INVOLUTORY REFLECTION GROUPS

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Symmetric groups

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and by the Robinson-Schensted correspondence

$$\sum_{\lambda \vdash n} f^{\lambda} = \# \text{ of involutions in } S_n$$

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B_n = signed permutations.
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Theorem (Frobenius-Schur)

Let G be finite. Then

$$\sum_{\phi \in Irr(G)} \dim \phi = \# \textit{ of involutions in } G$$

if and only if all irreducible complex representations of G can be realized over $\mathbb{R}$.

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Example

$G(r, n)$, the group of $n \times n$ monomial matrices whose non-zero entries are r-th roots of 1.

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \end{bmatrix} \in G(4,4)$$

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$G(r,p,n)$, the elements in $G(r,n)$ whose permanent is a r/p-th root of unity. The matrix above is an element in $G(4,2,4)$.

Involutory groups

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Question: which complex reflection groups are involutory?

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This duality plays a fundamental role in the study of the invariant theory of complex reflection groups (C. 2008).

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- If $G = G(r,1,1,n)$ then $G^* = G$. This holds in particular for $S_n = G(1,1,1,n)$ and $B_n = G(2,1,1,n)$.

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Lemma

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Proof by enumeration. No natural bijection.

The characterization

By the (projective) Robinson-Schensted correspondence

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Theorem (C, 2009)

The group $G(r, p, q, n)$ is involutory if and only if either $GCD(p, n) = 1, 2$ or $GCD(p, n) = 4$ and $r \equiv p \equiv q \equiv n \equiv 4 \mod 8$.

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Corollary

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- Baddeley for wreath products;
- Adin-Postnikov-Roichman for the groups $G(r, n)$.

The character of a model

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Theorem (Bump-Ginzburg)

Let G be finite, $z \in Z(G)$, $\tau \in Aut(G)$ such that $\tau^2 = 1$. Assume that

$$\sum_{\phi \in Irr(G)} \dim \phi = \#\{v \in G : v\tau(v) = z\}.$$

Then

$$\sum_{\phi \in Irr(G)} \chi^{\phi}(g) = \#\{v \in G : v\tau(v) = gz\}.$$

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If $G \subset GL(n, \mathbb{C})$ is involutory then

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Symmetric vs antisymmetric

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Two types of absolute involutions in G^*.

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- Antisymmetric elements: $A \in G(r, n)$ then

$$A\bar{A} = -I \iff A = -A^t$$

Example

$$A = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

then $A\bar{A} = -I = I \in G^*$.

Colors of generalized permutations

$$A = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \zeta_4^{3} & 0 \\ 0 & 0 & 0 & \zeta_4^{0} \\ \zeta_4^{1} & 0 & 0 & 0 \\ 0 & \zeta_4^{2} & 0 & 0 \end{bmatrix},$$

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If $A \in G^*$ then

$$z(A) \in \frac{(\mathbb{Z}/r\mathbb{Z})^n}{\Delta(\mathbb{Z}/p\mathbb{Z})}$$

Coefficients of the model

Let $g \in G$ and $v \in G^*$, for example

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Theorem (C. 2009)

Let $G = G(r,p,n)$ be involutory. Then the vector space M^* endowed with the above action of G extended by linearity is a model for G.

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Theorem

Using the same definition as before for the action, we have that $M(r,q,p,n)$ is a model for the projective reflection group $G(r,p,q,n)$.

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- $(\lambda, \mu) \downarrow_{D_n} = (\mu, \lambda) \downarrow_{D_n}$.
- if $\lambda \neq \mu$ then $(\lambda, \mu) \downarrow_{D_n}$ is irreducible while

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Feeling

The irreducible constituents of the submodule spanned by the elements in any symmetric conjugacy class are exactly those corresponding to the shapes of the elements in the class by the (projective) Robinson-Schensted correspondence.