FLAG ORDERS

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1. **Flag Weak Order** ABR

2. **Flag Strong Order** ACR

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1. Preliminaries

1.1 Classical Orders on S_n

Let

$$S := \{(i, i+1) : \ 0 \leq i < n\},$$

and

$$T := \{(i, j) : \ 1 \leq i < j \leq n\}.$$

The ***(right) weak order*** on S_n is
the reflexive and transitive closure of $\pi \preceq \cdot \ \sigma$ if

(1) $\sigma = \pi s$ for some $s \in S$; and

(2) $\mathrm{inv}(\pi) < \mathrm{inv}(\sigma)$.

The ***strong order*** on S_n is
the reflexive and transitive closure of $\pi \leq \cdot \ \sigma$ if

(1) $\sigma = \pi t$ for some $t \in T$; and

(2) $\mathrm{inv}(\pi) < \mathrm{inv}(\sigma)$.

1.2 Wreath Products

Consider $G(r,n) = \boldsymbol{Z}_r \wr S_n$

the wreath product of a cyclic group $\boldsymbol{Z}_r$ with a symmetric group S_n.

Example Let $\omega := e^{2\pi i/r}$.

$$v = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & \omega^0 \\ 0 & \omega^1 & 0 \end{pmatrix} \qquad |v| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2. Problems

Problem 1. Define weak and strong orders on $\boldsymbol{Z}_r \wr S_n$.

Problem 2. Find a "correct analogue" of the inversion number on the group $\boldsymbol{Z}_r \wr S_n$.

Problem 3. Find generating sets for $\boldsymbol{Z}_r \wr S_n$, which will be the counterpart of

$$S := \{(i, i+1) : \; 1 \le i < n\},$$

$$T := \{(i, j) : \; 1 \le i < j \le n\}.$$

Denote

$$(x;q,t)_k := \prod_{i=0}^{k-1} (x - [ti-1]_q).$$

Define **(q,t)-Stirling numbers** via

$$x^n = \sum_{k=0}^{n} S_{q,t}(n,k) \cdot (x;q,t)_k.$$

$$(x;q,t)_n = \sum_{k=0}^{n} s_{q,t}(n,k) \cdot x^k.$$

Problem 4. *[Remmel]* Find combinatorial interpretations of these Stirling numbers.

Foata-Han's flag inversion number

For $\pi \in \boldsymbol{Z}_r \wr S_n$ let

$$\operatorname{finv}(\pi) := r \cdot \operatorname{inv}(|\pi|) + \text{sum of exponents}.$$

Example

$$\operatorname{finv}\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & \omega^0 \\ 0 & \omega^1 & 0 \end{pmatrix} = r \cdot 1 + (2 + 0 + 1)$$

Proposition *[Foata-Han]*

$$\sum_{\pi \in \boldsymbol{Z}_r \wr S_n} q^{\operatorname{finv}(\pi)} = \prod_{i=1}^{n} \frac{q^{ri} - 1}{q - 1}.$$

3. The Flag Orders

Let

$$S_{r,n} := \{n_i : \ 1 \le i \le n\} \cup \{a_i : \ 1 \le i < n\},$$

where

$$n_i := \begin{pmatrix} 1 & \dots & i & \dots & n \\ 1 & \dots & i\omega & \dots & n \end{pmatrix}$$

and

$$a_i := \begin{pmatrix} 1 & \dots & i & i+1 & \dots & n \\ 1 & \dots & (i+1)\omega & i & \dots & n \end{pmatrix}.$$

Let

$$T_{r,n} := \{gS_{r,n}g^t : \ g \in B_n\}.$$

The ***flag (right) weak order*** on $\boldsymbol{Z}_r \wr S_n$, $\preceq$, is the reflexive and transitive closure of $\pi \preceq \cdot\ \sigma$ if

(1) $\sigma = \pi s$ for some $s \in S_{r,n}$;   and

(2) $\text{finv}(\pi) < \text{finv}(\sigma)$.

The ***flag strong order*** on $\boldsymbol{Z}_r \wr S_n$, $\leq$, is the reflexive and transitive closure of $\pi \leq \cdot\ \sigma$ if

(1) $\sigma = \pi t$ for some $t \in T_{r,n}$;   and

(2) $\text{finv}(\pi) < \text{finv}(\sigma)$.

Proposition The posets $(G(r,n),\preceq)$ and $(G(r,n),\leq)$ are

(i) ranked (by flag inversion number);

(ii) self-dual (with $\pi \mapsto \pi w_0$, where $w_0 := [\omega^{-1}n,\ldots,\omega^{-1}1]$ is the unique maximal element in both orders);

(iii) rank-symmetric and unimodal.

Proposition The poset $(G(r,n), \preceq)$ is a complemented lattice.

Theorem
Suppose that $\pi \prec \sigma$ and that $\mathrm{finv}(\sigma) - \mathrm{finv}(\pi) \geq 2$.
Then the order complex of the open interval (π, σ) is homotopy equivalent to the sphere $\mathbf{S}^{k-2}$ if σ is the join of k atoms of the interval $[\pi, w_0]$; and contractible otherwise.

Corollary For every $\pi, \sigma \in G(r,n)$

$$\mu(\pi,\sigma) = \begin{cases} (-1)^k & \sigma \text{ is a join of } k \text{ atoms in } [\pi, w_0]; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary (Tits Property)

Any two labelled maximal chains in $(G(r,n),\preceq)$ are connected via the following pseudo-Coxeter moves

$$n_i n_j = n_j n_i \qquad (i \neq j),$$

$$a_i n_j = n_j a_i \qquad (j \neq i, i+1),$$

$$a_i n_{i+1} = n_i a_i \qquad (1 \leq i < n),$$

and

$$a_i a_{i+1} n_{i+1} a_i = a_{i+1} n_{i+1} a_i a_{i+1} \qquad (1 \leq i < n).$$

Euler-Mahonian

Denote

$$S_n(q,t) := \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{des}(\pi)}.$$

For every $\pi \in G(r,n)$ let wdes(π) be the number of elements which are covered by π in the poset $(G(r,n), \preceq)$.

Proposition

$$\sum_{\pi \in G(r,n)} q^{\mathrm{finv}(\pi)} t^{\mathrm{wdes}(\pi)}$$

$$= (1 + qt[r-1]_q)^n S_n(q^r, \frac{t[r]_q}{1 + qt[r-1]_q}).$$

3. Colored Rook Monoid

The colored rook monoid $P(r,n)$ consists of partial permutations on n letters colored by $\{\omega^0, \ldots, \omega^{r-1}\}$.

Example

$$v = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & 0 \\ \omega^1 & 0 & 0 \end{pmatrix} \qquad |v| = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

For $\pi \in P(r,n)$ let

$$\mathrm{finv}(\pi) := \mathrm{rank}(\pi) \cdot \mathrm{inv}(\pi) + \text{exponents sum}$$
$$+ r \cdot \sum_{\text{nonzero row } i} (i + n + 1 - |\pi(i)|).$$

Example (cont.)

$$\mathrm{finv}(v) = 2 \cdot 1 + (2+1) + 3 \cdot (1+2+3+3) = 32.$$

Flag Strong Order on $P(r,n)$

The ***flag strong order*** on $P(r,n)$, $\leq$, is the reflexive and transitive closure of $\pi \leq \cdot\, \sigma$ if

(1) $\sigma = \pi t$ for some $t \in T_{r,n}$ and

$\text{finv}(\pi) < \text{finv}(\sigma)$;

or

(2) π is obtained from σ by replacing a nonzero entry by 0.

Remark 1. For $r = 1$ it coincides with Renner-Solomon's strong order.

2. The flag strong order on $G(r,n)$ is embedded as an upper interval.

(q,t) Stirling Numbers

Recall $(x;\ q,r)_k := \prod_{i=0}^{k-1} (x - [ri-1]_q)$.

Theorem

(1) $$x^n = \sum_{k=0}^{n} S_{q,t}(n,k) \cdot (x;\ q,t)_k$$

$$= \sum_{0 \le \pi \le \omega^{r-2} id} q^{\mathrm{finv}(\pi) - \binom{n-\mathrm{rank}(\pi)}{2}} (x;\ q,r)_{n-\mathrm{rank}(\pi)}.$$

(2) $$(x)_{n,q,r} = \sum_{k=0}^{n} s_{q,r}(n,k) \cdot x^k$$

$$= \pm \sum_{id \le \pi \le w_0} q^{\mathrm{finv}(\pi)} (\frac{x}{q})^{\mathrm{rlmax}(\pi)},$$

where $n - \mathrm{rlmax}(\pi) :=$
$\#\{$right to left maxima in $|\pi|$ colored by $r-1\}$.