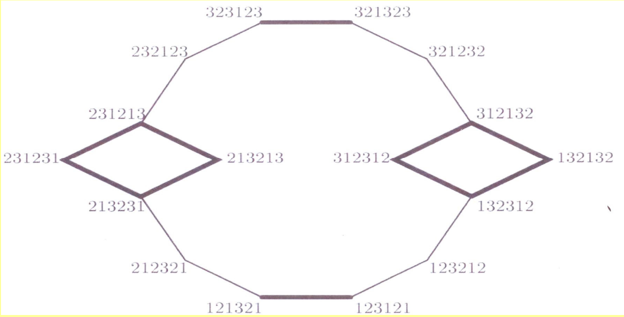
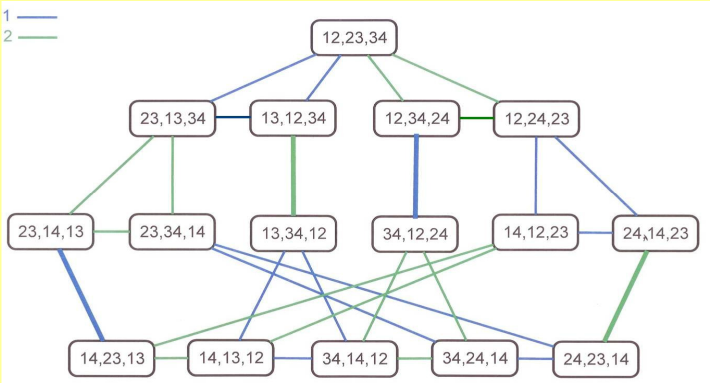
Graphs of Reduced Words

Yuval Roichman (BIU)



Part I – with Victor Reiner (UMN)



Par II – with Ron Adin (BIU)

Part I (with V. Reiner)

Notation:

S_n – symmetric group on n letters

$S = \{s_i := (i, i+1) \mid 1 \le i < n\}$ simple reflections

Braid Relations:

$$s_i s_j = s_j s_i \quad (|j-i|>1) \qquad\qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$\mathcal{W}0 := [n, n-1, \ldots, 1]$ the longest element

The Graph $G_S(w_0)$

- Vertices – all reduced words in *S* of w_0
- Edges – braid relations

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[Tits, Bjorner, Athanasiadis-Santos]

Higher Bruhat order

[Manin-Schechtman, Ziegler, Felsner, SSV]

The Graph $G_S(w_0)$

- Vertices – all reduced words in *S* of w_0
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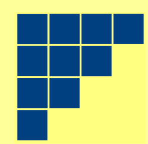
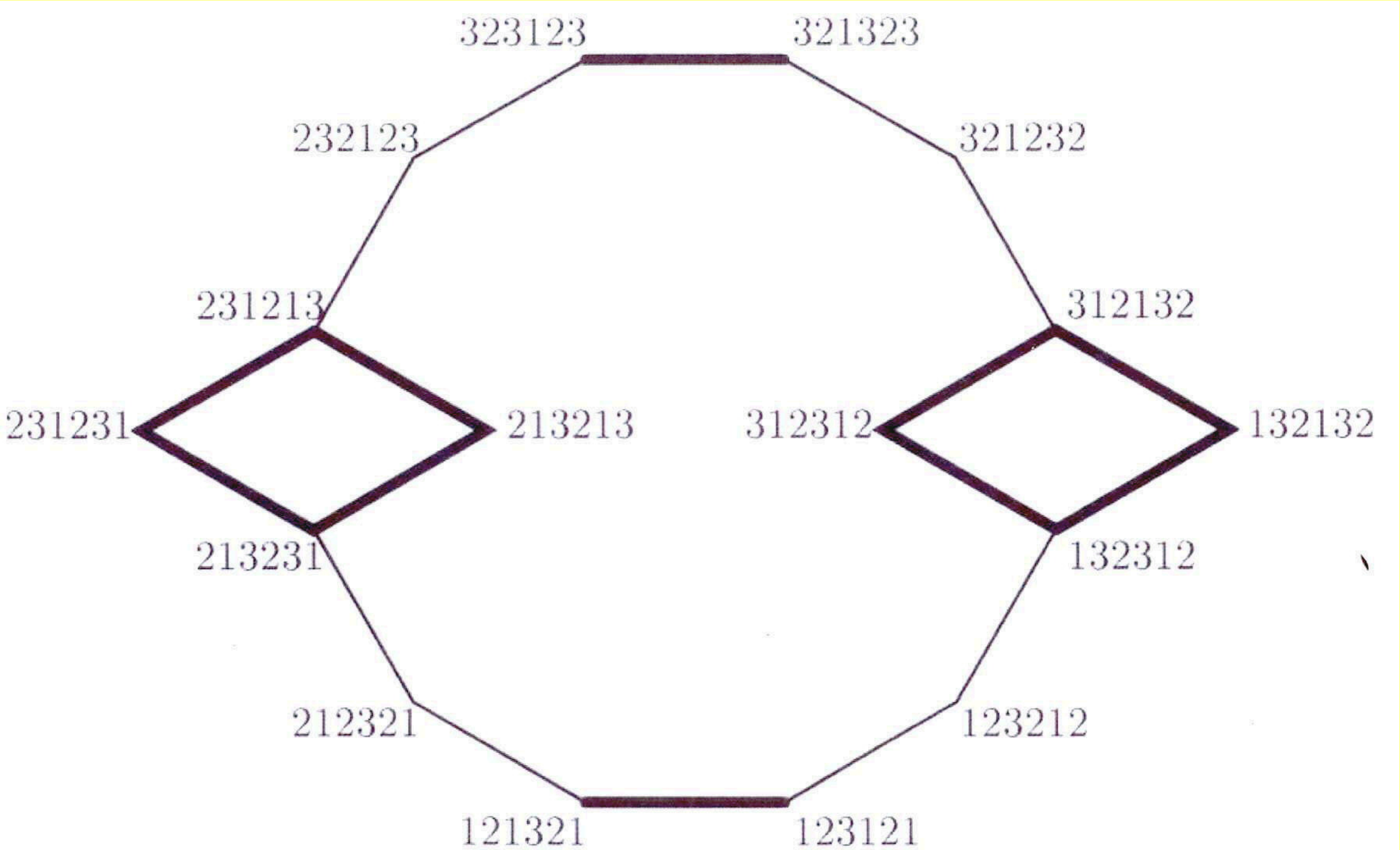
Theorem [Stanley]

\# vertices = # SYT of staircase shape 

Example.



The graph $G_S(w_0)$ for n=4

Properties of $G_S(w_0)$

- not vertex transitive
- 2 types of edges

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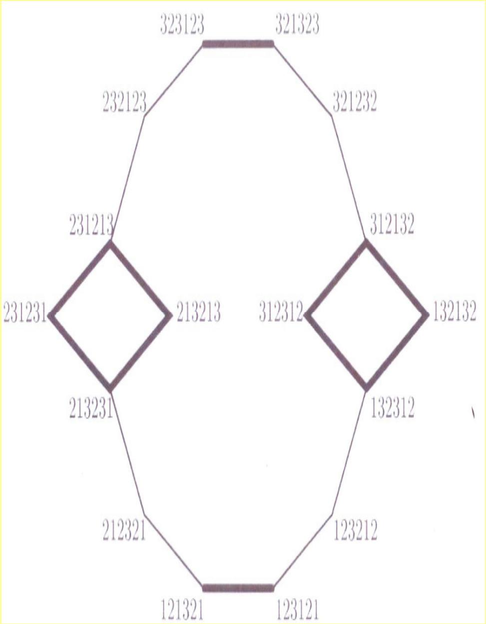
Theorem [Tits] $G_S(w_0)$ is connected.

Problem. Find its diameter.

[Autord-Dehoronoy]

Idea : Turn the graph into a poset



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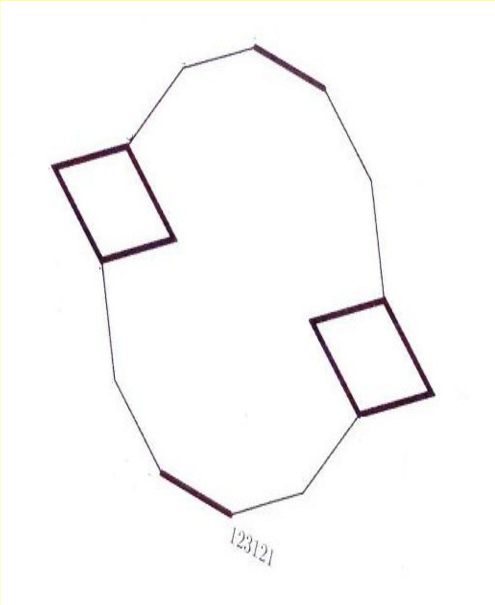

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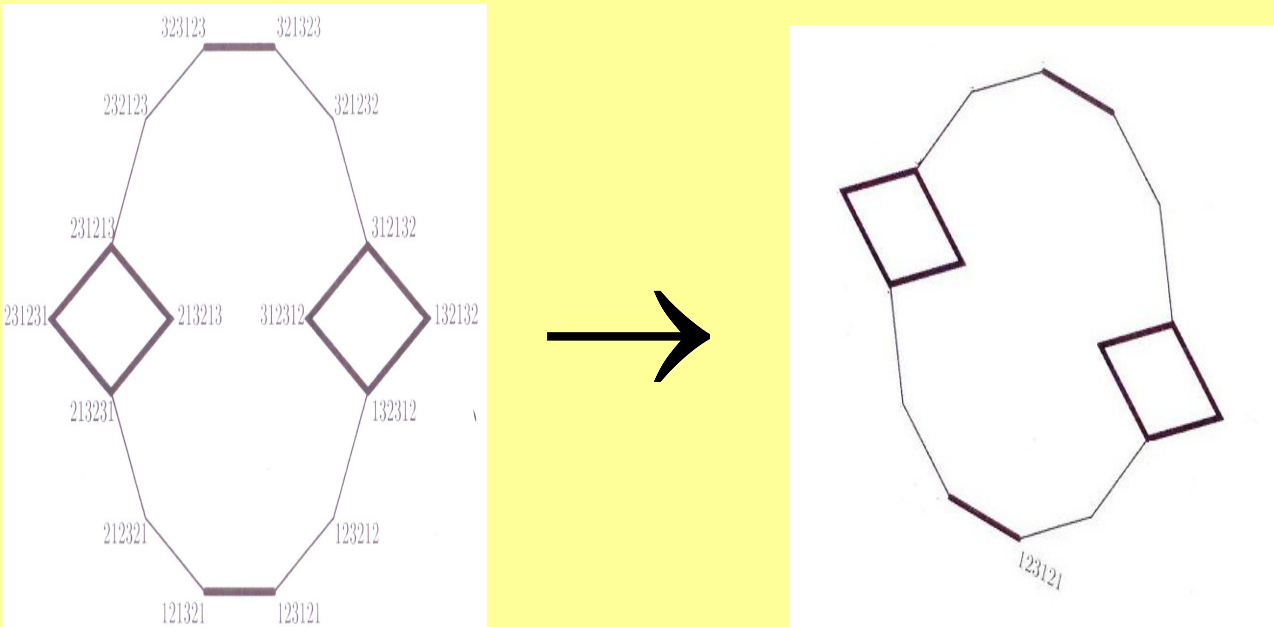

- **<u>Note:</u>** The poset depends on the choice of $\hat{0}$.
- Let $\hat{0}$ = 1234 123 12 1 .

Properties of the poset

- Prop. The poset is ranked.
- Prop. $\exists!$ maximum $\hat{1}$.

$$\hat{1}$$

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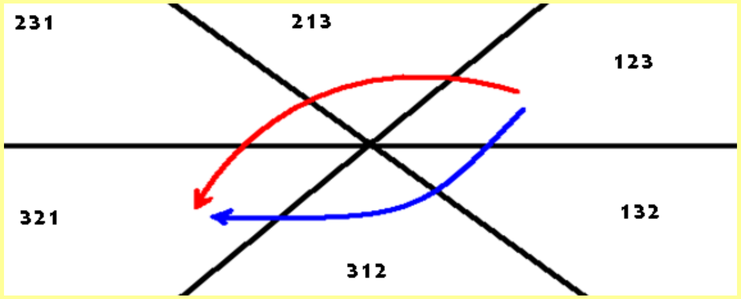
- Cor. $\mathcal{D}iameter = rank(\hat{1})$.

- Fact. For $n>5$ the poset is not a lattice.

reduced words of w_0 $\longleftrightarrow$ maximal chains in weak order

$\longleftrightarrow$ galleries in hyperplane arrangement of type A from c to $-c$

$\longrightarrow$ linear order on all reflections



Examples.

121 $\leftrightarrow$ (12), (13), (23)

212 $\leftrightarrow$ (23), (13), (12)

123 12 1 $\leftrightarrow$ (12), (13), (14), (23), (24), (34)

132312 $\leftrightarrow$ (12), (34), (14), (13), (24), (23)

Inversions

Definition Let

$$I_2(w) := \{(ij) >_{lex} (ab) \mid w = ...(ij)....(ab)....\}$$

$$I_3(w) := \{a < b < c \mid w = ...(bc)....(ac)....(ab)....\}$$

Prop. The pair $(I_2(w), I_3(w))$ determines w .

Prop. $rank(w) = \# I_2(w) \cup I_3(w)$.

Theorem [Reiner – R 09]

$$Diameter(G_S(w_0)) = \frac{1}{24} n(n-1)(n-2)(3n-5)$$

= *# codim-2 subspaces in the intersection lattice.*

Remark. Analogous result holds for type *B*
(and other supersolvable arrangements).

Lower bound holds for other types.
Upper bound open.

Part II (with R. Adin)

Notation:

S_n – Symmetric group on n letters

$T := \{(i, j) \mid 1 \leq i < j \leq n\}$ all reflections

Slide Relations:

$rt = (rtr)r$ $\qquad$ $rt = t(trt)$

$C := (1,2,\ldots,n)$ element of maximal length

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- Edges – slide relations

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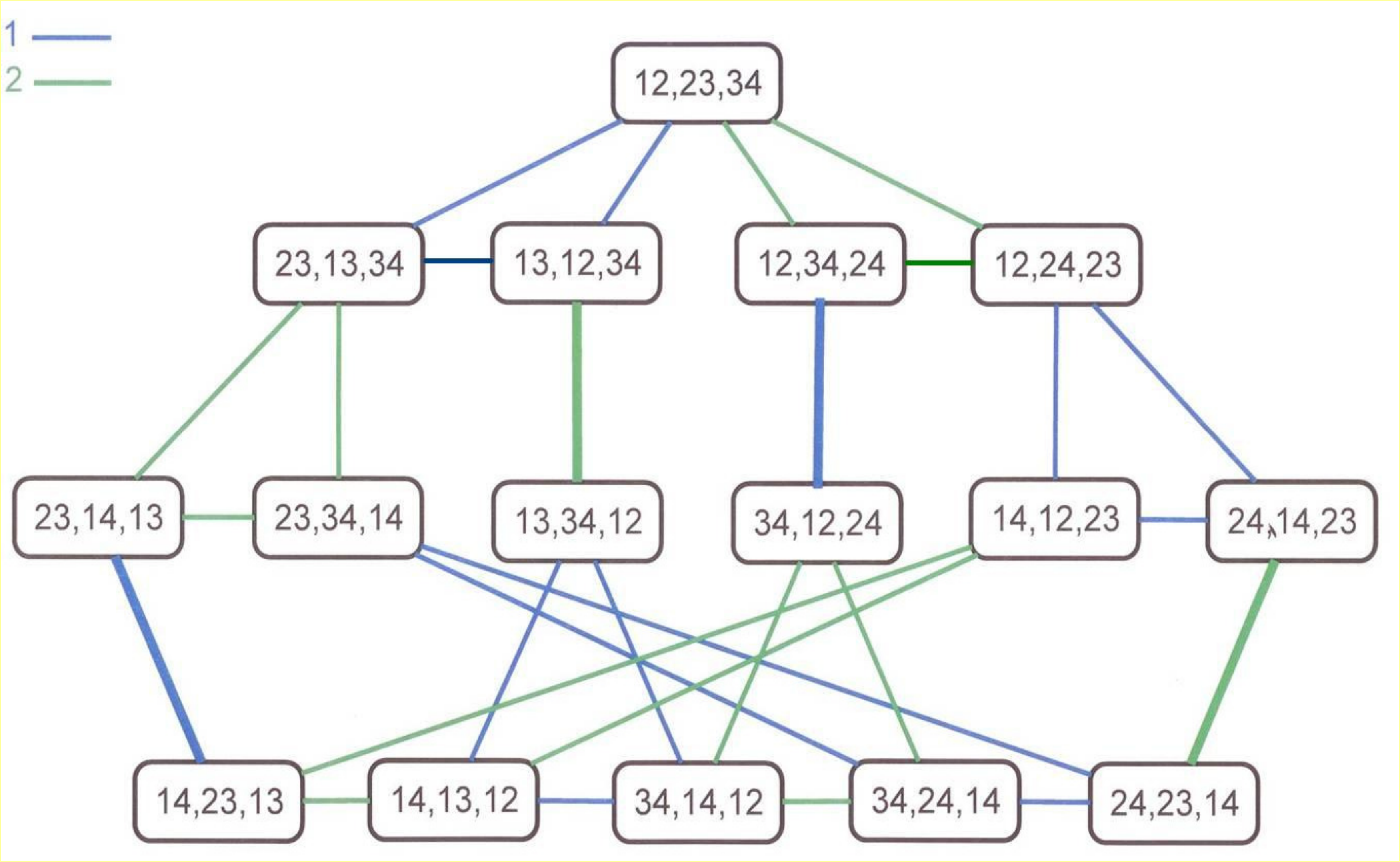
Theorem [Denes] # vertices = # labeled trees

$$= n^{n-2}$$

[Goulden-Yong, HHMMN]

Example.



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- 2 types of edges

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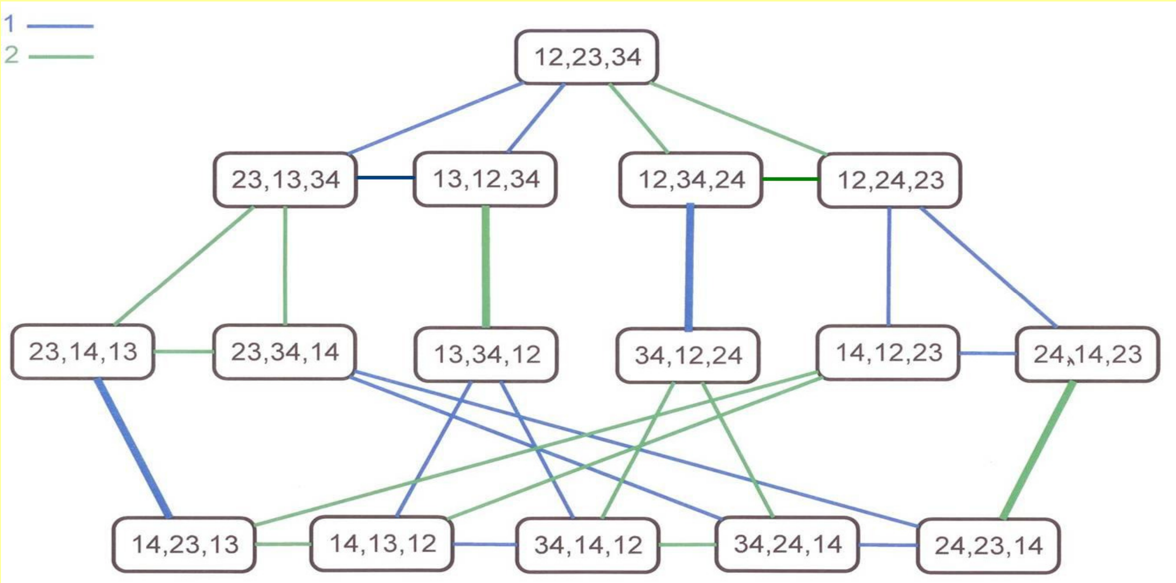
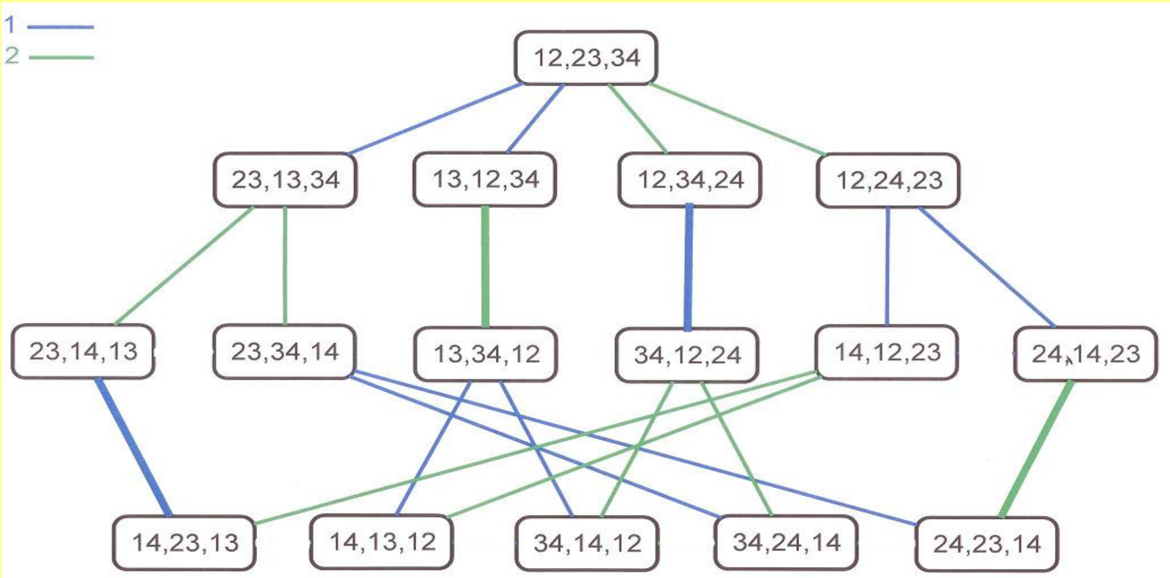


Idea : Turn the graph into a poset



- **<u>Note:</u>** The poset depends on the choice of $\hat{0}$.
- Let $\hat{0} := (s_1, s_2, \ldots, s_{n-1})$.

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- Theorem. # of local maxima = Catalan

  (all of same rank)

- Theorem. The poset is a semi-lattice.

Inversions

Definition. A pair (r,t) in $w=\ldots r\ldots t\ldots$
is an inversion if one the following holds:

(i) $r=(c,d)$ $t=(a,b)$;
(ii) $r=(a,c)$ $t=(a,b)$;
(iii) $r=(b,c)$ $t=(a,c)$;
(iv) $r=(a,d)$ $t=(b,c)$ + extra condition on the tree ;
(v) $r=(b,c)$ $t=(a,d)$ + extra condition on the tree ,

where $a<b<c<d$.

Prop. The pair $(I_{(i)\vee(iii)\vee(v)}(w), I_{(ii)\vee(iv)}(w))$ determines w .

Proposition $rank(w)=inv(w)$.

Theorem $\binom{n-1}{2} \leq Diameter(G_T(c)) \leq \frac{3}{4}n(n-1)$.

Theorem Every maximal interval is isomorphic to the weak order on S_{n-1} .

q,t-Catalan number

Define

$$C_n(q,t) := \sum_{k=0}^{n-1} q^k t^{n-1-k} C_k(q,t) C_{n-1-k}(q,t)$$

with

$$C_0(q,t) := 1$$

[Carlitz-Riordan, Butler-F, Haglund, Sagan-Savage, …]

Right & Left Inversions

A pair (r,t) in $w=\ldots r\ldots t\ldots$ is

a right inversion if it is of types (ii) or (iv) ;

a left inversion if it is of types (iii) or (v).

Theorem

$$\sum_{rank(w)=\binom{n-1}{2}} q^{right-inv(w)} t^{left-inv(w)} = C_{n-1}(q,t)$$

Open Problems

Mostly open!

e.g., other types ….

תודה

Thank you

Grazie

ולהתראות

Good bye

Arrivederci