r-Stirling numbers, Whitney numbers and their common generalization

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22th September, 2008.

Signed Stirling numbers of the first kind

$$x^{\underline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

Meaning of $(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}$: the number of permutations of n elements with k cycles.

Stirling numbers of the second kind

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} x^{\underline{k}}$$

Meaning of $\begin{Bmatrix} n \\ k \end{Bmatrix}$: the number of partitions of n elements into k subsets.

Unimodality. A sequence (a_n) of positive real numbers is unimodal if

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_k = a_{k+1} = \cdots = a_{k+l} \geq a_{k+l+1} \geq a_{k+1+2} \geq \cdots$$

for some indices k and l.

Log-concavity. (a_n) is log-concave if

$$a_k^2 \geq a_{k-1}a_{k+1} \quad (k \geq 1)$$

Log-concavity $\Rightarrow$ unimodality.

Hammersley, Erdős:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} < \begin{bmatrix} n \\ 2 \end{bmatrix} < \cdots < \begin{bmatrix} n \\ K_n - 1 \end{bmatrix} < \begin{bmatrix} n \\ K_n \end{bmatrix} > \begin{bmatrix} n \\ K_n + 1 \end{bmatrix} > \cdots > \begin{bmatrix} n \\ n \end{bmatrix}$$

with

$$\left[\log n - \frac{1}{2}\right] < K_n < [\log n].$$

Canfield, Dobson, Günter, Harborth, Kanold, Lieb, Menon, Pomerance, Rennie, Wegner:

$$\left\{ {n \atop 1} \right\} < \left\{ {n \atop 2} \right\} < \cdots < \left\{ {n \atop K_n - 1} \right\} \leq \left\{ {n \atop K_n} \right\} > \left\{ {n \atop K_n + 1} \right\} > \cdots > \left\{ {n \atop n} \right\}$$

with

$$\frac{n}{\log n} < K_n < \frac{n}{\log n - \log \log n}$$

Broder-Carlitz:

Signed r-Stirling numbers of the first kind

$$x^{\underline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r (x+r)^k$$

Meaning of $(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_r$: the number of permutations of n elements with k cycles *such that the first r elements are in distinct cycles*.

r-Stirling numbers of the second kind

$$(x+r)^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_r x^{\underline{k}}$$

Meaning of $\begin{Bmatrix} n \\ k \end{Bmatrix}_r$: the number of partitions of n elements into k subsets *such that the first r elements are in distinct subsets*.

Necessary tools to prove unimodality of r-Stirling numbers

Newton: If $p(x)=\sum_{k=1}^{n} a_k x^k$ has only real roots then

$$a_k^2 \geq a_{k+1}a_{k-1}\frac{k}{k-1}\frac{n-k+1}{n-k}.$$

Darroch: If $p(x)=\sum_{k=1}^{n} a_k x^k$ has only real roots and $p(1)>0$ then for the maximizing index K_n

$$|K_n-\mu|<1,$$

where

$$\mu=\frac{p'(1)}{p(1)}=\sum_{j=1}^{n}\frac{1}{r_j+1}.$$

Here $-r_j$'s are the roots of $p(x)$.

The case of the first kind – Mező 2007

$$p(x) = \sum_{k=1}^{n} (-1)^{n-k} \left[{n \atop k} \right]_r x^k = (x+r)(x+r+1)\cdots(x+r+n-1).$$

Newton: $(-1)^{n-k} \left[{n \atop k} \right]_r$ is log-concave,

Darroch:

$$\left| K_{n,r} - \left(\frac{1}{r+1} + \frac{1}{r+2} + \cdots + \frac{1}{r+n} \right) \right| < 1,$$

that is,

$$\left| K_{n,r} - \left(r + \log\left(\frac{n-1}{r-1} \right) \right) \right| < 1.$$

The case of the second kind – Mező 2007

$$\sum_{k=1}^{n} {n \brace k}_r x^k = B_{n,r}(x)$$

For $r = 0$, we get the usual Bell polynomials.

The problem: we do not know the root-structure of these polynomials so the above theorems cannot be applied.

Theorem.

$$B_{n,r}(x) = x\left(B'_{n-1,r}(x) + B_{n-1,r}(x)\right) + rB_{n-1,r}(x)$$

Rolle theorem $\Rightarrow$ $B_{n,r}(x)$ has only real and negative roots $\Rightarrow$ ${n \brace k}_r$ is log-concave.

We shall call these polynomials as r-Bell polynomials.

Estimations for the maximizing index

Bonferroni inequality:

$$\frac{(m+r)^n}{m!} - \frac{(m-1+r)^n}{(m-1)!} < \left\{ {n+r \atop m+r} \right\}_r < \frac{(m+r)^n}{m!},$$

and a corollary:

$$\left\{ {n+r \atop m+r} \right\}_r \sim \frac{(m+r)^n}{m!} \quad (n \to \infty).$$

Theorem – Mező 2007

$$\frac{n-r}{\log(n-r)} < K_{n,r} < \frac{n-r}{\log(n-r) - \log\log(n-r)},$$

Some words again on the r-Bell numbers and polynomials.

$$B_{n,r} := B_{n,r}(1) = \sum_{k=1}^{n} {n \brace k}_r.$$

Meaning: $B_{n,r}$ gives the number of partitions of an n-set with the restriction that *the first r elements are in distinct subsets*.

A surprising occurrence: in a paper of Whitehead he made a table of the coefficients of the polynomial $x^r(x)_{n-r}$ according to the "complete graph base". These are exactly the r-Bell numbers.

Exponential generating function:

$$\sum_{n=0}^{\infty} B_{n,r}(x)\frac{z^n}{n!} = e^{x(e^z-1)+rz}.$$

Ordinary generating function:

$$\sum_{n=0}^{\infty} B_{n,r}(x)z^n = \frac{-1}{rz-1}\,\frac{1}{e^x}\,{}_1F_1\left(\begin{array}{c}\frac{rz-1}{z}\\ \frac{rz+z-1}{z}\end{array}\,\middle|\, x\right).$$

Summation formula:

$$B_{n,r}(x) = \frac{1}{e^x}\sum_{k=0}^{\infty}\frac{(k+r)^n}{k!}x^k.$$

Integral formula (Cesàro: $r = 0$ in 1885):

$$B_{n,r} = \frac{2n!}{\pi e}\,\mathrm{Im}\int_0^{\pi} e^{e^{e^{i\theta}}} e^{re^{i\theta}} \sin(n\theta)d\theta.$$

The ordered Bell numbers (1968)

Ordered Bell numbers (R. D. James, M. Tanny):

$$F_n = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}.$$

Meaning: F_n gives the number of *ordered* partitions of an n-set.

Some identities:

$$F_n = \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}}.$$

$$F_n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle 2^k.$$

The ordered r-Bell numbers – Mező 2007

$$F_{n,r} = \sum_{k=0}^{n} (k+r)! \left\{ {n+r \atop k+r} \right\}_r .$$

Meaning: $F_{n,r}$ gives the number of *ordered* partitions of an n-set such that the first r elements are in distinct subsets.

Some identities:

$$F_n = \sum_{k=0}^{\infty} \frac{(k+r)^n}{2^{k+r+1}} \frac{(k+r)!}{k!}.$$

$$F_n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_r 2^k.$$

The r-Eulerian numbers – Joint work with G. Nyul

Meaning: the permutation $\begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix}$ has an r-ascent (i_j, i_{j+1}) if $i_j < i_{j+1}$ *and* $\{i_j, i_{j+1}\} \not\subseteq \{1, \ldots, r\}$.

$\left\langle {n \atop k} \right\rangle_r$ gives the number of permutations of an n-set having k r-ascents.

Recursion:

$$\left\langle {n \atop m} \right\rangle_r = (m+1)\left\langle {n-1 \atop m} \right\rangle_r + (n-m+r)\left\langle {n-1 \atop m-1} \right\rangle_r.$$

The r-Eulerian triangles are no longer symmetric (if $r > 1$) but log-concavity is preserved.

Generalized Worpitzky-identity – G. Nyul

Original identity (Worpitzky – 1883):

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} = x^n.$$

The generalized identity involving r-Eulerian numbers is:

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_r \binom{x+k}{n+r} = x^n (x)_r.$$

Whitney numbers – Dowling 1973

Whitney numbers of the first kind:

$$m^n x^{\underline{n}} = \sum_{k=0}^{n} w_m(n,k)(mx+1)^k.$$

Whitney numbers of the second kind:

$$(mx+1)^n = \sum_{k=0}^{n} m^k W_m(n,k) x^{\underline{k}}.$$

They connected to the so-called Dowling lattices.

$$w_1(n,k) = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}, \quad W_1(n,k) = \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}.$$

Dowling numbers – M. Benoumhani (1996)

$$D_m(n) = \sum_{k=0}^{n} W_m(n,k)$$

They satisfy the same identities as the Bell numbers.

But the integral representation formula were not presented before.

Mező 2008:

$$D_m(n) = \frac{2n!}{\pi \sqrt[m]{e}} \mathrm{Im} \int_0^{\pi} e^{e^{it}} \sqrt[m]{e^{e^{meit}}} \sin(nt) dt.$$

A question of Benoumhani

Ordered Dowling numbers:

$$F_m(n) = \sum_{k=0}^{n} k! W_m(n,k)$$

Benoumhani: it is known that for the ordered Bell numbers $F_n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_r 2^k$. Can we construct "Eulerian-like" numbers to get a similar identity? I gave the answer. If we define these new "Eulerian-like numbers" as

$$A_m(n,k) = \sum_{i=0}^{n} m^i i! W_m(n,i) \binom{n-i}{k} (-1)^{n-i-k},$$

then

$$F_m(n) = \sum_{k=1}^{n} A_m(n,k) 2^k \quad \text{and} \quad A_1(n,k) = \left\langle {n+1 \atop k+1} \right\rangle.$$

The crucial point

It seems that the two way of generalizations (r-Stirling and Whitney) can be unified.

Stirling numbers

Bell and ordered Bell

Eulerian

r-Stirling numbers

r-Bell and ordered r-Bell

r-Eulerian

Whitney numbers

Downing and ordered Downing

“Whitney-Eulerian”

r-Whitney numbers

⋮

Thank you for your attention