A bijection between vexillary involutions and Motzkin paths

William J. Keith, Maria Torres & Fátima Rodrigues, CELC, Univ. Lisbon

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Vexillary permutations: 2143-avoiding.

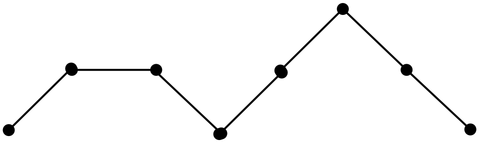
Wilf-equivalent patterns: 2134, 3421, 1243, 1234, 4321, others.

All also involution Wilf-equivalent.

1234-avoiding permutations are mapped by the Robinson-Schensted correspondence to pairs of tableaux of shapes with at most three columns.

Motzkin numbers: $m_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} \frac{(2i)!}{(i+1)!i!}$

Enumerate *Motzkin paths*, paths from $(0,0)$ to $(n,0)$ not going below the x axis, using steps $(1,1)$, $(1,0)$, and $(1,-1)$:



They count many other objects, including 1-2 trees, three-column tableaux, and vexillary involutions.

Current proof of this latter claim uses *generating trees*: vexillary involutions of length n and 1-2 trees with n branches are both shown to be equinumerous with the number of nodes at the n-th level of a tree whose nodes are labeled, and whose branches are written by a *succession rule* on the labels at each node:

Proposition 2.2. *The generating trees of vexillary involutions and* 1-2 *trees can both be characterized by the following succession system*:

$$\begin{cases} (1) & \\ (t) & \overset{1}{\rightsquigarrow} (t+1) \\ & \overset{2}{\rightsquigarrow} (2,t+1),(3,t+1),\ldots,(t+1,t+1) \\ (p,t) & \overset{1}{\rightsquigarrow} (p,p) \\ & \overset{2}{\rightsquigarrow} (2,t+1),(3,t+1),\ldots,(p+1,t+1),(p,t),(p,t-1),\ldots,(p,p+1) \end{cases}$$

Figure 4 describes the first five levels of the generating tree specified by the succession system of Proposition 2.2.

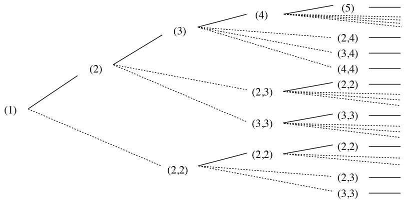


Figure 4: The generating tree of the succession system given in Proposition 2.2.

O. Guibert, E. Pergola, and R. Pinzani. Vexillary Involutions are Enumerated by Motzkin Numbers, Annals of Combinatorics 5 (2001) 153-174.

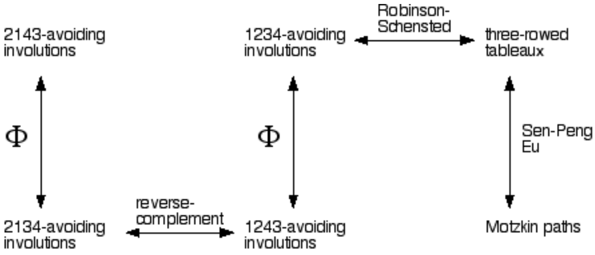
Guibert, Pergola, and Pinzani complete the proof that vexillary involutions are counted by the Motzkin numbers, by noting that 1-2 trees with n branches are known to be counted by the Motzkin numbers.

Open Problem: Marilena Barnabei (of Barnabei, Bonetti, and Silimbani) noted that this means that we don't actually have a map taking a particular vexillary involution to a particular Motzkin path, or any other object counted by the Motzkin numbers. It's easy to associate a specific vexillary involution to a node on the tree, but the association of 1-2 trees is via a counting formula, and a specific association is not obvious.

The goal of this project was to produce such a map, which we will illustrate today.

Overview

We interpreted the question as "Produce a bijection between 2143-avoiding involutions and Motzkin paths." The map we produced works as displayed in the schematic below.



Acknowledgement

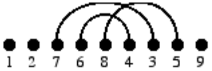
We had believed that the map Φ was new, but a careful look at the literature has led us to believe (without rigorous proof yet) that our algorithm has the same effect as the simplest case of a map in the Ph.D. thesis of Julian West:

J. West, Permutations with forbidden subsequences and stack-sortable permutations, Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, 1990.

Thus an alternative approach might have been to show that West's map respects involutions in the 2143 and 1243 cases, which is not at all clear. Our map definitely respects involutions; had we been able to show that it was West's map, the proof would follow. Instead, we will show that our map is well-defined and one-to-one in both directions, thus confirming that it is a bijection.

The map Φ

Φ consists of manipulations of the *arcs* that define an involution:



Any element is the top or the bottom of an arc, or a fixedpoint, which may be considered both. Denote these by t, b, or fp, pairs as (fp,t), etc.
The permutation 127684359 is 2143-avoiding but not 2134-avoiding. If we *exchange* the top of the 8-5 arc and the fp 9:

we get 127694385, which is 2134-avoiding but not 2143-avoiding.

The map Φ

Φ manipulates arcs involved in 2143, 2134, 1243, and 1234 patterns in permutations, with two algorithms:
Algorithm 1: Input: A 2143-avoiding (resp., 1243-avoiding) involution.

1. Find the lexicographically earliest 34 of a 2134 (resp., 1234) pattern. If none exists, exit.
2. If this 34 is of type (fp,fp), exchange the two places to create a cycle.
3. If this 34 is of type (b,b), exchange the two tops of the arcs.
4. If this 34 is of type (b,fp), exchange the top of the arc and the fp.
5. Repeat.

Output: a 2134-avoiding (resp., 1234-avoiding) involution.

The map Φ

Φ manipulates arcs involved in 2143, 2134, 1243, and 1234 patterns in permutations, with two algorithms:

Algorithm 2: Input: A 2134-avoiding (resp., 1234-avoiding) involution.

1. Find the lexicographically latest 43 of a 2143 (resp., 1243) pattern. If none exists, exit.
2. If this 43 is a cycle, exchange the two places to create two fixed points.
3. If this 43 is of type (t,t), exchange the two tops of the arcs.
4. If this 43 is of type (fp,t), exchange the top of the arc and the fp.
5. Repeat.

Output: a 2143-avoiding (resp., 1243-avoiding) involution.

Example

Before proving that Φ is a bijection, let's take a look at an example to see how it works, starting from a vexillary involution and ending with a 1234-avoiding involution. We'll underscore the moved elements at each step.

Example

Begin with the 2143-avoiding involution:

$$8\ 7\ 4\ 3\ 5\ 6\ 2\ 1\ 9\ 10$$

We look for 2134 patterns, and find many: the earliest 2134 pattern is (8,7,9,10), but the earliest 34 is the (5,6) of (4,3,5,6).

Example

Apply Rule 2 of Algorithm 1:

$$8\ 7\ 4\ 3\ \underline{6}\ \underline{5}\ 2\ 1\ 9\ 10$$

The elements 5 and 6 are fixedpoints, so we convert them in to a cycle.

Example

On the next run-through, use Rule 4:

$$8\ 7\ 4\ 3\ \underline{9}\ \underline{6}\ 2\ 1\ \underline{5}\ 10$$

Now the earliest 34 pattern is (6,9), which is a (b,fp), so we exchange the top (that's the 5) with the fixed 9.

Example

Next, we use Rule 4 again:

$$8\ 7\ 4\ 3\ \underline{10}\ 6\ 2\ 1\ \underline{9}\ \underline{5}$$

And (9,10) is another (b,fp). Notice that the top of the arc is still the 5.

Example

Now (6,9) is the earliest 34. These are (fp,fp), so we make a cycle.

$$8\ 7\ 4\ 3\ 10\ \underline{9}\ 2\ 1\ \underline{6}\ 5$$

There are no more 34 patterns: the involution is now 2134-avoiding.

Example

Next, we apply reverse-complement:

$$6\ 5\ 10\ 9\ 2\ 1\ 8\ 7\ 4\ 3$$

The resulting involution is 1243-avoiding. It also happens to be 1234-avoiding, so we are done.

Proof strategy

In case anyone is conjecturing this from the example, it's not always the case that the involution after reverse-complement is 1234-avoidig. The simplest counterexample is the vexillary involution 1234, which maps by Φ to 1234, reverse-complements to 1234, and then must be mapped by Algorithm 1 again to 1243.

Our algorithm has a Tower of Hanoi flavor, which we believe will be instrumental in helping prove that Φ ends up being West's bijection. Building it from manipulations of arcs assures us that it respects involutions, which is important to answering the question posed.

Proof strategy

The first thing to show is that these algorithms are well-defined: the lexicographically earliest 34 or latest 43 patterns are only of the types listed. We will do so by showing that this is the case in the input permutations, and application of any of the rules never changes this situation.

We must also show that each direction of both cases of the algorithms are one-to-one. This shows that they are a bijection, and proves the theorem.

(It should be noted, of course, that if it could be shown that our algorithms eventually yield West's map, these proofs are superfluous.)

Proof

We will argue these two properties in detail for the case of the map from 2143-avoiding to 2134-avoiding permutations.

Only minor changes are necessary for the other three cases. Indeed, the **34 to **43-avoiding directions seem somewhat simpler.

Lemma

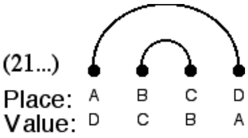
In a 2143-avoiding involution, or such an involution subjected to a number of steps of Algorithm 1, the only types of place pairs for the 34 elements of 2134 patterns are (fp, fp), (b,b), or (b,fp).

Proof: We must consider the remaining six of nine cases of types of pairs, with special attention to the case in which a (b,t) is or is not a cycle. Of course, the latter case is irrelevant when considering 34 patterns, since a cycle is a descent.

Proof
Well-definedness

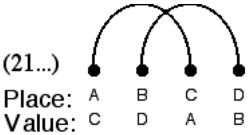
Pairs (b,b) or (b,fp) are permitted. But the lexicographically earliest 34 can never be (b,t). Consider the two arcs in which such a (b,t) could appear:

If the two arcs are nested, all (b,t) pairs are descents:



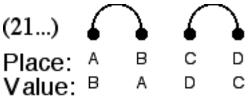
Proof
Well-definedness

Pairs (b,b) or (b,fp) are permitted. But the lexicographically earliest 34 can never be (b,t). Consider the two arcs in which such a (b,t) could appear:
If the two arcs are crossed, all (b,t) pairs are descents:



Proof
Well-definedness

Pairs (b,b) or (b,fp) are permitted. But the lexicographically earliest 34 can never be (b,t). Consider the two arcs in which such a (b,t) could appear:
If the two arcs are consecutive, only the first bottom and last top are an ascent. But then the (b,b) is a lexicographically earlier 34 than the (b,t), and qualifies for the same 21**.



It is clear that the preceding arguments hold regardless of the nature of the underlying involution, including whether it is 2143-avoiding or if any step of the algorithm has been applied.

The same will hold for pairs (t,t), (t,fp), and (t,b).

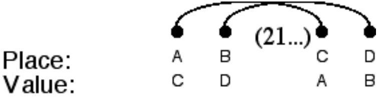
Proof
Well-definedness

If a (t,t) is an ascent, so is the (b,b) consisting of the bottoms of the associated arcs, whether crossed or consecutive (if nested, the (t,t) is a descent).

If a (t,t) is a 34 pattern for some 21, the associated (b,b) is also a 34 pattern, either for the same or an earlier 21. If the (b,b) qualifies for the same 21, we are done; if not, what happened?

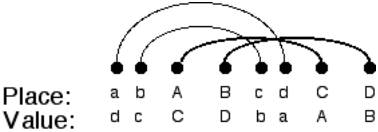
Proof

Well-definedness



CD is still an ascent, and it is an ascent of higher value than AB. So the only way it can fail to qualify as a 34 for the 21 of the 21AB is if the 1, or possibly both the 2 and the 1, follow C or possibly C and D in the permutation.

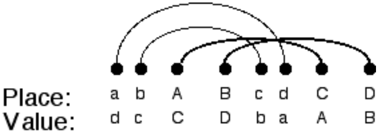
Proof
Well-definedness



But observe that A and B are the values of a place or places preceding the 1 or the 2 and 1 in such a case. Yet the 2 and 1 must be of lower value than A and B. This can only be the case if they are the tops of arcs that have bottoms preceding places A and B.

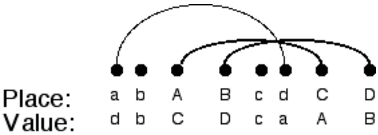
Proof

Well-definedness



Then the displayed values at the tops of these arcs – here d and c – are a descent preceding, and lower in value, than the bottoms CD. We have illustrated this with the case in which both the 2 and the 1 follow both the C and the D, but the the other cases are similar.

Proof
Well-definedness



Here, for example, baAB forms a 2134 pattern, but a lies after C and D. This means that the bottom of its arc, d, lies before C and D, so dbCD forms a 2134 pattern with an earlier 34. (We can think of fp as simultaneously tops and bottoms when convenient.)

Proof

Well-definedness

For (t,fp) or (t,b), the fp and b are always of higher value than the bottom associated to the t, so the associated (b,fp) and (b,b) would be a lexicographically earlier 34. If it does not qualify for the same 21, an argument similar to the preceding shows that it qualifies for an earlier 21 consisting of the associated bottoms.

That covers all three cases beginning with bottoms (two are allowed) and the three beginning with tops (none are). We still need to show that (fp,t) and (fp,b) are impossible.

Proof
Well-definedness

If (fp,t) is an ascent, the fp is of lower value than the top, so it must precede the associated bottom. Hence the associated (fp,b) would be earlier, and obviously qualify for the same 21. So all we need to show is that (fp,b) is impossible for the lexicographically earliest 34 in a 2143-avoiding involution, or one to which any number of steps of the algorithm has been applied.

Unlike the previous arguments, it *is* now important to take the nature of the input involution into account. In 21354, the lexicographically earliest 34 is 35, a (fp,b), but this involution is not vexillary.

Proof
Well-definedness

The existence of a (fp,b) would imply the existence of a 21f354, by which we mean a 21354 pattern in which the 3 is a fixed point. Since 2154 is a 2143 pattern, the input vexillary involution contains no such patterns.

We will show that if such a pattern exists after the application of any of the three rules, such a pattern existed prior to application of the rule, and hence by induction no such pattern ever exists in the involutions under study.

Proof

Well-definedness

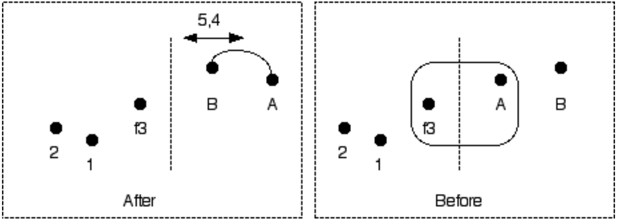
Trivial case: if a 21f354 pattern exists after an application of a rule, and involves no point with value changed by application of the rule, then it existed before application of the rule.

Thus we can confine our attention to the cases when at least one element of the 21f354 is one that had its value changed by the rule.

Proof
Well-definedness

Rule 2 exchanges two fixed points to create a cycle. Suppose it exchanges A and B to create a cycle BA. It cannot create f3.

If after application B is the 5 or 4, f3 would have been an earlier fixed point than A and B, so (f3,A) would have been an earlier (fp,fp). The 21 of the 21f354 would qualify 21f3A as a 2134 pattern. This contradicts the application of Rule 2 to (A,B).



Proof

Well-definedness

Rule 2 exchanges two fixed points to create a cycle. Suppose it exchanges A and B to create a cycle BA. It cannot create f3.

- If A (after application) is the 5 or the 4, either (f3,A) or (A,f3) would be such a pattern, depending on the location of B.

- If either A or B is the 2 or the 1 of the 21f354, the 21 of the 21AB for which Rule 2 was applied would have been part of a 21f354 before application of the rule.

Rule 3 also creates no fixed points. It exchanges the top ends of two arcs, creating two nested arcs. There are five places the f3 could be with respect to these two arcs:

If the f3 is in the first or second place, a 21f354 would have existed with the 21 that invoked the rule, the f3, and the later of the original two arcs.

Proof
Well-definedness

Rule 3 also creates no fixed points. It exchanges the top ends of two arcs, creating two nested arcs. There are five places the f3 could be with respect to these two arcs:

If the f3 is in the third place in the middle, none of the elements moved can be the 2, 1, 5, or 4 because of their relative values, and so the pattern would have existed before the application of the rule.

Proof

Well-definedness

Rule 3 also creates no fixed points. It exchanges the top ends of two arcs, creating two nested arcs. There are five places the f3 could be with respect to these two arcs:

And if the f3 is in the fourth or fifth place, a 21f354 would have existed with the earlier-ending of the two arcs, the f3, and the 54 (which, again, cannot include the later top).

Proof

Well-definedness

Rule 4 applies to (b,fp) pairs.

If the moved fixedpoint is the f3 in a 21f354 after application of the rule, the 5 cannot be the moved top (it has a lower value). If the 5 is later than this top, in which case it was unaltered, and the 21 which invoked the rule would serve as the start of a 21f354 with the original fixedpoint.

Rule 4 applies to (b,fp) pairs.

If the 5 of a 21f354 is between the new fixedpoint and the moved top, its displayed value is higher than that of the the bottom in the (b,fp), and so the (b,b) would have been a lexicographically earlier 34 than the (b,fp).

Rule 4 applies to (b,fp) pairs.

So the f3 of any 21f354 is not the one moved by the rule. If it was before the b of the (b,fp), or after the fp, then a the 21f354 would have existed before application of the rule with the same 21 or 54 respectively.

Proof
Well-definedness

Rule 4 applies to (b,fp) pairs.

Otherwise, it is within the arc created by the rule, so neither end can be the 2, 1, 5, or 4. Then none of the five elements of the pattern were moved by the rule, so the 21f354 existed previously.

Proof
Well-definedness

So if any rule in Algorithm 1 creates a 21f354 pattern, one existed before the rule was applied. But the original 2143-avoiding involution had no such pattern, so none can be created.

Hence the lexicographically earliest 34 pattern is never a (fp,t).

But then the only possible earliest 34 patterns are the ones listed in the algorithm.

The algorithm always increases the lowest element of a 34 pattern. To create a new 34 pattern in which the lowest element is the new 4, there would have to have been a prior element higher than the original 3, following a prior 21, and lower than the original 4. This would have been a lexicographically earlier 34 pair than the pair acted upon by the algorithm.

Thus the algorithm is well-defined. □

Proof
Well-definedness

The preceding portion of the argument is the one that requires the most changes when dealing with 1243-avoiding involutions, but the changes are straightforward and the resulting argument is still well in the same vein.

We will show that if two involutions differ in earliest place i, they will differ in a related place after application of the algorithm. If i is not a manipulated place in either involution, this is trivial, so we will assume it is manipulated in at least one of the two involutions.

In replacing ascents with descents, it is possible to create new 2143 patterns and 2134 patterns in an involution. However, there are 21-descents which survive the process and which can be used as markers to track the differences in the ascending and descending sequences of which place i is a part.

Note that if place i is the earliest differing place in the two involutions, it cannot be a top because then the earlier associated bottoms would be in different places.

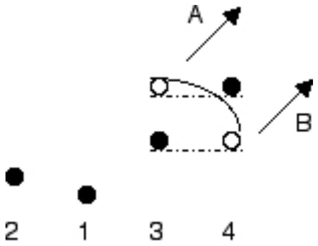
Hence place i is a bottom or a fixed point, and so if it is manipulated by the rules, it is part of the actual 34 sequence after a 21.

Our markers will be based on the following lemma:

Lemma

If a pair of places not part of the manipulation performed by a rule is a 34 pattern for a 21 after an application of that rule, it is a 34 for some 21 consisting of places not altered by the rule.

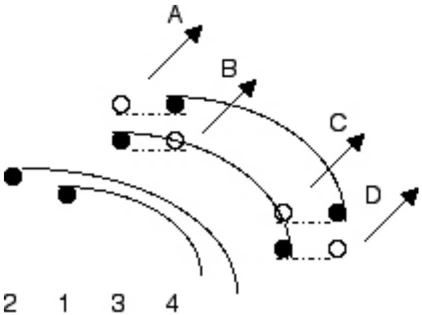
If a pair of places is a 34 for a 21 pattern not involving an element moved by a rule, the lemma holds trivially, so we assume that the pair is a 34 for a 21 involving a place moved by the rule, and show that the requisite earlier 21 exists in each of the three cases.



Here we diagrammatically display the elements involved in an application of Rule 2, the exchange of two fixed points. The arrows A and B represent the possible places of 34 pairs later and higher than the first or second of the two points exchanged by the rule.

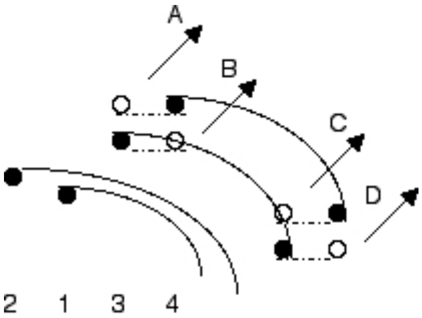
In either case, it's clear that A and B are both 34 patterns for the 21 that invoked the rule in the first place.

Proof
One-to-oneness

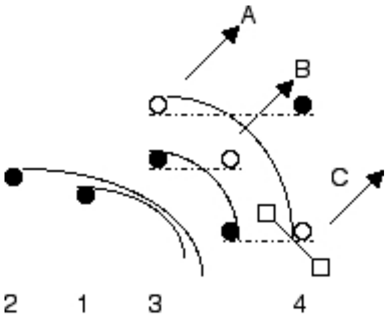


Here we display Rule 3, the exchange of the tops of two arcs. Sequences 34 in possible positions A and B are later and higher than the original 21 which invoked the rule. That thus serves as the required 21.

Proof
One-to-oneness



Positions C and D may be lower than that 21, but by arguments similar to the previous section, the only way this can happen is if that 2 and 1 are the bottoms of arcs which end with lower values, before the positions displayed. Those tops are then the required 21.



Similarly for Rule 4, sequences 34 in positions A and B permit use of the 21 that invoked the rule. If a sequence in position C later and higher than the new top does not, it does so only if the 2 and/or 1 are the tops of arcs that end with low enough values, before the new top, to serve as the required 21.

This means that if place i is the earliest differing place in the two involutions, and it is an element of an ascending 34... sequence after a 21 in both, then there is some descent prior to and lower than i which survives the algorithm, and is necessarily the same in both, being earlier than i.

The input sequence of ascending elements later and higher than that 21 will be reversed to become a descending sequence, which will differ in the s-th lowest place if place i was the s-th element of the ascending sequence.

Proof

One-to-oneness

If place i is manipulated in one involution but not the other, then after the algorithm exits, place i contains a value that is part of a descending sequence of length at least 2 after a 21 in one of the involutions, but not the other.

And if place i is not manipulated in either involution, then it remains different after the algorithm exits. Hence, we have one-to-oneness.

Proof

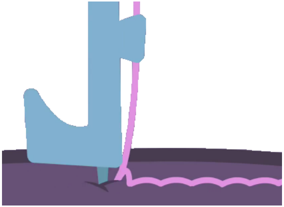
The arguments for the reverse algorithm in the 2134-avoiding to 2143-avoiding case are similar, indeed somewhat simpler.

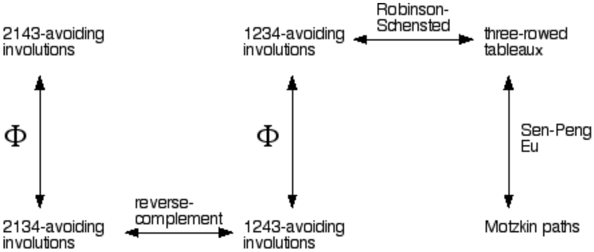
Two well-defined one-to-one maps (probably inverses, especially if they're West's map) constitute a bijection.

And similar arguments apply for the 1243-avoiding and 1234-avoiding cases.

Stitching it together

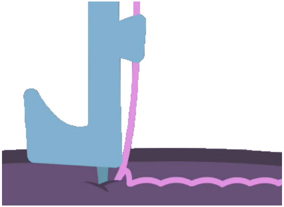
Let's pause to take stock of where we are.





Stitching it together

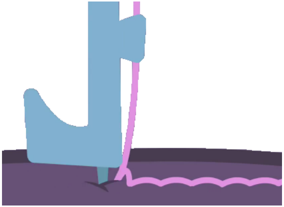
We've shown the hard part – that the map Φ is a bijection that respects involutions.

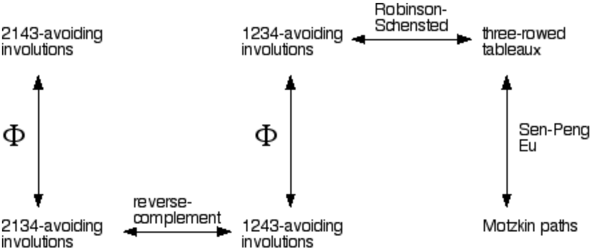


$$
\begin{tikzcd}
\text{2143-avoiding involutions} \arrow[d, leftrightarrow, swap, "\Phi"] & \text{1234-avoiding involutions} \arrow[r, leftrightarrow, "\text{Robinson-Schensted}"] \arrow[d, leftrightarrow, swap, "\Phi"] & \text{three-rowed tableaux} \arrow[d, leftrightarrow, "\text{Sen-Peng Eu}"] \\
\text{2134-avoiding involutions} \arrow[r, leftrightarrow, "\text{reverse-complement}"] & \text{1243-avoiding involutions} & \text{Motzkin paths}
\end{tikzcd}
$$

Stitching it together

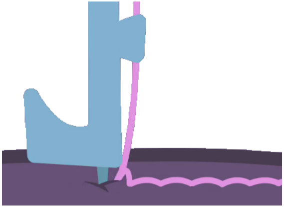
So does reverse-complement, so their composition gives us a bijection between vexillary and 1234-avoiding involutions.

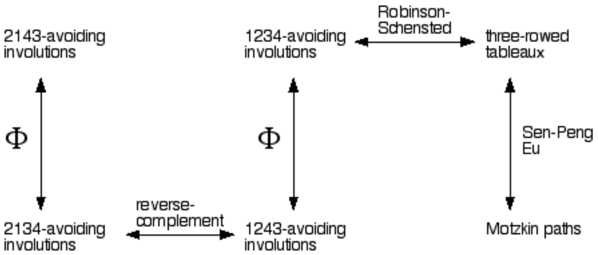




Stitching it together

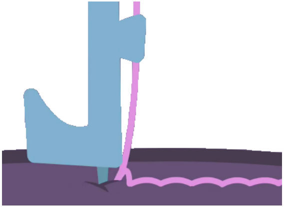
We can complete the project by mapping 1234-avoiding involutions to Motzkin paths, and there are a variety of ways to do this.

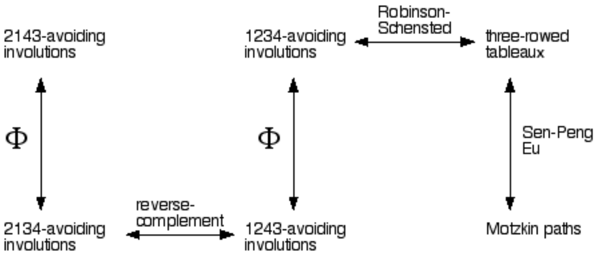




Stitching it together

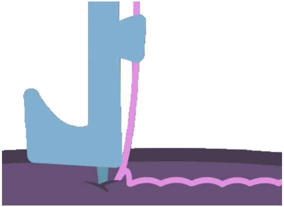
We map 1234-avoiding involutions to tableaux with at most three rows using the Robinson-Schensted correspondence.

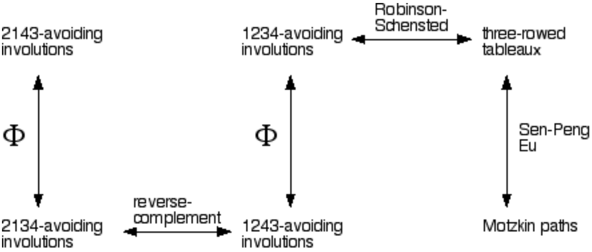




Stitching it together

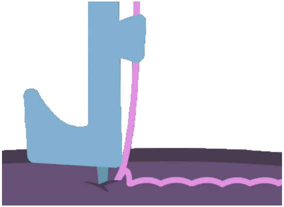
We then use a nice map from the reading words of such tableaux to Motzkin paths, found in a 2010 paper of Sen-Peng Eu, available online with arXiv id 1002.4060.

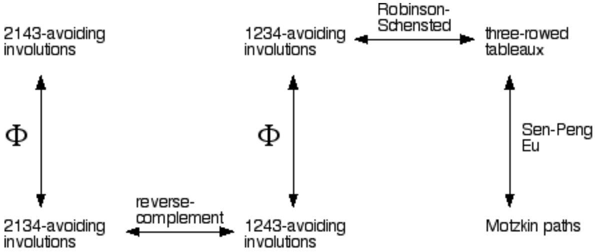




Stitching it together

Let's conclude with a look at some examples of the complete map from vexillary involutions to Motzkin paths.





Examples

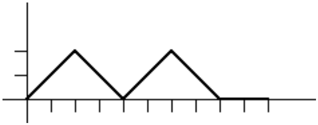
Our starter example was the vexillary involution $(8, 7, 4, 3, 5, 6, 2, 1, 9, 10)$. We mapped it to the 1234-avoiding involution $(6, 5, 10, 9, 2, 1, 8, 7, 4, 3)$. Indeed, this is 123-avoiding; Robinson-Schensted sends it to a pair of the tableaux

$$\begin{array}{cc} 10 & \\ 9 & \\ 6 & 8 \\ 5 & 7 \\ 2 & 4 \\ 1 & 3 \end{array} .$$

The reading word for this tableaux is only 1s and 2s: 1122112211.

Examples

Eu's map sends this word to



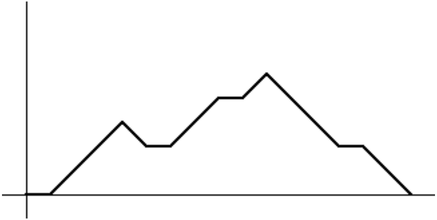
(Eu's map sends a 1-2 reading word to a Dyck path when it has exactly as many 2s as 1s.)

Examples

Here's a bigger example. The vexillary involution $(12, 2, 11, 4, 5, 6, 13, 8, 14, 15, 3, 1, 7, 9, 10, 16)$ is mapped to the 1234-avoiding involution
$(10, 8, 7, 4, 16, 14, 3, 2, 9, 1, 15, 13, 12, 6, 11, 5)$.
This gives us the tableaux and path:

10		
8	16	
7	14	
4	13	
3	9	15
2	6	12
1	5	11

$\rightarrow$



Conclusion

We're sitting down to work out the details as to whether our map is West's, since that would simplify the resulting paper.

We'd also like to perhaps come up with the map we originally intended, which would be a direct bijection from vexillary involutions to Motzkin paths.

To this end we've assembled a Mathematica notebook with all of the maps and a library of the starting involutions and image paths for vexillary involutions up to length 10. We would be happy to make these available to other investigators who ask.

Thank you!