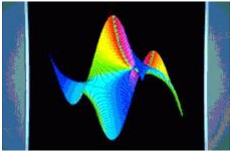
Asymptotics for reflectable lattice walks in a Weyl chamber of type B



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Random turns model of vicious walkers

In general: k walkers on $\mathbb{N} \times \mathbb{N}$ with steps from the set $\{\rightarrow, \nearrow, \searrow\}$. At each step exactly one walker makes a step from the set $\{\nearrow, \searrow\}$.
Non-intersecting: At no time any two paths share a vertex.

This corresponds to a walk in $0 < x_1 < \cdots < x_k$ with steps of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$.

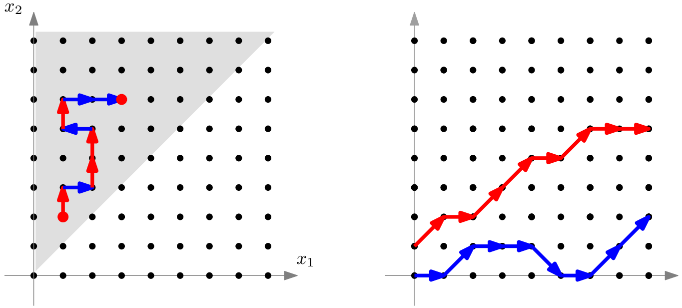

Figure: Correspondence between a walk in $0 < x_1 < x_2$ from $(1, 2)$ to $(3, 6)$ and two vicious walkers from $(0, 0) \rightarrow (8, 2)$ and $(0, 1) \rightarrow (8, 5)$

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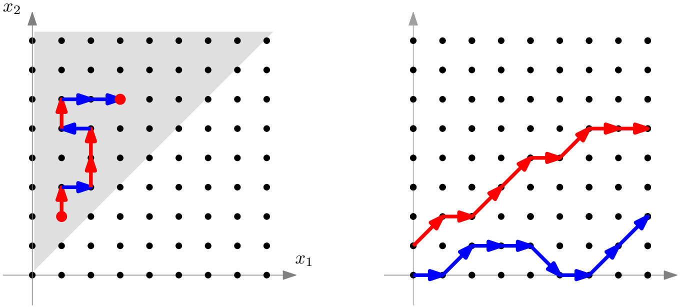

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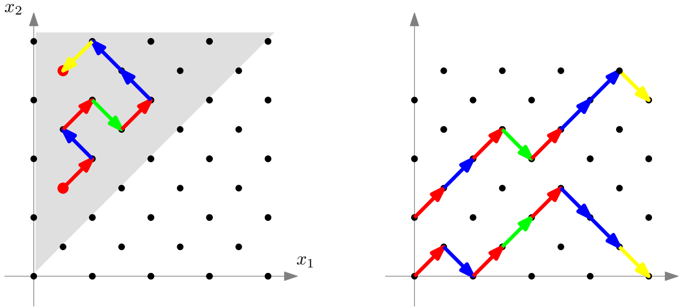

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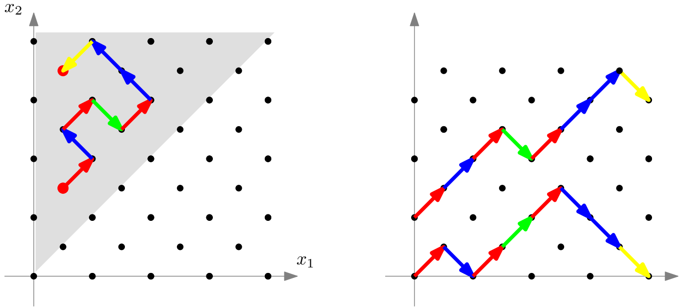

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k-non-crossing tangled diagrams

$(k+1)$-non-crossing tangled diagrams on the set $\{1, 2, \ldots, n\}$ correspond to walks of length n in $0 < x_1 < \cdots < x_k$ with either steps from the set

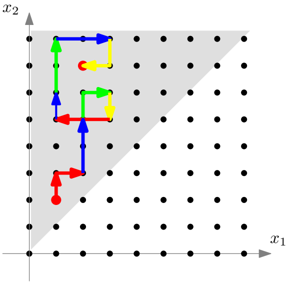
$$\{\mathbf{0}\} \cup \mathcal{A} \cup \mathcal{A}^2 \qquad \text{(with isolated points)}$$

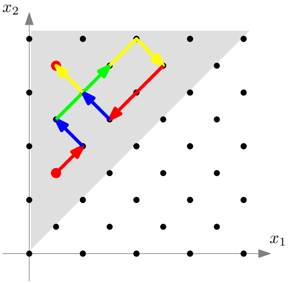
or with steps from the set

$$\mathcal{A} \cup \mathcal{A}^2 \qquad \text{(without isolated points)},$$

where $\mathcal{A} = \{\nearrow, \searrow\}^k$.

The model





We consider lattice walks on a regular lattice $\mathcal{L} \subset \mathbb{R}^k$ that are confined to the region

$$\mathcal{W}^0 = \{(x_1, \ldots, x_k) \in \mathcal{L} \ : \ 0 < x_1 < \cdots < x_k\}.$$

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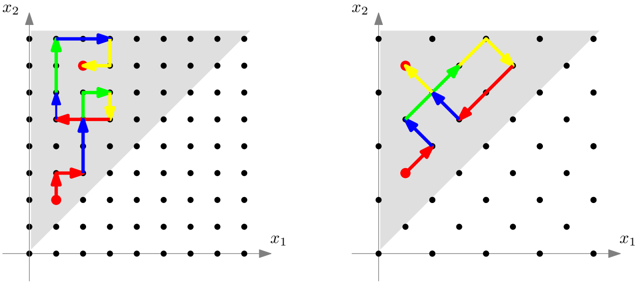

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Skip

Some notation

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$$\mathcal{W} = \left\{(x_1, \dots, x_k) \in \mathbb{R}^k \ : \ 0 \le x_1 \le \cdots \le x_k\right\}$$

Let $\left\{\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(k)}\right\}$ denote the canonical basis in $\mathbb{R}^k$, and set

$$\Delta = \left\{\mathbf{b}^{(j+1)} - \mathbf{b}^{(j)} \ : \ 1 \le j < k\right\} \cup \left\{\mathbf{b}^{(1)}\right\}.$$

The set Δ is a *root system* of the reflection group of type B_k generated by the reflections in the hyperplanes

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Atomic step sets and composite step sets

Definition

Let $\mathcal{A} \subseteq \mathbb{R}^k$ be a finite set and denote by $\mathcal{L}$ the $\mathbb{Z}$-lattice spanned by $\mathcal{A}$. Then the set $\mathcal{A}$ is said to be an atomic step set if and only if

- If $\mathbf{a} \in \mathcal{A}$ then $r_\alpha(\mathbf{a}) \in \mathcal{A}$ for all $\alpha \in \Delta$.
- If $\mathbf{u} \in \mathcal{W}^0 \cap \mathcal{L}$ and $\mathbf{a} \in \mathcal{A}$ then $\mathbf{u} + \mathbf{a} \in \mathcal{W}$.

Definition

A finite set $\mathcal{S}$ consisting of finite sequences of elements of an atomic step set is said to be an composite step set if and only if

$$(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}) \in \mathcal{S} \implies (r_\alpha(\mathbf{a}^{(1)}), \ldots, r_\alpha(\mathbf{a}^{(j)}), \mathbf{a}^{(j+1)}, \ldots, \mathbf{a}^{(m)}) \in \mathcal{S}$$

for all $\alpha \in \Delta$ and $j = 1, \ldots, m$.

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Statement of the problem

Let $\mathbf{u}, \mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$. We are interested in

- $P_n^+(\mathbf{u} \to \mathbf{v})$, the generating function of n-step walks from $\mathbf{u}$ to $\mathbf{v}$ confined to $\mathcal{W}^0$.

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Known results

- *Krattenthaler et al.:* The number of vicious walkers in the lock step model starting at $(0,0),(0,2),\ldots,(2,2k-2)$ and ending in $(2n,0),(2n,2),\ldots,(2n,2k-2)$ is asymptotically equal to

$$4^{kn}2^{k^2-k}\pi^{-k/2}n^{-k^2-k/2}\prod_{j=1}^{k}(2j-1)!.$$

- *Chen, Zeilberger et al.*: The number of k-noncrossing tangled diagrams behaves like

$$\mathrm{const}\cdot n^{-(k-1)^2+(k-1)/2}\left(4(k-1)^2+2(k-1)+1\right)^n.$$

Asymptotics for $P_n^+(\mathbf{u} \to \mathbf{v})$

Theorem

Let $\mathcal{M}$ we denote the set of maximal points of $(\varphi_1, \dots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \dots, e^{i\varphi_k})|$.
We have the asymptotics

$$P_n^+(\mathbf{u}\to\mathbf{v}) = |\mathcal{M}| S(1,\dots,1)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{S(1,\dots,1)}{nS''(1,\dots,1)}\right)^{k^2+k/2} \times \frac{\left(\prod_{1\le j<m\le k} (v_m^2 - v_j^2)(u_m^2 - u_j^2)\right)\left(\prod_{j=1}^{k} v_j u_j\right)}{\left(\prod_{j=1}^{k}(2j-1)!\right)} \left(1 + O(n^{-1/4})\right)$$

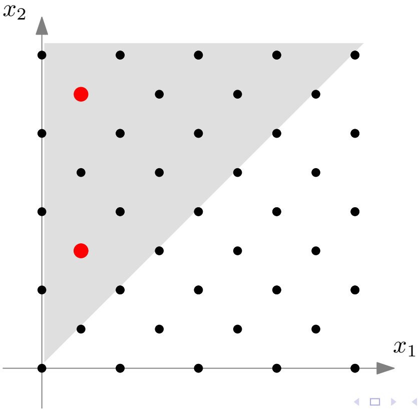
as $n \to \infty$ *in the set* $\{n \ : \ P_n^+(\mathbf{u}\to\mathbf{v}) > 0\}$.

The reflection principle

Theorem (Gessel, Zeilberger)

Under certain assumptions on the step set, we have

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \sum_{r \in B_k} (-1)^{l(r)} P_n(r(\mathbf{u}) \to \mathbf{v}).$$



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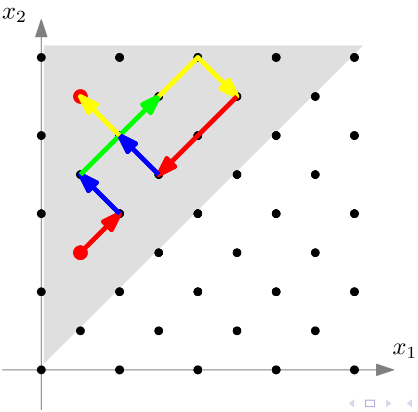

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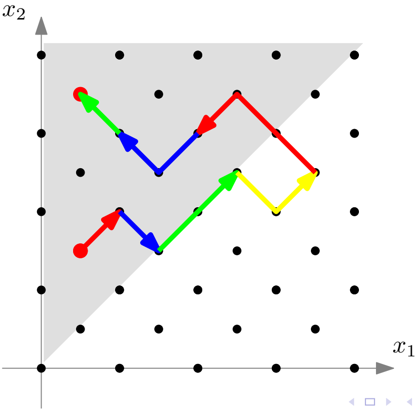

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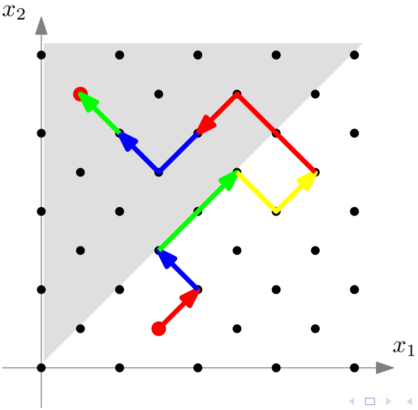

The step generating function

We associate

$$\mathbf{s} = (\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}) \in \mathcal{S} \qquad \longleftrightarrow \qquad w(\mathbf{s})\mathbf{z}^{\delta \mathbf{s}},$$

where $\delta \mathbf{s} = \mathbf{a}^{(1)} + \cdots + \mathbf{a}^{(k)}$. The step generating function $S(z_1, \ldots, z_k)$ is defined by

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The generating function for n-step walks $\mathbf{u} \to \mathbf{v}$ is given by

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The step generating function - Properties

Lemma (Grabiner and Magyar)

For type B_k, the only reflectable sets are

$$\left\{\pm\mathbf{b}^{(1)}, \pm\mathbf{b}^{(2)}, \ldots, \pm\mathbf{b}^{(k)}\right\} \qquad and \qquad \left\{\sum_{j=1}^{k} \varepsilon_j \mathbf{b}^{(j)} \;:\; \varepsilon_j \in \{\pm 1\}\right\}.$$

Corollary

The composite step generating function $S(z_1, \ldots, z_k)$ is either equal to

$$P\left(\sum_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right)\right) \qquad or \qquad P\left(\prod_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right)\right).$$

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An exact counting formula

Lemma

For any two lattice points $\mathbf{u}, \mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$ *we have*

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \frac{1}{(2i)^2 \pi^k k!} \times \int \cdots \int_{|z_1| = \cdots = |z_k| = 1} \det_{1 \le j,m \le k} \left(z_j^{u_m} - z_j^{-u_m} \right) \det_{1 \le j,m \le k} \left(z_j^{v_m} - z_j^{-v_m} \right) \times S(z_1, \ldots, z_k)^n \left(\prod_{j=1}^{k} \frac{dz_j}{iz_j} \right).$$

An exact counting formula - Proof

The reflection principle gives us for $P_n^+(\mathbf{u} \to \mathbf{v})$ the expression

$$\sum_{\substack{\sigma\in\mathfrak{S}_k \\ \varepsilon_1,\dots,\varepsilon_k\in\{-1,+1\}}} \left(\prod_{j=1}^{k} \varepsilon_j\right) \operatorname{sgn}(\sigma) \left[z_1^{v_1-\varepsilon_1 u_{\sigma(1)}} \dots z_k^{v_k-\varepsilon_k u_{\sigma(k)}}\right] S(z_1,\dots,z_k)^n,$$

which, by virtue of Cauchy's formula, turns into

$$\frac{1}{(2\pi i)^k} \int \cdots \int_{|z_1|=\cdots=|z_k|=1}$$

$$\times\, S(z_1,\dots,z_k)^n \left(\prod_{j=1}^{k} \frac{dz_j}{z_j^{v_j+1}}\right).$$

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Asymptotics

The substitution $z_j \mapsto e^{i\varphi_j}$ gives us

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \frac{1}{\pi^k k!} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \det_{1 \le j,m \le k} (\sin(u_m \varphi_j)) \det_{1 \le j,m \le k} (\sin(v_m \varphi_j)) \times S(e^{i\varphi_1}, \dots, e^{i\varphi_k})^n \prod_{j=1}^{k} d\varphi_j.$$

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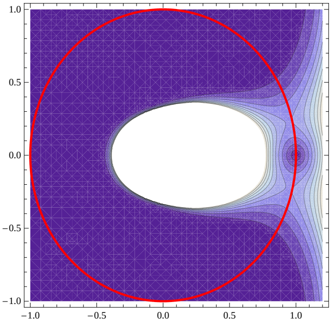
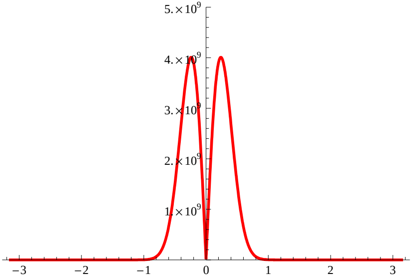
Example: 2-noncrossing tangled diagrams

For 2-noncrossing tangled diagrams, the integral derived on the previous two pages is given by $(\mathbf{a} = (1))$

$$P_n^+(\mathbf{a} \to \mathbf{a}) = \int_{-\pi}^{\pi} \sin(\varphi)^2 \left(1 + 2\cos(\varphi) + 4\cos(\varphi)^2\right)^n d\varphi.$$





Saddlepoint asymptotics

Hence, we know that $\mathcal{M}$, the set of maximal points of

$$(\varphi_1, \ldots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|$$

is a subset of $\{0, \pi\}^k$.
Further, it is seen that

$$P_n^+(\mathbf{u} \to \mathbf{v}) \approx \frac{|\mathcal{M}|}{k!} \int_{-\varepsilon}^{\varepsilon} \ldots \int_{-\varepsilon}^{\varepsilon} \det_{1\le j,m\le k}(\sin(u_m\varphi_j)) \det_{1\le j,m\le k}(\sin(v_m\varphi_j)) \times S(e^{i\varphi_1}, \ldots, e^{i\varphi_k})^n \prod_{j=1}^{k} d\varphi_j,$$

where we choose $\varepsilon = \varepsilon(n) = n^{-5/12}$.

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Saddlepoint asymptotics

It remains to asymptotically evaluate

$$\int_{-\varepsilon}^{\varepsilon} \dots \int_{-\varepsilon}^{\varepsilon} \det_{1\le j,m\le k}(\sin(u_m\varphi_j)) \det_{1\le j,m\le k}(\sin(v_m\varphi_j))\, S(e^{i\varphi_1},\dots,e^{i\varphi_k})^n \prod_{j=1}^{k} d\varphi_j$$

Simple calculations show that

$$S(e^{i\varphi_1},\dots,e^{i\varphi_k})^n = S(1,\dots,1)^n e^{-n\Lambda\sum_{j=1}^{k}\varphi_j^2/2}\left(1+O\left(n^{-5/3}\right)\right)$$

for $max_j|\varphi_j| < n^{-5/12}$, where $\Lambda = \frac{S''(1,\dots,1)}{S(1,\dots,1)}$.

But how do we expand $\det_{1\le j,m\le k}(\sin(u_m\varphi_j))$?

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Determinants and asymptotics: Technique

Lemma

Let $A_m(x_j, y_m)$ *be analytic for* $\max_j |x_j| < R$. *Then we have*

$$\det_{1\le j,m\le k}\left(A_m(x_j,y_m)\right) = \left(\prod_{1\le j<m\le k}(x_m-x_j)\right)\det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int_{|\xi|=R}\frac{A_m(\xi,y_m)\,d\xi}{\prod_{\ell=1}^{j}(\xi-x_\ell)}\right).$$

Proof.

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Proof.

$$\int_{|\xi|=R}\frac{A_m(\xi,y_m)d\xi}{(\xi - x_j)} - \int_{|\xi|=R}\frac{A_m(\xi,y_m)d\xi}{(\xi - x_t)} = (x_t - x_j)\int_{|\xi|=R}\frac{A(\xi,y)d\xi}{(\xi - x_j)(\xi - x_t)}.$$

Determinants and asymptotics: $\det(\sin(u_m\varphi_j))$

Lemma

For all $u_1, \ldots, u_k \in \mathbb{R}$ we have as $(\varphi_1, \ldots, \varphi_k) \to (0, \ldots, 0)$ the asymptotics

$$\det_{1\le j,m\le k}(\sin(u_m\varphi_j)) = \left(\prod_{j=1}^{k}\varphi_j\right)\left(\prod_{1\le j<m\le k}(\varphi_m^2-\varphi_j^2)\right)\left(\prod_{j=1}^{k}\frac{(-1)^{j-1}}{(2j-1)!}\right)$$

$$\times\left(\left(\prod_{j=1}^{k}u_j\right)\left(\prod_{1\le j<m\le k}(u_m^2-u_j^2)\right)+O\left(\max_j|\varphi_j|^2\right)\right).$$

Determinants and asymptotics: Proof

We have to take into account the symmetry

$$\sin(u_m\varphi_j) = \frac{1}{2}\left(\sin(u_m\varphi_j) - \sin(-u_m\varphi_j)\right)$$

Now, we plug this into the determinant and obtain by the same series of operations as before

$$\det_{1\le j,m\le k}(\sin(u_m\varphi_j)) = \left(\prod_{j=1}^{k}\varphi_j\right)\left(\prod_{1\le j<m\le k}(\varphi_m^2-\varphi_j^2)\right) \times \det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int_{|\eta|=1}\frac{\sin(u_m\eta)d\eta}{\prod_{\ell=1}^{j}(\eta^2-\varphi_\ell^2)}\right)$$

Determinants and asymptotics: Proof

We have to take into account the symmetry

$$\sin(u_m\varphi_j) = \frac{1}{2}\left(\frac{1}{2\pi i}\int_{|\xi|=R}\frac{\sin(u_m\xi)d\xi}{\xi-\varphi_j} - \frac{1}{2\pi i}\int_{|\xi|=R}\frac{\sin(-u_m\xi)d\xi}{\xi-\varphi_j}\right)$$

Now, we plug this into the determinant and obtain by the same series of operations as before

$$\det_{1\le j,m\le k}(\sin(u_m\varphi_j)) = \left(\prod_{j=1}^{k}\varphi_j\right)\left(\prod_{1\le j<m\le k}(\varphi_m^2-\varphi_j^2)\right) \times \det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int_{|\eta|=1}\frac{\sin(u_m\eta)d\eta}{\prod_{\ell=1}^{j}(\eta^2-\varphi_\ell^2)}\right)$$

Determinants and asymptotics: Proof

We have to take into account the symmetry

$$\sin(u_m\varphi_j) = \frac{\varphi_j}{2\pi i}\int_{|\xi|=R}\frac{\sin(u_m\xi)d\xi}{\xi^2-\varphi_j^2}$$

Now, we plug this into the determinant and obtain by the same series of operations as before

$$\det_{1\le j,m\le k}(\sin(u_m\varphi_j)) = \left(\prod_{j=1}^{k}\varphi_j\right)\left(\prod_{1\le j<m\le k}(\varphi_m^2-\varphi_j^2)\right)$$
$$\times \det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int_{|\eta|=1}\frac{\sin(u_m\eta)d\eta}{\prod_{\ell=1}^{j}(\eta^2-\varphi_\ell^2)}\right)$$

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Consider again

$$\int_{-\varepsilon}^{\varepsilon}\cdots\int_{-\varepsilon}^{\varepsilon} \det_{1\le j,m\le k}(\sin(u_m\varphi_j)) \det_{1\le j,m\le k}(\sin(v_m\varphi_j))\, S(e^{i\varphi_1},\ldots,e^{i\varphi_k})^n \prod_{j=1}^{k} d\varphi_j$$

This is asymptotically equal to

$$\left(\prod_{j=1}^{k}\frac{u_j v_j}{(2j-1)!^2}\right)\left(\prod_{1\le j<m\le k}(u_m^2-u_j^2)(v_m^2-v_j^2)\right) \times \int_{-\varepsilon}^{\varepsilon}\cdots\int_{-\varepsilon}^{\varepsilon}\left(\prod_{1\le j<m\le k}(\varphi_m^2-\varphi_j^2)\right)^2\left(\prod_{j=1}^{k}\varphi_j^2 e^{-n\Lambda\varphi_j^2/2}d\varphi_j\right) \times\left(1+O\left(n^{-2/3}\right)\right)$$

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$$\int_{-\varepsilon}^{\varepsilon}\cdots\int_{-\varepsilon}^{\varepsilon}\det_{1\le j,m\le k}(\sin(u_m\varphi_j))\det_{1\le j,m\le k}(\sin(v_m\varphi_j))\,S(e^{i\varphi_1},\ldots,e^{i\varphi_k})^n\prod_{j=1}^{k}d\varphi_j$$

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$$\times\int_{-\varepsilon}^{\varepsilon}\cdots\int_{-\varepsilon}^{\varepsilon}\left(\prod_{1\le j<m\le k}(\varphi_m^2-\varphi_j^2)\right)^2\left(\prod_{j=1}^{k}\varphi_j^2e^{-n\Lambda\varphi_j^2/2}d\varphi_j\right)$$
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This is asymptotically equal to

$$\left(\prod_{j=1}^{k}\frac{u_j v_j}{(2j-1)!^2}\right)\left(\prod_{1\le j<m\le k}(u_m^2-u_j^2)(v_m^2-v_j^2)\right)$$
$$\times (n\Lambda)^{-k^2-k/2}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\left(\prod_{1\le j<m\le k}(\varphi_m^2-\varphi_j^2)\right)^2\left(\prod_{j=1}^{k}\varphi_j^2 e^{-\varphi_j^2/2}d\varphi_j\right)$$
$$\times\left(1+O\left(n^{-2/3}\right)\right)$$

Asymptotics for $P_n^+(\mathbf{u} \to \mathbf{v})$

Theorem

Let $\mathcal{M}$ we denote the set of maximal points of $(\varphi_1, \ldots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|$.
We have the asymptotics

$$P_n^+(\mathbf{u} \to \mathbf{v}) = |\mathcal{M}| S(1, \ldots, 1)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{S(1, \ldots, 1)}{nS''(1, \ldots, 1)}\right)^{k^2+k/2} \times \frac{\left(\prod_{1\le j<m\le k} (v_m^2 - v_j^2)(u_m^2 - u_j^2)\right)\left(\prod_{j=1}^{k} v_j u_j\right)}{\left(\prod_{j=1}^{k}(2j-1)!\right)} \left(1 + O(n^{-1/4})\right)$$

as $n \to \infty$ in the set $\{n \ : \ P_n^+(\mathbf{u} \to \mathbf{v}) > 0\}$.