathbb{N}$.

Extension of Bochner's differential equation

Corollary

Let $\{P_n\}_{n\in\mathbb{N}}$ be a classical MOPS and k a positive integer. Consider the differential operator $\mathcal{F} = \Phi(x)D^2 - \Psi(x)D$ where Φ is a monic polynomial with $\deg \Phi \leqslant 2$, and Ψ a polynomial such that $\deg \Psi = 1$.
Then, for any set $\{c_{k,\mu} \,:\, 0 \leqslant \mu \leqslant k\}$ of complex numbers not depending on n, each element of $\{P_n\}_{n\in\mathbb{N}}$ fulfils the differential equation given by

$$\sum_{\mu=0}^{k} c_{k,\mu}\, \mathcal{F}^{\mu} P_n(x) = \sum_{\mu=0}^{k} c_{k,\mu}\, (\chi_n)^{\mu}\; P_n(x)\;, \quad n \in \mathbb{N},$$

where $\{\chi_n\}_{n\geqslant 1}$ represents a sequence of nonzero complex numbers and $\mathcal{F}^k$ is recursively defined through $\mathcal{F}^k[y](x) = \mathcal{F}\big(\mathcal{F}^{k-1}[y](x)\big)$, for $k \in \mathbb{N}^*$ with $\mathcal{F}^0$ denote the identity operator.

Relation between the two generalisations

Corollary

Let $\{P_n\}_{n\in\mathbb{N}}$ be a classical sequence and k a positive integer. If there exist coefficients $d_{k,\mu}$ and $\widetilde{d}_{k,\mu}$ $0\leqslant\mu\leqslant k$, not depending on n, such that

$$\begin{aligned}\Xi_{n-k}(k) &= \textstyle\sum_{\tau=0}^{k} d_{k,\tau}\,(\chi_n)^{\tau}\,, \quad n\geqslant 0,\\ (\chi_n)^k &= \textstyle\sum_{\tau=0}^{k} \widetilde{d}_{k,\tau}\,\Xi_{n-\tau}(\tau)\,, \quad n\geqslant 0,\end{aligned}$$

then the two following equalities hold:

$$\sum_{\nu=0}^{k}\Lambda_k(k;x)D^{k+\nu} = \sum_{\tau=0}^{k} d_{k,\tau}\,\mathcal{F}^{\tau}$$

$$\mathcal{F}^k = \sum_{\tau=0}^{k}\widetilde{d}_{k,\tau}\left\{\sum_{\nu=0}^{\tau}\Lambda_\nu(\tau;x)D^{\tau+\nu}\right\}$$

canonical elements for each one of the classical families

$n \in \mathbb{N}$	**Hermite**	**Laguerre** $\alpha \neq -(n+1)$	**Bessel** $\alpha \neq -\frac{n}{2}$	**Jacobi** $\alpha, \beta \neq -(n+1)$, $\alpha+\beta \neq -(n+2)$
$\Phi(x)$	1	x	x^2	x^2-1
$\Psi(x)$	$2x$	$x-\alpha-1$	$-2\,(\alpha x+1)$	$-(\alpha+\beta+2)\,x+(\alpha-\beta)$
$(\chi_n)^k$	$(-2)^k\ n^k$	$(-1)^k\ n^k$	$n^k\ (n+2\alpha-1)^k$	$n^k\ (n+\alpha+\beta+1)^k$
$\Xi_{n-k}(k)$	$(-2)^k\{n\}_{(\mathbf{k})}$	$\dfrac{(-1)^k}{(\alpha+1)_k}\{n\}_{(\mathbf{k})}$	$C_\alpha^k(2\alpha-1+n)_k\{n\}_{(\mathbf{k})}$	$C_{\alpha,\beta}^k(\alpha+\beta+1+n)_k\{n\}_{(\mathbf{k})}$
where			$C_\alpha^k=4^{-k}(2\alpha)_{2k}$	$C_{\alpha,\beta}^k=\frac{(-4)^{-k}\ (\alpha+\beta+2)_{2k}}{(\alpha+1)_k\ (\beta+1)_k}$

Stirling numbers

Representing by $s(k,\nu)$ and $S(k,\nu)$, with $k,\nu \in \mathbb{N}$, the **Stirling numbers of first and second kind**, respectively, the following equalities hold:

$$\{x\}_{(\mathbf{k})} = \sum_{\nu=0}^{k} s(k,\nu)\, x^{\nu} \, .$$

and

$$x^k = \sum_{\nu=0}^{k} S(k,\nu)\, \{x\}_{(\nu)} \, ,$$

where $\{x\}_{(\mathbf{k})} = x(x-1)\ldots(x-k+1)$ represent the **falling factorial** of x
Such numbers fulfil a "*triangular*" recurrence relation, more precisely...

$$\begin{cases} s(k+1,\nu+1) = s(k,\nu) - k\, s(k,\nu+1) \\ s(k,0) = s(0,k) = \delta_{k,0} \\ s(k,\nu) = 0 \, , \quad \nu \geqslant k+1 \end{cases}$$

and

$$\begin{cases} S(k+1,\nu+1) = S(k,\nu) + (\nu+1)\, S(k,\nu+1) \\ S(k,0) = S(0,k) = \delta_{k,0} \\ S(k,\nu) = 0 \, , \quad \nu \geqslant k+1 \end{cases}$$

A-modified factorial of a number

Definition

Let A be a number (possibly complex) and $k \in \mathbb{N}$. For any number x we define

$$\{x\}_{(\mathbf{k};\mathbf{A})} := \begin{cases} 1 & \textit{if} \quad k = 0, \\ \prod_{\nu=0}^{k-1} (x - \nu(\nu + A)) & \textit{if} \quad k \in \mathbb{N}^*, \end{cases} \tag{1}$$

to be the **A-modified falling factorial (of order k)**.

As a result, there exist two unique sequences of numbers $\{\widehat{s}_A(k,\nu)\}_{k,\nu\in\mathbb{N}}$ and $\{\widehat{S}_A(k,\nu)\}_{k,\nu\in\mathbb{N}}$ such that

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A-modified Stirling numbers

Proposition

The set of numbers $\{\widehat{s}_A(k,\nu)\}_{\nu,k\geqslant 0}$ *satisfy the following "triangular" recurrence relation*

$$\widehat{s}_A(k+1,\nu+1) = \widehat{s}_A(k,\nu) - k(k+A)\,\widehat{s}_A(k,\nu+1)\,,$$
$$\widehat{s}_A(k,0) = \widehat{s}_A(0,k) = \delta_{k,0}\,,$$
$$\widehat{s}_A(k,\nu) = 0\,,\nu \geqslant k+1\,,$$

whereas the set of numbers $\{\widehat{S}_A(k,\nu)\}_{\nu,k\geqslant 0}$ *satisfy the "triangular" relation*

$$\widehat{S}_A(k+1,\nu+1) = \widehat{S}_A(k,\nu) + (\nu+1)(\nu+1+A)\widehat{S}_A(k,\nu+1)\,,$$
$$\widehat{S}_A(k,0) = \widehat{S}_A(0,k) = \delta_{k,0}\,,$$
$$\widehat{S}_A(k,\nu) = 0\,,\nu \geqslant k+1\,,$$

for $k,\nu \in \mathbb{N}$*.*

A-modified Stirling numbers: some properties

- $$\widehat{S}_A(k,\nu) = \frac{1}{\nu!}\sum_{\sigma=1}^{\nu}\binom{\nu}{\sigma}(-1)^{\nu+\sigma}\,\frac{(A+2\,\sigma)\,\Gamma(A+\sigma)}{\Gamma(A+\sigma+\nu+1)}\,\Big(\sigma(\sigma+A)\Big)^k,$$

  for $k,\nu\in\mathbb{N}$ and $1\leqslant\nu\leqslant k$.

- When $x=n(n+A)$ for $n\in\mathbb{N}$ and $A\in\mathbb{C}$, its A-modified factorial (of order k) is given by:

  $$\{n(n+A)\}_{(\mathbf{k};\mathbf{A})} = \prod_{\nu=0}^{k-1}\Big(n(n+A)-\nu(\nu+A)\Big) = \prod_{\nu=0}^{k-1}\Big((n-\nu)(n+A+\nu)\Big)$$

  which, in accordance with the definition of falling or rising factorial, may be expressed like

  $$\{n(n+A)\}_{(\mathbf{k};\mathbf{A})} = \{n\}_{(\mathbf{k})}\,(n+A)_k. \qquad (2)$$

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list of the first *A-modified Stirling* numbers of

first kind: $\widehat{s}_A(k,\nu)$

ν / k	1	2	3	4	5
1	1	0	0	0	0
2	$-(1+A)$	1	0	0	0
3	$2\,(1+A)_2$	$-5-3A$	1	0	0
4	$-6\,(1+A)_3$	$49+A(48+11A)$	$-2(7+3A)$	1	0
5	$24\,(1+A)_4$	$-2(410+515\,A)$ $-2\,A^2(202+25A)$	$273+5A(40+7A)$	$-10(3+A)$	1

second kind: $\widehat{S}_A(k,\nu)$

ν / k	1	2	3	4	5
1	1	0	0	0	0
2	$1+A$	1	0	0	0
3	$(1+A)^2$	$5+3A$	1	0	0
4	$(1+A)^3$	$21+A(24+7A)$	$14+6A$	1	0
5	$(1+A)^4$	$(5+3A)(17+A(18+5A))$	$147+5A(24+5A)$	$10(3+A)$	1

Hermite Case

we have

$$\mathcal{F} = D^2 - 2xD$$

$$\Lambda_\nu(k;x) = \binom{k}{\nu} (-2)^{k-\nu}\, P_{k-\nu}(x)\,,\ 0 \leqslant \nu \leqslant k,$$

therefore ...

$$\begin{cases} \displaystyle\sum_{\nu=0}^{k} \Lambda_\nu\,(k;x)\, D^{k+\nu} = \sum_{\tau=0}^{k} (-2)^{k-\tau}\, s(k,\tau)\, \mathcal{F}^\tau \\ \displaystyle\mathcal{F}^k = \sum_{\tau=0}^{k} (-2)^{k-\tau}\, S(k,\tau) \sum_{\nu=0}^{\tau} \Lambda_\nu\,(\tau;x)\, D^{\tau+\nu} \end{cases},$$

Laguerre Case

we have

$$\mathcal{F} = x\,D^2 - (x-\alpha-1)D$$

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\,\frac{(-1)^{k-\nu}}{(\alpha+1)_k}\,x^\nu\,P_{k-\nu}(x;\alpha+\nu)$$

therefore ...

$$\begin{cases} \displaystyle\sum_{\nu=0}^{k}\Lambda_\nu\,(k;x)\,D^{k+\nu} = \sum_{\tau=0}^{k}\frac{(-1)^{k-\tau}}{(\alpha+1)_k}\,s(k,\tau)\,\mathcal{F}^\tau \\ \displaystyle\mathcal{F}^k = \sum_{\tau=0}^{k}(-1)^{k-\tau}\,(\alpha+1)_\tau\,S(k,\tau)\,\sum_{\nu=0}^{\tau}\Lambda_\nu\,(\tau;x)\,D^{\tau+\nu} \end{cases},$$

Bessel Case

we have

$$\mathcal{F} = x^2\, D^2 + 2(\alpha x + 1) D$$

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\; C_\alpha^k\; (2\alpha - 1 + k + \nu)_{k-\nu}\;\; x^{2\nu}\; P_{k-\nu}(x;\alpha+\nu),\; 0 \leqslant \nu \leqslant k,$$

therefore ...

$$\begin{cases} \displaystyle\sum_{\nu=0}^{k} \Lambda_\nu\,(k;x)\, D^{k+\nu} = \sum_{\tau=0}^{k} C_\alpha^k\, \widehat{s}_{2\alpha-1}(k,\nu)\, \mathcal{F}^\tau \\[2ex] \displaystyle\mathcal{F}^k = \sum_{\tau=0}^{k} \left(C_\alpha^\tau\right)^{-1} \widehat{S}_{2\alpha-1}(k,\tau) \sum_{\nu=0}^{\tau} \Lambda_\nu\,(\tau;x)\, D^{\tau+\nu} \end{cases}$$

Jacobi Case

we have

$$\mathcal{F} = (x^2-1)D + \Big((\alpha+\beta+2)\,x - (\alpha-\beta)\Big)D$$

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\, C^k_{\alpha,\beta}\,(\alpha+\beta+1+k+\nu)_{k-\nu}\,(x^2-1)^\nu\, P_{k-\nu}(x;\alpha+\nu,\beta+\nu)$$

therefore ...

$$\begin{cases} \displaystyle\sum_{\nu=0}^{k} \Lambda_\nu\,(k;x)\,D^{k+\nu} = \sum_{\tau=0}^{k} C^k_{\alpha,\beta}\,\widehat{s}_{\alpha+\beta+1}(k,\tau)\,\mathcal{F}^\tau \\ \displaystyle\mathcal{F}^k = \sum_{\tau=0}^{k} \left(C^\tau_{\alpha,\beta}\right)^{-1} \widehat{S}_{\alpha+\beta+1}(k,\tau)\,\sum_{\nu=0}^{\tau} \Lambda_\nu\,(\tau;x)\,D^{\tau+\nu} \end{cases},$$

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