Zeros of multivariate polynomials in combinatorics

Petter Brändén (KTH)

Séminaire Lotharingien de Combinatoire 69

Strobl, September 10–12, 2012

Multivariate polynomials with prescribed zero restrictions

- Statistical mechanics: Lee–Yang program on phase transitions, correlation inequalities.
- Probability theory: Negative dependence, symmetric exclusion process.
- Matrix theory: Matrix inequalities for Hermitian matrices, Horn's problem.
- Control theory: Stability of solutions systems of equations.
- Complex analysis: Dynamics of zeros of polynomials and entire functions.
- PDE: Hyperbolic PDE, fundamental solution of PDE's with constant coefficients.
- Optimization: Convex optimization generalizing semidefinite programming.
- Combinatorics: Unimodality, log-concavity, graph polynomials, matroid theory.

Outline

- Stable polynomials; a multivariate analog of real-rooted polynomials.
- Inequalities (Negative dependence).
- Symmetric exclusion process.
- Linear operators preserving real-rootedness/stability.
- Multivariate Eulerian polynomials.
- "Stability" in the algebra of free quasi-symmetric functions.
- Infinite log-concavity.
- Stable polynomials and matroid theory.
- Generalized Lax conjecture in convex optimization.

Real-rooted polynomials

Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with positive coefficients.

- $\{a_k\}_{k=0}^n$ is unimodal if for some m:

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_{n-1} \geq a_n.$$

$\Longleftarrow$ $\{a_k\}_{k=0}^n$ is log-concave:

$$a_k^2 \geq a_{k-1} a_{k+1}, \qquad \text{for all } 1 \leq k \leq n-1.$$

$\Longleftarrow$ $\{a_k\}_{k=0}^n$ is ultra-log-concave:

$$\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}, \qquad \text{for all } 1 \leq k \leq n-1.$$

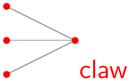
$\Longleftarrow$ $P(x)$ is real-rooted.

Examples of real–rooted polynomials

- Eulerian polynomials (and generalizations):

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)+1},$$

  where $\mathrm{des}(\sigma) = |\{i \in [n-1] : \sigma(i) > \sigma(i+1)\}|$.
- Matching polynomials: Generating polynomial of matchings in a graph. A matching is a subset M of pairwise disjoint edges.
- Independence polynomials of claw-free graphs: Generating polynomial of independent sets of vertices. A graph is claw free if it contains no induced claw.



- Orthogonal polynomials.
- Characteristic polynomials of hermitian matrices.

Multivariate analog of real-rootedness

- Let $P(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ and $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.
- P is stable if
$$\mathbf{x} \in H^n \quad \Longrightarrow \quad P(\mathbf{x}) \neq 0.$$

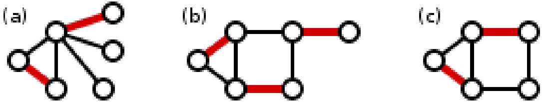
- $(1 - x_1 x_2)(1 + 2x_1 + 4x_2 + 3x_3)$ is stable.
- If $P \in \mathbb{R}[x_1]$, then P is stable iff it is real-rooted.
- If $P \in \mathbb{R}[\mathbf{x}]$ is stable, then $P(x, x, \ldots, x)$ is real-rooted.
- By convention call the zero polynomial stable.
- The space of stable polynomial in n variables and of degree at most d is closed. (Hurwitz' theorem on the continuity of zeros).
- This space has nonempty interior.

Examples

- Helmann–Lieb theorem
  Let $\mathbf{x} = (x_i)_{i \in V}$ be variables and $\lambda = (\lambda_e)_{e \in E}$ nonnegative weights. Then

$$P_{G,\lambda}(\mathbf{x}) = \sum_{M} (-1)^{|M|} \prod_{e=ij \in M} \lambda_e x_i x_j,$$

where the sum is over all matchings is stable.



  - In particular, the generating polynomial

$$\sum_{M} x^{|M|}$$

is real-rooted.

Determinantal polynomials

- Let $A_0, \ldots, A_n$ be hermitian $m \times m$ matrices. If $A_1, \ldots, A_n$ are positive semidefinite, then

$$P(\mathbf{x}) = \det(A_0 + A_1 x_1 + \cdots + A_n x_n)$$

is stable.

Proof. We may assume that A_1 is positive definite. Let $\mathbf{x} + i\mathbf{y} \in H^n$. We need to prove that $P(\mathbf{x} + i\mathbf{y}) \neq 0$.

$$\begin{aligned} P(\mathbf{x} + i\mathbf{y}) &= \det\left(A_0 + \sum_{k=0}^{n} x_k A_k + i \sum_{k=0}^{n} y_k A_k \right) \\ &= \det(A + iB) = \det(B)\det(B^{-1/2} A B^{-1/2} + iI) \end{aligned}$$

- $B^{-1/2} A B^{-1/2}$ is hermitian, so $-i$ is not an eigenvalue. Thus $P(\mathbf{x} + i\mathbf{y}) \neq 0$.

Determinantal polynomials

For $n = 2$ there is a converse which follows from seminal work of Helton and Vinnikov which solves a conjecture of P. Lax from 1958:

Theorem

Let $P(x, y)$ be a real polynomial of degree at most d. TFAE

- P is stable;
- There exist three symmetric real $d \times d$ matrices A, B, C such that A, B are positive semidefinite and

$$P(x, y) = \det(xA + yB + C).$$

- The exact converse fails for more than three variables by a count of parameters: $\mathrm{Det}_{n,d} \leq n\binom{d+1}{2}, \mathrm{Stable}_{n,d} = \binom{n+d}{n}$.

Spanning tree polynomials

- Let $G = (V, E)$ be a connected graph with $V = \{1, \ldots, n\}$.
- The spanning tree polynomial (in $\mathbf{x} = (x_e)_{e \in E}$) is

$$P_G(\mathbf{x}) = \sum_T \prod_{e \in T} x_e,$$

  where the sum is over all spanning trees of G.
- The weighted Laplacian of G is the linear matrix polynomial

$$L_G(\mathbf{x}) = \sum_{e \in E} x_e (\delta_{e_1} - \delta_{e_2})(\delta_{e_1} - \delta_{e_2})^T,$$

  where $\{\delta_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$ and e_1, e_2 are the vertices incident to the edge e.

- Kirchhoff's matrix-tree theorem
  Let $L_G(\mathbf{x})_i$ be the matrix obtained by deleting row and column i in $L_G(\mathbf{x})$. Then $P_G(\mathbf{x}) = \det(L_G(\mathbf{x})_i)$.
  - Spanning tree polynomials are stable.

Inequalities

- Let

$$P(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha)\mathbf{x}^\alpha, \qquad \text{where } \mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

  be a stable polynomial with non-negative coefficients.

- For all $\alpha, \beta \in \mathbb{N}^n$

$$a(\alpha)a(\beta) \geq a(\alpha \vee \beta)a(\alpha \wedge \beta).$$

- Thm. (Gurvits): If P is homogeneous of degree n, then

$$a(1, 1, \dots, 1) \geq \frac{n!}{n^n}\text{Cap}(P),$$

  where

$$\text{Cap}(P) = \inf_{x_1, \dots, x_n > 0} \frac{P(x_1, \dots, x_n)}{x_1 \cdots x_n}.$$

Inequalities

- Recall that a matrix $A = (a_{ij})_{i,j=1}^n$ with nonnegative entries is doubly stochastic if each row and each column sums to one.
- Let $P(\mathbf{x}) = \prod_{i=1}^n (\sum_{j=1}^n a_{ij} x_j) = \sum_\alpha a(\alpha) \mathbf{x}^\alpha$.
- Then
$$a(1, \ldots, 1) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n a_{i\sigma(i)} = \operatorname{per}(A),$$
where $\mathfrak{S}_n$ is the symmetric group on $\{1, \ldots, n\}$.
- Also P is stable and $\operatorname{Cap}(P) = 1$.
- Hence
$$\operatorname{per}(A) \geq \frac{n!}{n^n}$$
- which was conjectured by Van der Waerden in 1926 and proved by Egorychev/Falikman in 1979/1980.

Inequalities: Positive dependence

- Let S be a finite set and μ a discrete probability measure on $\{0,1\}^S$ i.e.,

$$\mu : \{0,1\}^S \to [0,\infty), \qquad \sum_{\eta \in \{0,1\}^S} \mu(\eta) = 1$$

- Think of S as sites that can be occupied by particles.
- μ is pairwise positively correlated if for all distinct $i, j \in S$

$$\mu\Big(\eta : \eta(i) = \eta(j) = 1\Big) \geq \mu\Big(\eta : \eta(i) = 1\Big)\mu\Big(\eta : \eta(j) = 1\Big)$$

- μ is positively associated if for all increasing $f, g : \{0,1\}^S \to \mathbb{R}$

$$\int fg d\mu \geq \int f d\mu \int g d\mu$$

- Let $i \neq j$ and

$$f(\eta) = \begin{cases} 1 \text{ if } \eta(i) = 1, \\ 0 \text{ if } \eta(i) = 0 \end{cases} \quad \text{and} \quad g(\eta) = \begin{cases} 1 \text{ if } \eta(j) = 1, \\ 0 \text{ if } \eta(j) = 0 \end{cases}$$

- Then

$$\int f d\mu = \mu\Big(\eta : \eta(i) = 1\Big) \text{ and } \int fg d\mu = \mu\Big(\eta : \eta(i) = \eta(j) = 1\Big).$$

- Hence positive association is stronger than pairwise positive correlation.

- FKG Theorem (Fortuin, Kasteleyn, Ginibre)

  μ is positively associated if

$$\mu(\alpha)\mu(\beta) \leq \mu(\alpha \vee \beta)\mu(\alpha \wedge \beta), \text{ for all } \alpha, \beta \in \{0,1\}^S.$$

Inequalities: Negative dependence

- μ is pairwise negatively correlated if for all distinct $i, j \in S$

$$\mu\Big(\eta : \eta(i) = \eta(j) = 1\Big) \leq \mu\Big(\eta : \eta(i) = 1\Big)\mu\Big(\eta : \eta(j) = 1\Big)$$

- μ is negatively associated (NA) if for all increasing $f, g : \{0, 1\}^S \to \mathbb{R}$ depending on disjoint sets of variables

$$\int fgd\mu \leq \int fd\mu \int gd\mu$$

- Negative association is a desirable property implying for example central limit theorems, but hard to prove for specific examples.
- There is no known FKG theorem for negative dependence. Find a "useful" property that implies NA!

Examples of NA measures

- The uniform spanning tree measure associated to a connected graph $G = (V, E)$ is the discrete probability measure on $\{0,1\}^E$, that puts all mass and equal mass to the spanning trees of G.
- Thm. (Feder and Mihail): Uniform spanning tree measures are negatively associated.
- Determinantal measures: Let A be a positive semidefinite $n \times n$ matrix with all eigenvalues ≤ 1. A defines a measure by

$$\mu(\eta : \xi \leq \eta) = \det(A[\xi]),$$

  where $A[\xi]$ is the principal minor with rows and columns indexed by ξ.
- Thm. (R. Lyons): Determinantal measures are negatively associated.

Strong Rayleigh measures

- The partition function of μ is the multivariate polynomial

$$Z_\mu(\mathbf{x}) = \sum_{\eta \in \{0,1\}^S} \mu(\eta)\mathbf{x}^\eta, \quad \text{where} \quad \mathbf{x}^\eta = \prod_{i \in S} x_i^{\eta(i)}.$$

- Strong Rayleigh measures: μ is strong Rayleigh if Z_μ is stable.

- Theorem (Borcea, B., Liggett)

  Strong Rayleigh measures are negatively associated.

  - The proof uses a general form of the Feder–Mihail theorem and theorems in analysis due to Grace–Walsh–Szegő and Gårding.
  - Uniform spanning tree measures are strong Rayleigh.
  - Determinantal measures are strong Rayleigh.
  - Strong Rayleigh measures have nonempty interior in the space of all discrete probability measures on $\{0,1\}^S$.

The Symmetric Exclusion Process (SEP)

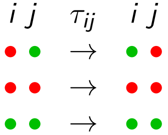
- Finite (countable) set S of sites.
- Configuration of particles $\eta \in \{0,1\}^S$.

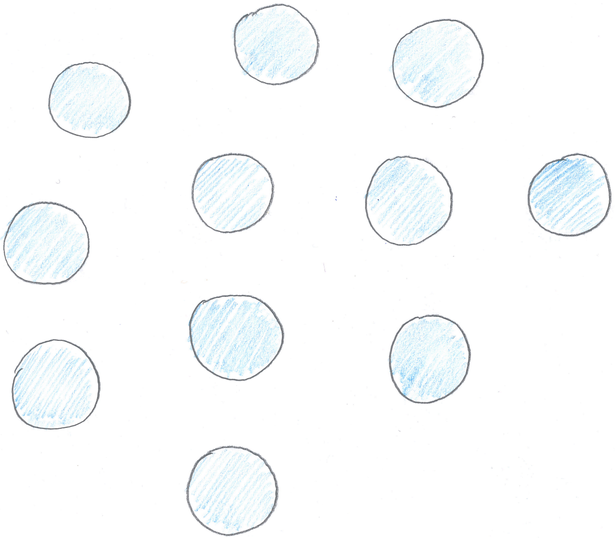
$$\eta(i) = 0 \text{ vacant} \qquad \eta(i) = 1 \text{ occupied}$$

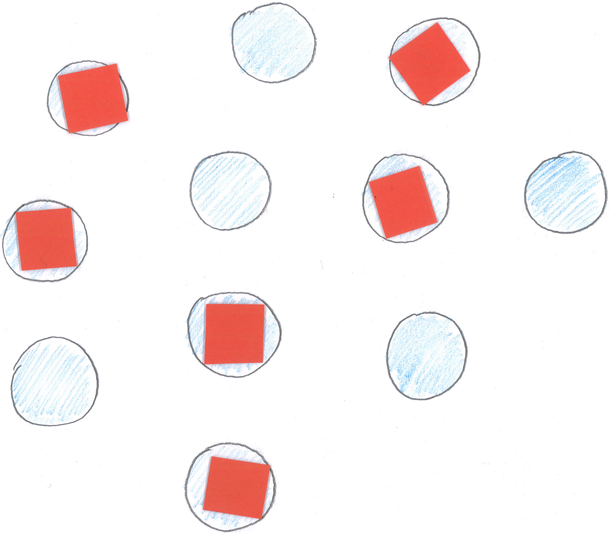
- Nonnegative symmetric $S \times S$ matrix $Q = (q_{ij})_{i,j=1}^n$
- The Symmetric Exclusion Process is the continuous time Markov process on $\{0,1\}^S$, $t \mapsto \eta_t$, with transitions described by:

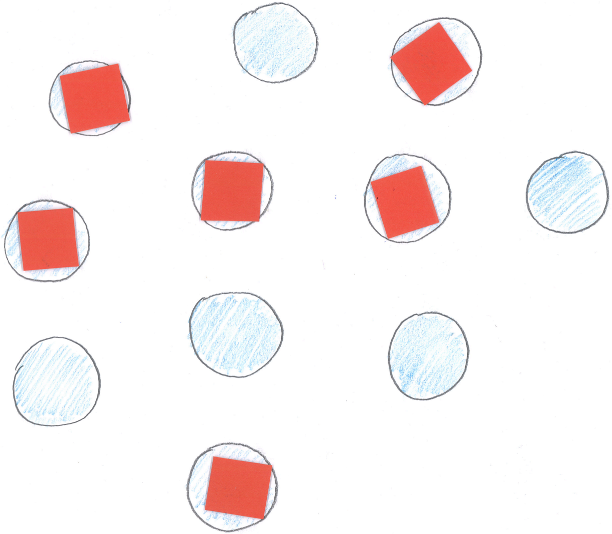
$$\eta \to \tau_{ij}(\eta) \quad \text{at rate } q_{ij}$$

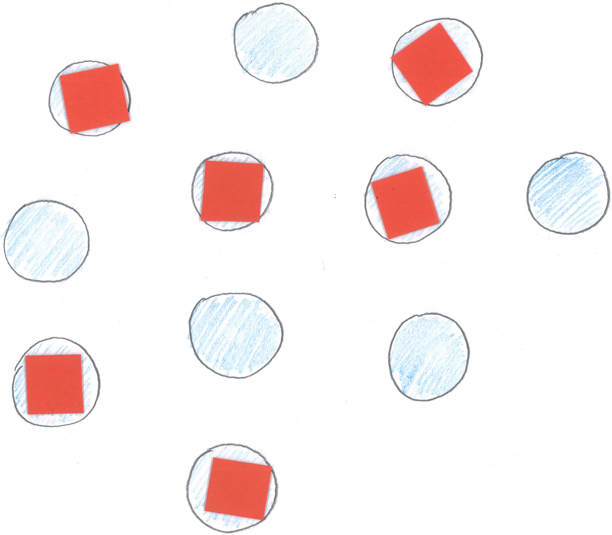
  where τ_{ij} is the transposition that interchanges the coordinates $\eta(i)$ and $\eta(j)$.

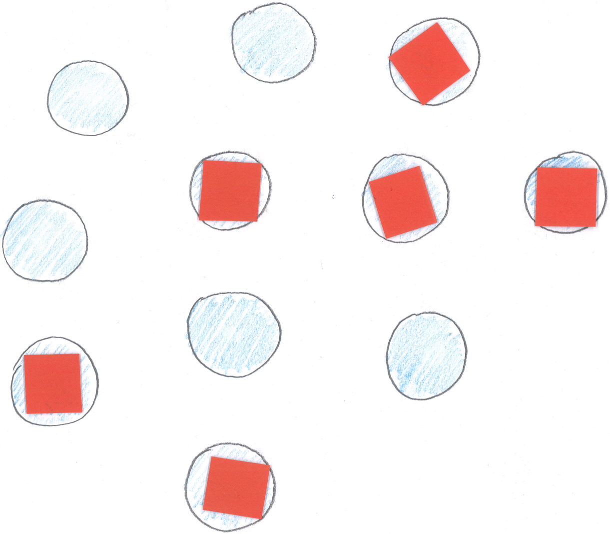


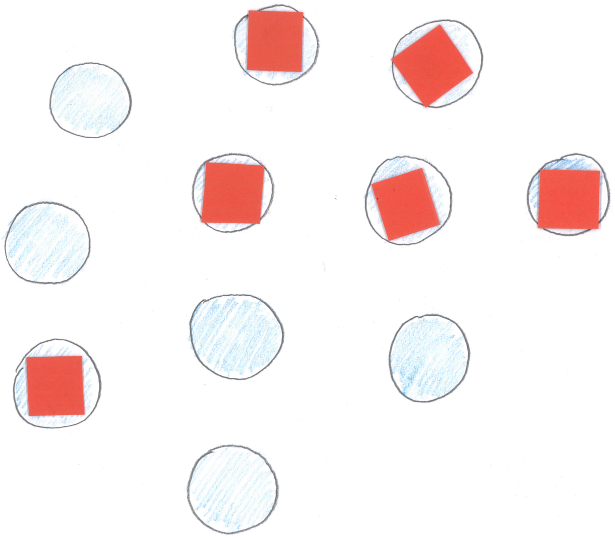












Recall that a product measure is a measure μ of the form

$$Z_\mu(\mathbf{x}) = \prod_{i=1}^{n}(1 - p_i + p_i x_i), \quad \text{where} \quad 0 \le p_i \le 1.$$

Theorem (Liggett 1970's)

Suppose that the initial distribution is a product measure then for any finite $A \subseteq S$ and $t \ge 0$

$$\mathbb{P}(\eta_t \equiv 1 \text{ on } A) \le \prod_{i \in A} \mathbb{P}(\eta_t(i) = 1)$$

Theorem (Andjel 1985)

Suppose that the initial distribution is a product measure then for any finite disjoint sets $A, B \subseteq S$ and $t \ge 0$

$$\mathbb{P}(\eta_t \equiv 1 \text{ on } A \cup B) \le \mathbb{P}(\eta_t \equiv 1 \text{ on } A)\mathbb{P}(\eta_t \equiv 1 \text{ on } B)$$

- Conjecture (Liggett, Pemantle)

  Suppose that the initial distribution in SEP is a product measure, then the distribution is negatively associated for all $t \geq 0$.

  - Unfortunately NA is not preserved by SEP.

- Problem

  Find a negative dependence property P satisfying

  (1) P is preserved by SEP,

  (2) P $\Longrightarrow$ NA,

  (3) Product measures have property P.

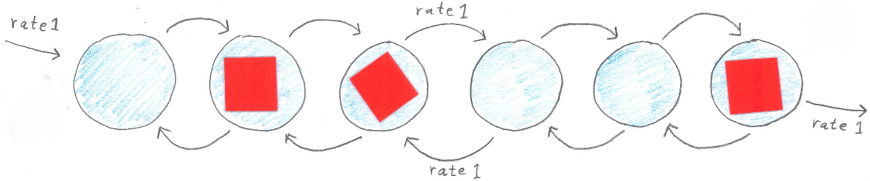
  - Strong Rayleigh measures satisfy (2) and (3).
  - Thm. (Borcea, B., Liggett): The strong Rayleigh property is preserved by SEP.

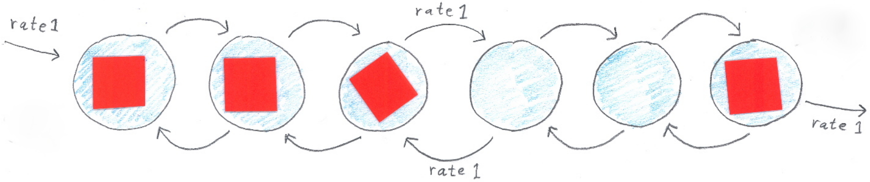
A refined particle process

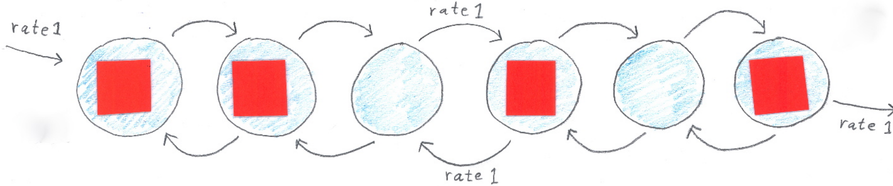
- Consider SEP with particle creation and annihilation allowed.
- At each site a particle is created at a certain rate (provided that the site is empty).
- At each site a particle is annihilated at a certain rate (provided that the site is occupied).

Observation (Wagner)

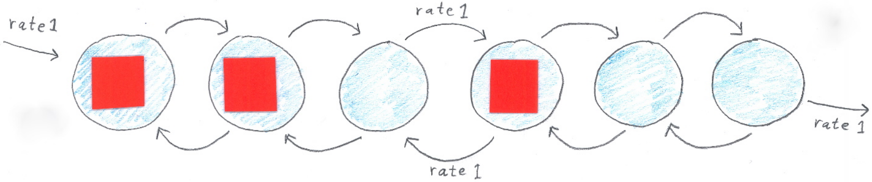
SEP with particle creation and annihilation preserves the strong Rayleigh property.


rate1
rate 1
rate 1
rate 1

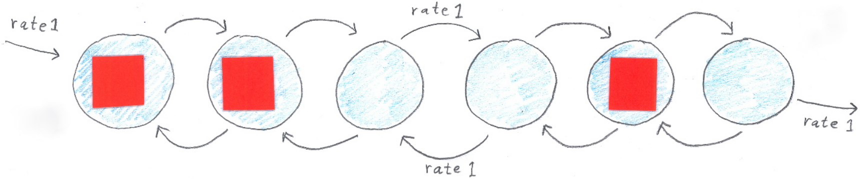

rate1
rate 1
rate 1
rate 1



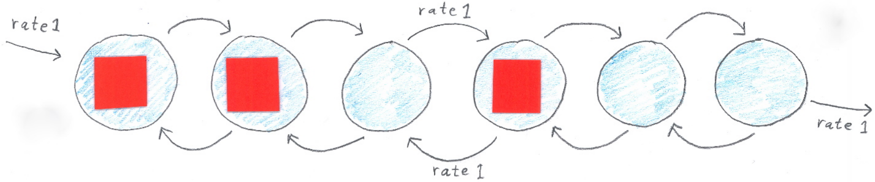
rate1
rate 1
rate 1
rate 1

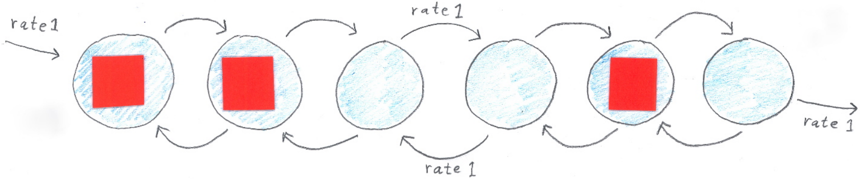


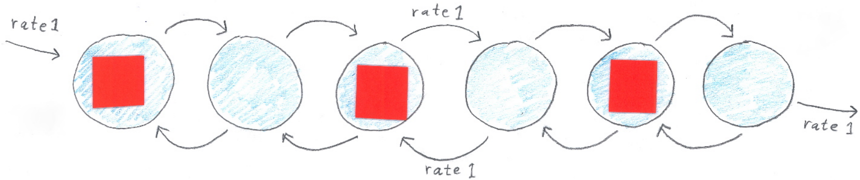
rate1
rate 1
rate 1
rate 1

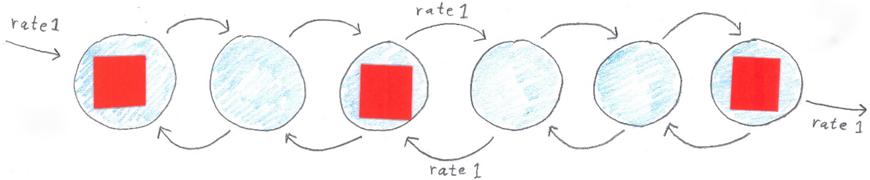


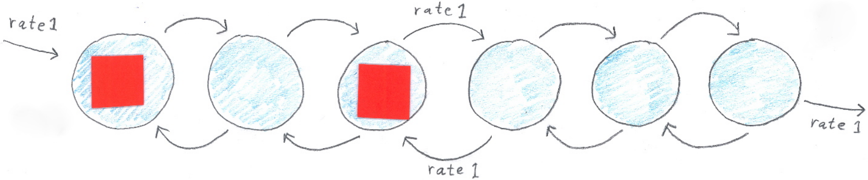
rate1
rate 1
rate 1
rate 1

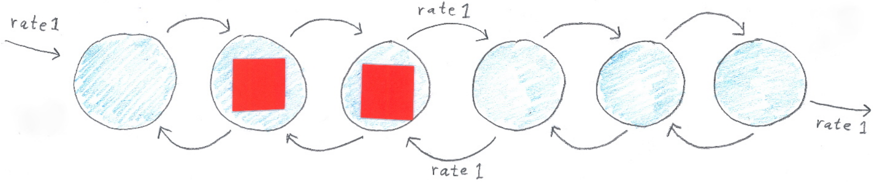

rate 1
rate 1
rate 1
rate 1


rate1
rate 1
rate 1
rate 1

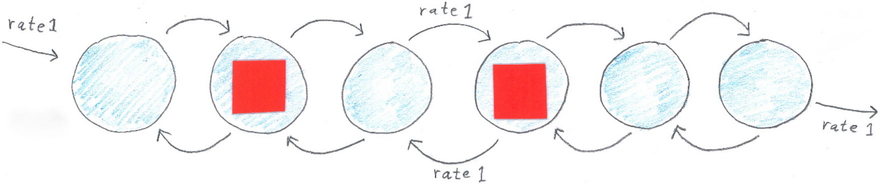

rate1
rate 1
rate 1
rate 1


rate1
rate 1
rate 1
rate 1


rate1
rate 1
rate 1
rate 1



rate1
rate 1
rate 1
rate 1



rate 1
rate 1
rate 1
rate 1

- What is the stationary distribution?
- Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{n+1} \in \mathfrak{S}_{n+1}$ be a permutation.
- Let $\mathrm{DB}(\sigma) = (\eta_1, \dots, \eta_n) \in \{0,1\}^n$ be defined by $\eta_{\sigma_i} = 1 \iff \sigma_{i-1} > \sigma_i$.
- $\mathrm{DB}(37284156) = (1,1,0,1,0,0,0)$.
- Let μ_n be the distribution of DB, i.e.,

$$\mu_n(\eta) = \frac{|\{\sigma \in \mathfrak{S}_{n+1} : \mathrm{DB}(\sigma) = \eta\}|}{(n+1)!}.$$

Theorem (Corteel and Williams)

μ_n is the stationary distribution for the above process.

- Hence μ_n is strong Rayleigh.
- Its partition function Z_n satisfies

$$x(n+1)!Z_n(x,\dots,x) = A_{n+1}(x),$$

where $A_n(x)$ is the nth Eulerian polynomial.

- Problem. Find the stationary distribution for other graphs.
- It is necessarily strong Rayleigh.

Linear operators preserving stability

Problem.

Characterize linear operators $T : \mathbb{R}[x] \to \mathbb{R}[x]$ preserving real rootedness.

- An old problem that goes back to the work of Laguerre, Hermite, Jensen, Pólya, Schur who wanted to prove that all zeros of the entire function

$$\xi(x) = \frac{1}{2}\left(x^2 + \frac{1}{2}\right)\pi^{ix/2-1/4}\Gamma\left(\frac{1}{4} - \frac{ix}{2}\right)\zeta\left(\frac{1}{2} - ix\right)$$

are real.
- and more recently to Craven, Csordas, Saff, Iserles, Nørsett, Brenti, Wagner, ...
- Gauss and Lucas: $T = d/dx$
- Hermite, Poulain, Jensen: $T = \sum_{k=0}^{n} a_k (d/dx)^k$ preserves real–rootedness iff $\sum_{k=0}^{n} a_k x^k$ is real–rooted.
- A sequence $\{\lambda_k\}_{k=0}^{\infty} \subset \mathbb{R}$ is a multiplier sequence, if the (diagonal) operator $T(x^k) = \lambda_k x^k$ preserves real–rootedness.
- Hence $\lambda_k = k$ is a multiplier sequence ($T = xd/dx$).

Theorem (Pólya and Schur, 1914). TFAE

(i) $\{\lambda_k\}_{k=0}^{\infty}$ is a multiplier sequence.

(ii) For each $n \in \mathbb{N}$, all zeros of

$$T\left((1+x)^n\right) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k x^k$$

are real and of the same sign.

(iii) The exponential generating function

$$T(e^x) = \sum_{k=0}^{\infty} \frac{\lambda_k}{k!} x^k$$

is an entire function, which is the limit, uniform on compact sets, of polynomials with only real zeros which are all of the same sign.

General Characterization

- Let $\mathbb{R}_n[x] = \{P \in \mathbb{R}[x] : \deg P \leq n\}$.
- The symbol of a linear operator $T : \mathbb{R}_n[x] \to \mathbb{R}[x]$ is the bivariate polynomial

$$G_T(x, y) = T\left((x + y)^n\right) = \sum_{k=0}^{n} \binom{n}{k} T(x^k) y^{n-k}.$$

- Call T degenerate if its range is at most two-dimensional.

Theorem (Borcea and B.). A nondegenerate linear operator $T : \mathbb{R}_n[x] \to \mathbb{R}[x]$ preserves real-rootedness iff

$$G_T(x, y) \text{ or } G_T(x, -y) \text{ is stable.}$$

Example

- Let $A_n(x) = \sum_{k=0}^{n} A(n,k)x^k$ be the Eulerian polynomial of degree n.
- $A(n+1,k) = kA(n,k) + (n+2-k)A(n,k-1)$

$$A_{n+1}(x) = x(1-x)\frac{d}{dx}A_n(x) + (n+1)xA_n(x) = T(A_n(x))$$

$$T = x(1-x)\frac{d}{dx} + (n+1)x$$

- We want to prove that $T : \mathbb{R}_n[x] \to \mathbb{R}[x]$ preserves real-rootedness.

$$T\left((x+y)^n\right) = x(x+y)^{n-1}(x+(d+1)y+d),$$

- which is stable

- The symbol of a linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is the formal power series

$$\mathcal{G}_T(x,y) = T(e^{-xy}) = \sum_{k=0}^{\infty} \frac{T(x^k)}{k!}(-y)^k.$$

- The Laguerre–Pólya class of entire functions in n variables, $\mathcal{L}\text{-}\mathcal{P}_n(\mathbb{R})$, consists of all entire functions that are the uniform limit on compact sets of real stable polynomials in n variables.
- $e^{-xy} = \lim_{n\to\infty}(1 - xy/n)^n$

Theorem (Borcea and B.). A nondegenerate linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ preserves real-rootedness iff

$\mathcal{G}_T(x,y)$ or $\mathcal{G}_T(x,-y)$ is in the Laguerre–Pólya class.

Example: Differential operators

- Let $T = \sum_{k=0}^{n} Q_k(x) d^k/dx^k$ be a differential operator. Then

$$\mathcal{G}_T(x,y) = T(e^{-xy}) = e^{-xy} \sum_{k=0}^{n} Q_k(x)(-y)^k.$$

- Hence T preserves real-rootedness iff $\sum_{k=0}^{n} Q_k(x)(-y)^k$ is stable iff
  there exist three symmetric real matrices A, B, C such that A, B are positive semidefinite and

$$\sum_{k=0}^{n} Q_k(x) y^k = \det(xA - yB + C).$$

- Preserving real stability in one variable $\iff$ Symbol is real stable in two variables.
- For $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n$, let

  $$\mathbb{C}_\kappa[x_1, \ldots, x_n] = \{P \in \mathbb{C}[x_1, \ldots, x_n] : \deg_{x_j}(P) \le \kappa_j \text{ for all } j\}.$$

- The symbol of a linear operator $T : \mathbb{C}_\kappa[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ is the $2n$-variate polynomial

  $$G_T(\mathbf{x}, \mathbf{y}) = T\left((x_1 + y_1)^{\kappa_1} \cdots (x_n + y_n)^{\kappa_n}\right),$$

  where T only acts on the x-variables.

Theorem (Borcea and B.). Suppose that the range of $T : \mathbb{C}_\kappa[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ has dimension at least two. Then T preserves stability if and only if $G_T(\mathbf{x}, \mathbf{y})$ is stable.

- The complex Laguerre–Pólya class of entire functions in n variables, $\mathcal{L}\text{-}\mathcal{P}_n(\mathbb{C})$, consists of all entire functions that are the uniform limit on compact sets of stable polynomials in n variables.
- The symbol of a linear operator $T : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ is the formal power series

$$\mathcal{G}_T(\mathbf{x}, \mathbf{y}) = T(e^{-\mathbf{x}\cdot\mathbf{y}}) = \sum_{\alpha \in \mathbb{N}^n} \frac{T(\mathbf{x}^\alpha)}{\alpha!}(-\mathbf{y})^\alpha,$$

  where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$.

Theorem (Borcea and B.). Suppose that the range of $T : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ has dimension at least two. Then T preserves stability if and only if $\mathcal{G}_T(\mathbf{x}, \mathbf{y})$ is in the complex Laguerre–Pólya class.

Example

- Helmann–Lieb theorem
  Let $\mathbf{x} = (x_i)_{i \in V}$ be variables and $\lambda = (\lambda_e)_{e \in E}$ nonnegative weights. Then
  $$P_{G,\lambda}(\mathbf{x}) = \sum_{M} (-1)^{|M|} \prod_{e=ij \in M} \lambda_e x_i x_j,$$
  where the sum is over all partial matchings is stable.
  - Proof following Choe, Oxley, Sokal and Wagner:
  - Let $\mathrm{MAP} : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ be the linear operator that maps P to its multi-affine part:
    $$\mathrm{MAP}\left(\sum_{\alpha \in \mathbb{N}^n} a(\alpha)\mathbf{x}^\alpha\right) = \sum_{\alpha \in \{0,1\}^n} a(\alpha)\mathbf{x}^\alpha.$$

- MAP preserves stability:

$$\mathrm{MAP}(e^{-\mathbf{x}\cdot\mathbf{y}}) = (1 - x_1 y_1)\cdots(1 - x_n y_n).$$

- The Heilmann–Lieb theorem follows from

$$\mathrm{MAP}\left(\prod_{ij\in E}(1 - \lambda(ij)x_i x_j)\right) = P_{G,\lambda}(\mathbf{x}).$$

Example: Eulerian polynomials

- Consider the homogenized Eulerian polynomials:

$$A_n(x, y) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)+1} y^{\mathrm{asc}(\sigma)+1} = y^{n+1} A_n(x/y).$$

$$A_{n+1}(x, y) = xy \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) A_n(x, y) = T(A_n).$$

- To prove that $A_n(x, y)$ is stable and thus that $A_n(x)$ real-rooted, we want to prove that T preserves stability:

$$T(e^{-xz-yw}) = -xy(z + w)e^{-xz-yw} \in \mathcal{L}\text{-}\mathcal{P}_4(\mathbb{C}).$$

Multivariate Eulerian polynomials

- Define the descent bottom set and ascent bottom set of $\sigma \in \mathfrak{S}_n$ as

$$\mathrm{DB}(\sigma) = \{\sigma(i) : \sigma(i-1) > \sigma(i)\} \text{ and}$$
$$\mathrm{AB}(\sigma) = \{\sigma(i) : \sigma(i) < \sigma(i+1)\},$$

where $\sigma(0) = \sigma(n+1) = \infty$.

- Define the weight of σ as

$$w(\sigma) = \prod_{i \in \mathrm{DB}(\sigma)} x_i \prod_{j \in \mathrm{AB}(\sigma)} y_j.$$

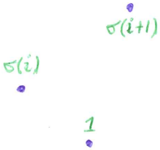
$$w(5762413) = x_5 x_6 x_2 x_1 y_2 y_1 y_3 y_5.$$

- Define a multivariate Eulerian polynomial by

$$A_n(\mathbf{x}, \mathbf{y}) = \sum_{\sigma \in \mathfrak{S}_n} w(\sigma).$$

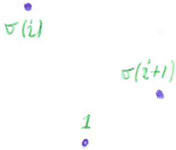
Multivariate Eulerian polynomials

- $A_n(\mathbf{x}, \mathbf{y})$ is multi-affine and homogeneous of degree $n+1$.
- For $i = 0, \ldots, n$, the ith slot of σ is the space between $\sigma(i)$ and $\sigma(i+1)$.
- Let σ be a permutation of $\{2, \ldots, n+1\}$ and insert the letter 1 in the slot i of σ.
- If $\sigma(i) < \sigma(i+1)$, then the new weight is



$$x_1 y_1 \frac{\partial}{\partial y_{\sigma(i)}} w(\sigma)$$

- If $\sigma(i) > \sigma(i+1)$, then the new weight is



$$x_1 y_1 \frac{\partial}{\partial x_{\sigma(i+1)}} w(\sigma)$$

- Inserting 1 in all slots has the effect:

$$x_1 y_1 \left(\sum_{k=2}^{n+1} \frac{\partial}{\partial x_k} + \sum_{k=2}^{n+1} \frac{\partial}{\partial y_k} \right) w(\sigma)$$

- Lemma

$$A_{n+1}(\mathbf{x}, \mathbf{y}) = x_1 y_1 \left(\sum_{k=2}^{n+1} \frac{\partial}{\partial x_k} + \sum_{k=2}^{n+1} \frac{\partial}{\partial y_k} \right) A_n(\mathbf{x}^*, \mathbf{y}^*),$$

where $\mathbf{x}^* = (x_2, x_3, \ldots)$.

- Corollary

  $A_n(\mathbf{x}, \mathbf{y})$ is stable.

  - Proof. It suffices to prove that operators of the form $T = \sum_{i=1}^{n} \lambda_i \partial/\partial x_i$, where $\lambda_i \geq 0$, preserves stability.

$$T(e^{-\mathbf{x}\cdot\mathbf{y}}) = -e^{-\mathbf{x}\cdot\mathbf{y}} \left(\sum_{i=1}^{n} \lambda_i y_i \right)$$

Stability of free quasi-symmetric functions

- Let $\mathrm{FQSym} = \oplus_{n=0}^{\infty} \mathrm{FQSym}_n$ be a formal $\mathbb{C}$-linear vector space with FQSym_n having bases $\mathfrak{S}_n$.
- The product in FQSym is defined on the bases elements:

$$231 \cdot 21 = 23154 + 23514 + 23541 + 25314 + 25341 + 52314 + 25431 + 52341 + 52431 + 54231$$

- FQSym is called the algebra of free quasi-symmetric functions or the Malvenuto–Reutenauer (Hopf-) algebra.
- Let as before

$$w(\sigma) = \prod_{i \in \mathrm{DB}(\sigma)} x_i \prod_{j \in \mathrm{AB}(\sigma)} y_j$$

and extend w linearly to FQSym.
- Call a weight $w' : \mathrm{FQSym} \to \mathbb{R}[t_1, t_2, \ldots]$ good if it is of the form

$$w'(\xi) = w(\xi)(t_{f(1)}, t_{f(2)}, \ldots, t_{g(1)}, t_{g(2)}, \ldots),$$

where $f, g : \mathbb{Z}_+ \to \mathbb{Z}_+$ are arbitrary.

- In particular $w_1(\sigma) = t_1^{\mathrm{des}(\sigma)+1} t_2^{|\sigma|-\mathrm{des}(\sigma)}$ is good.

- **Lemma (B., Leander).**

  Let w' be a good weight and $\eta, \xi \in \mathsf{FQSym}$. Then $w'(\eta \cdot \xi)$ only depends on $w'(\eta)$ and $w'(\xi)$.

  - Hence each good w' defines an (descent bottom) algebra.

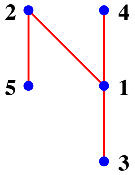
- **Theorem (B., Leander).**

  Let w' be a good weight and $\eta, \xi \in \mathsf{FQSym}$ be such that $w'(\eta)$ and $w'(\xi)$ are stable. Then $w'(\eta \cdot \xi)$ is stable.

  - Note that $A_n(\mathbf{x}, \mathbf{y}) = w(1^n)$
  - The case of the theorem when $w' = w_1$ is a reformulation of conjecture of Brenti, first proved by Wagner.

P-Eulerian polynomials

- Let P be a partially ordered set on $\{1, \ldots, n\}$.



- Let $\mathcal{L}(P)$ be the linear extensions of P.

σ	$\mathrm{des}(\sigma)$
35124	1
35142	2
31524	2
31452	2
53124	2
31542	3
53142	3

$$A_P(x) = \sum_{\sigma \in \mathcal{L}(P)} x^{\mathrm{des}(\sigma)+1} = x^2 + 4x^3 + 2x^4$$

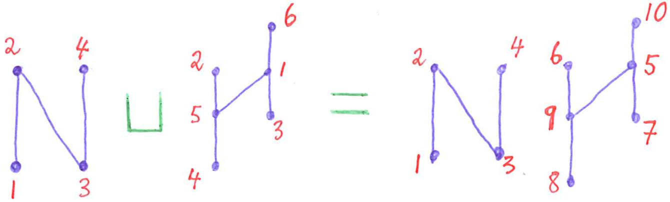
Neggers–Stanley conjecture: All zeros of $A_P(x)$ are real.

- Disproved in 2004 by B.
- However it holds (or is open) for many important classes of permutations.
- We may define a multivariate analog by

$$A_P(\mathbf{x}, \mathbf{y}) = w\left(\ell(P)\right), \quad \text{where} \quad \ell(P) = \sum_{\sigma \in \mathcal{L}(P)} \sigma.$$

- If P is the anti-chain on $[n]$, then $A_P(\mathbf{x}, \mathbf{y}) = A_n(\mathbf{x}, \mathbf{y})$.
- Question: For which P is $A_P(\mathbf{x}, \mathbf{y})$ stable?

- The disjoint union $P \sqcup Q$ of two posets P and Q:



- Corollary

  If $A_P(\mathbf{x}, \mathbf{y})$ and $A_Q(\mathbf{x}, \mathbf{y})$ are stable, then so is $A_{P \sqcup Q}(\mathbf{x}, \mathbf{y})$.

Proof. $\ell(P \sqcup Q) = \ell(P) \cdot \ell(Q)$.

- Corollary

  $A_P(\mathbf{x}, \mathbf{y})$ is stable for naturally labelled trees.

Peaks

- We have an analogous version for peaks in permutations.
- Let

$$\Lambda(\sigma) = \{\sigma(i) : 2 \leq i \leq n-1 \text{ and } \sigma(i-1) < \sigma(i) > \sigma(i+1)\}.$$

- Define $w_\Lambda : \text{FQSym} \to \mathbb{R}[x_2, x_3, \ldots]$ by

$$w_A(\sigma) = \prod_{j \in \Lambda(\sigma)} x_j$$

- Again say that $w'_\Lambda : \text{FQSym} \to \mathbb{R}[t_1, t_2, \ldots]$ is good if it is obtained from w_Λ by renaming and identifying some (or none) of the variables.

- Lemma (B., Leander).

  Let w'_Λ be a good weight and $\eta, \xi \in$ FQSym. Then $w'_\Lambda(\eta \cdot \xi)$ only depends on $w'_\Lambda(\eta)$ and $w'_\Lambda(\xi)$.

  - A polynomial is Hurwitz stable if it non-vanishing whenever all variables are in the open right half-plane.

- Theorem (B., Leander).

  Let w'_Λ be a good weight and $\eta, \xi \in$ FQSym be such that $w'_\Lambda(\eta)$ and $w'(\xi)$ are Hurwitz stable. Then $w'_\Lambda(\eta \cdot \xi)$ is Hurwitz stable.

- Corollary.

$$\sum_{\sigma \in \mathfrak{S}_n} \prod_{j \in \Lambda(\sigma)} x_j \qquad \text{is Hurwitz stable.}$$

  - Corollary. Let $\mathcal{A}_n$ be the alternating permutations of length n.

$$\sum_{\sigma \in \mathcal{A}_n} \prod_{j \in \Lambda(\sigma)} x_j \qquad \text{is stable.}$$

Multivariate Eulerian polynomials for Coxeter groups

- Let W be a finite Coxeter group with generators S:

$$W = \langle S : (ss')^{m(s,s')} = 1, m(s,s) = 1 \rangle.$$

- The descent set of $w \in W$ is

$$D(w) = \{s \in S : \ell(ws) < \ell(w)\}.$$

- The W-Eulerian polynomial is

$$A_W(x) = \sum_{w \in W} x^{|D(w)|+1}.$$

- Conjecture (Brenti)

  For any finite Coxeter group W, $A_W(x)$ is real-rooted.

  - The only remaining case is type D? (Solution proposed by Shi-Mei Ma).

Multivariate Eulerian polynomials for Coxeter groups

- $A_n(\mathbf{x}, \mathbf{y})$ is a multivariate stable analog for type A.
- Recall that B_n may be realized as signed permutations

$$B_n = \{\sigma_1 \cdots \sigma_n : \sigma_i \in \mathbb{Z}, |\sigma_1| \cdots |\sigma_n| \in \mathfrak{S}_n\}.$$

$$D(\sigma) = \{i \in [n] : \sigma_{i-1} > \sigma_i\}, \quad \text{where} \quad \sigma_0 := 0.$$

- Visontai and Williams proposed a multivariate analog:

$$\mathrm{DT}(\sigma) = \{\max(|\sigma_{i-1}|, |\sigma_i|) : i \in [n] \text{ and } \sigma_{i-1} > \sigma_i\},$$
$$\mathrm{AT}(\sigma) = \{\max(|\sigma_{i-1}|, |\sigma_i|) : i \in [n] \text{ and } \sigma_{i-1} < \sigma_i\},$$
$$B_n(\mathbf{x}, \mathbf{y}) = \sum_{\sigma \in B_n} \prod_{i \in \mathrm{DT}(\sigma)} x_i \prod_{j \in \mathrm{DT}(\sigma)} y_j.$$

- Theorem (Visontai and Williams)

  $B_n(\mathbf{x}, \mathbf{y})$ is stable.

- Question

  Is there a case-free stable multivariate W-Eulerian polynomial?

  - Stable multivariate analogs of real-rooted Eulerian polynomials for various classes of permutations have been obtained by Haglund and Visontai.

- The set of descent bottoms is equidistributed with the excedence set $E(\sigma) = \{i : \sigma(i) > i\}$.
- Note that

$$\sum_{\sigma \in \mathfrak{S}_n} \prod_{i \in E(\sigma)} x_i = \operatorname{per} \begin{pmatrix} 1 & 1 & 1 & 1 & \\ x_1 & 1 & 1 & 1 & \\ x_1 & x_2 & 1 & 1 & \cdots \\ x_1 & x_2 & x_3 & 1 & \\ & & \vdots & & \end{pmatrix}$$

- Consider a shape λ that fits into an $n \times n$ box

$$\begin{pmatrix} x & y & y & y & y & y \\ x & x & x & y & y & y \\ x & x & x & y & y & y \\ x & x & x & x & y & y \\ x & x & x & x & x & y \\ x & x & x & x & x & y \end{pmatrix}$$

$$\lambda = (5, 5, 4, 3, 3, 1)$$

- Assign variables as

$$B_\lambda = \begin{pmatrix} x_1 & y_1 & y_1 & y_1 & y_1 & y_1 \\ x_1 & x_2 & x_3 & y_2 & y_2 & y_2 \\ x_1 & x_2 & x_3 & y_3 & y_3 & y_3 \\ x_1 & x_2 & x_3 & x_4 & y_4 & y_4 \\ x_1 & x_2 & x_3 & x_4 & x_5 & y_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 & y_6 \end{pmatrix}$$

- Theorem (B., Haglund, Visontai, Wagner)

  The permanent of B_λ is stable.

  - Using this we proved the

- Monotone Column Permanent Conjecture (Haglund, Ono, Wagner (1999))

  If A is a real matrix which is weakly increasing down columns and J is the all ones matrix, then $\mathrm{per}(A + xJ)$ is real-rooted.

SEP preserves SR

- It will be convenient to view a Markov chain on measures on $\{0,1\}^n$ as acting on the partition functions of the measures.
- Hence we view a Markov chain as a family of linear operators T_t, $t \geq 0$, acting on the space, $\mathcal{M}_n$, of multi-affine complex polynomials in n variables.
- The Markov property translates as

$$\frac{d}{dt} T_t = \mathcal{L} T_t, \qquad \text{for all } t \geq 0,$$

  where $\mathcal{L} : \mathcal{M}_n \to \mathcal{M}_n$ is the (linear) generator.
- In the case of SEP

$$\mathcal{L} = \sum_{i<j} q_{ij}(\tau_{ij} - \epsilon),$$

  where $q_{ij} \geq 0$ are the jump-rates, τ_{ij} is the transposition that interchanges coordinates i and j, and ϵ is the identity.

Infinite log-concavity

- Define an operator, $\mathcal{L}$, on sequences by
$$\mathcal{L}(\{a_k\}_{k=0}^n) = \{b_k\}_{k=0}^n$$
where
$$b_k = a_k^2 - a_{k-1}a_{k+1},$$
and $a_{-1} = a_{n+1} = 0$.
- $\{a_k\}_{k=0}^n$ is *i-fold log-concave* if $\mathcal{L}^i(\{a_k\})$ is non-negative.
- $\{a_k\}_{k=0}^n$ is infinitely log-concave $\mathcal{L}^i(\{a_k\})$ is non-negative for all i.
- For $k, n \in \mathbb{N}$ let
$$d_k(n) = 2^{-2n} \sum_{j=k}^{n} 2^j \binom{2n-2j}{n-j} \binom{n+j}{n} \binom{j}{k}$$
- $d_k(n)$ is the kth Taylor coefficient of the polynomial
$$P_n(a) = \frac{2^{n+3/2}(a+1)^{n+1/2}}{\pi} \int_0^\infty \frac{1}{(x^4+2ax^2+1)^{n+1}} dx.$$

- Boros–Moll Conjecture 1

  $\{d_k(n)\}_{k=0}^n$ is infinitely log-concave

  - Log-concavity proved by Kauers and Paule.
  - 2-log-concavity proved by Chen and Xia.

- Conjecture (B.)

  The polynomials

  $$R_n(x) = \sum_{k=0}^{n} \frac{d_k(n)}{(k+2)!} x^k$$

  are real-rooted.

- The conjecture implies 3-log-concavity of $\{d_k(n)\}_{k=0}^n$.
  - Proved by Chen, Dou and Yang by establishing a recursion which preserves real-rootedness.

- Boros–Moll Conjecture 2

  $\{\binom{n}{k}\}_{k=0}^n$ is infinitely log-concave.

  - $\binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1} = \frac{1}{n+2}\binom{n+1}{k}\binom{n+1}{k+1}$, Narayana numbers.
  - Proved for $n \leq 1450$ by Sagan and McNamara.

Conjecture (Fisk, Sagan–McNamara, Stanley)

If $\sum_{k=0}^{n} a_k x^k$ has only real and nonpositive zeros, then so does

$$\sum_{k=0}^{n} (a_k^2 - a_{k-1}a_{k+1})x^k.$$

$\Longrightarrow$ Boros–Moll Conjecture 2.

Grace–Walsh–Szegő Coincidence Theorem

Let $K \subset \mathbb{C}$ be a disk or a half-plane and let $f(x_1, \ldots, x_n)$ be a symmetric and multiaffine polynomial. For any $\zeta_1, \ldots, \zeta_n \in K$, there is a $\zeta \in K$ such that

$$f(\zeta_1, \ldots, \zeta_n) = f(\zeta, \ldots, \zeta).$$

A Catalan symmetric function identity

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and

$$e_k(\mathbf{x}) = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} \prod_{i \in S} x_i,$$

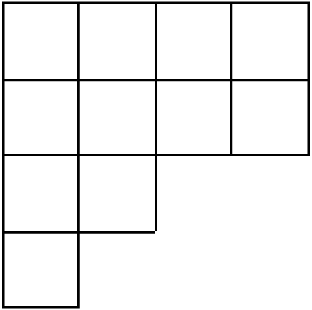
be the kth elementary symmetric polynomial in $\mathbf{x}$.

Lemma

$$\sum_{k=0}^{n} e_k(\mathbf{x})^2 - e_{k-1}(\mathbf{x}) e_{k+1}(\mathbf{x}) = x_1 \cdots x_n \sum_{k=0}^{\lfloor n/2 \rfloor} C_k e_{n-2k} \left(\mathbf{x} + \frac{1}{\mathbf{x}} \right),$$

where $1/\mathbf{x} = (1/x_1, \ldots, 1/x_n)$ and

$$C_k = \frac{1}{k+1} \binom{2k}{k} \quad \text{is a Catalan number.}$$



1	1	2	3
2	4	4	5
5	6		
8			

Shape $\lambda = (4, 4, 2, 1)$ Semi-standard Young tableau of shape λ

Schur function of shape λ

$$s_\lambda(x_1, \ldots, x_n) = \sum_T \prod_{t \in T} x_t = \cdots + x_1^2 x_2^2 x_3 x_4^2 x_5^2 x_6 x_8 + \cdots$$

summed over all SSYT of shape λ and entries in $\{1, \ldots, n\}$

$$e_k(\mathbf{x})^2 - e_{k+1}(\mathbf{x})e_{k-1}(\mathbf{x}) = s_{2^k}(\mathbf{x}),$$

where $2^k = (2, \ldots, 2)$. We want to prove

$$\sum_{k=0}^{n} s_{2^k}(\mathbf{x}) = \sum_{k=0}^{\lfloor n/2 \rfloor} C_k \sum_{|S|=2k} \mathbf{x}^S \prod_{j \notin S} (1 + x_j^2).$$

$$\begin{matrix} 1 & \color{red}{2} \\ \color{red}{2} & 5 \\ 4 & 6 \\ \color{red}{7} & \color{red}{7} \\ 8 & 9 \end{matrix} \qquad \Longrightarrow \qquad \begin{matrix} 1 & 5 \\ 4 & 6 \\ 8 & 9 \end{matrix}\ , \qquad \{2, 7\}$$

Number of standard Young tableaux of shape 2^k is C_k.

Proof of the SSMF Conjecture

- Let $P(x) = \sum_{k=0}^{n} a_k x^k = \prod_{k=0}^{n}(1 + \rho_k x)$, where $\rho_k > 0$.
- Suppose

$$Q(\zeta) = \sum_{k=0}^{n}(a_k^2 - a_{k-1}a_{k+1})\zeta^k = 0$$

  for some $\zeta \in \mathbb{C}$, with $\zeta \notin \{x \in \mathbb{R} : x \leq 0\}$.
- Write $\zeta = \xi^2$, where $\mathrm{Re}(\xi) > 0$.

$$Q(\zeta) = \sum_{k=0}^{n} e_k(\mathbf{z})^2 - e_{k+1}(\mathbf{z})e_{k-1}(\mathbf{z}), \qquad \text{where } \mathbf{z} = (\rho_1\xi, \ldots, \rho_n\xi).$$

- Hence

$$Q(\zeta) = a_n\xi^n \sum_{k=0}^{\lfloor n/2 \rfloor} C_k e_{n-2k}\left(\rho_1\xi + \frac{1}{\rho_1\xi}, \ldots, \rho_n\xi + \frac{1}{\rho_n\xi}\right) = 0.$$

$$\mathrm{Re}\left(\rho_k\xi + \frac{1}{\rho_k\xi}\right) > 0.$$

Proof of the SSMF Conjecture

- The Grace–Walsh–Szegő Theorem provides a number $\zeta \in \mathbb{C}$, with $\text{Re}(\zeta) > 0$, such that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} C_k e_{n-2k}(\zeta, \ldots, \zeta) = \sum_{k=0}^{\lfloor n/2 \rfloor} C_k \binom{n}{2k} \zeta^{n-2k} = 0$$

- If $\text{Re}(\zeta) > 0$ then $1/\zeta^2$ is not a negative real number.
- We are done if we can prove that all zeros of

$$p_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} C_k \binom{n}{2k} x^k$$

are negative.
- $p_n(x)$ is essentially a Jacobi orthogonal polynomial!!

Extending the SSMF conjecture

- Conjecture (Fisk)

  Suppose $a_0 + a_1 x + \cdots + a_d x^d$ has only real and negative zeros. Then so does

$$\sum_{n=0}^{d} \begin{vmatrix} a_n & a_{n-1} & a_{n-2} \\ a_{n+1} & a_n & a_{n-1} \\ a_{n+2} & a_{n+1} & a_n \end{vmatrix} x^n.$$

Extension

- Let $\alpha = \{\alpha_j\}_{j=0}^{\infty} \subset \mathbb{R}$ and consider the operator T_α defined by

$$a_k \mapsto \sum_{j=0}^{\infty} \alpha_j a_{k-j} a_{k+j}.$$

- Above we studied the case $\alpha = 1, 0, -1, 0, \ldots$.

- Theorem (B.)

T_α preserves the property of having only nonpositive zeros iff $T_\alpha(e^x)$ is in the Laguerre-Pólya class and has nonnegative coefficients.

Immanants

- Let χ_λ be a character of the symmetric group, indexed by the partition λ.
- The corresponding immanant is the matrix function defined by

$$\mathrm{im}_\lambda(A) = \sum_{\sigma \in \mathfrak{S}_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}, \quad \text{where} \quad A = (a_{ij})_{i,j=1}^{n}$$

- For the trivial character we get the permanent, and for the alternating the determinant.

- Theorem (Schur)

If A is positive semidefinite, then

$$\mathrm{im}_\lambda(A) \geq f^\lambda \det(A),$$

where $f^\lambda = \chi_\lambda(id)$ is the number of standard Young tableaux of shape λ.

Permanent on top

- Lieb's "Permanent–on–top" conjecture

  If A is positive semidefinite, then

$$\mathrm{im}_\lambda(A) \leq f^\lambda \,\mathrm{per}(A).$$

- Theorem (B.)

  Let A be a $n \times n$ matrix, then the polynomial

$$\sum_{\lambda \vdash n} \mathrm{im}_{\lambda'}(A) s_\lambda(\mathbf{x})$$

  is stable.

- Question

  What inequalities are satisfied for stable, homogeneous and symmetric polynomials?

- If the coefficients in the monomial bases of such a polynomial are nonnegative are also the coefficients in the Schur bases nonnegative?

- Theorem (Borcea, B.)

  Let $P(\mathbf{z}) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha)\mathbf{z}^\alpha/\alpha!$ be a stable polynomial with nonnegative coefficients. Then
- $a(\alpha)^2 \geq a(\alpha + e_i - e_j)a(\alpha - e_i + e_j)$, where $\{e_i\}_{i=1}^n$ is the standard bases.
- Recall that if $\lambda, \mu \vdash n$, then λ is majorized by μ if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_j \leq \mu_1 + \mu_2 + \cdots + \mu_j, \text{ for all } j \geq 1.$$

- If P is symmetric and $\lambda \leq \mu$ in the majorization order, then $a(\lambda) \geq a(\mu)$.

Stable polynomials and Matroid theory

- Let E be a finite set. A collection $\mathcal{B} \subset 2^E$ is the set of bases of a matroid if for all $B_1, B_2 \in \mathcal{B}$

$$e \in B_1 \setminus B_2 \Longrightarrow \exists f \in B_2 \setminus B_1 \text{ s.t. } B_1 \setminus \{e\} \cup \{f\} \in \mathcal{B}.$$

- If $v_1, \ldots, v_m$ are vectors in a vector space V over k that span V, then the set

$$\{\{i_1, \ldots, i_k\} : v_{i_1}, \ldots, v_{i_k} \text{ is a basis of } V\}$$

  is a bases of matroid representable over k.

- The support of a polynomial $P(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha)\mathbf{x}^\alpha$ is

$$\operatorname{supp}(P) = \{\alpha : a(\alpha) \neq 0\}.$$

- Theorem (Choe, Oxley, Sokal, Wagner)

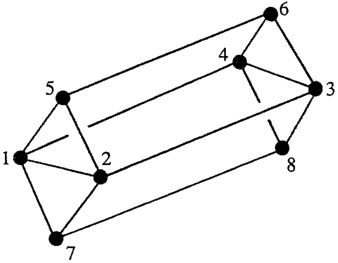
  The support of a homogeneous, multiaffine and stable polynomial is the set of bases of a matroid.

- Such matroids are called WHPP-matroids (weak half-plane property).
- Question. Which matroids are WHPP?
- Recall that the spanning tree polynomial is stable. Hence graphic matroids are WHPP.
- All matroids representable over $\mathbb{C}$ are WHPP: If $A = [v_1, \ldots, v_m] \in \mathbb{C}^{r \times m}$, then

$$\det(x_1 v_1 v_1^* + \cdots + x_m v_m v_m^*) = \sum_{|B|=r} |\det(A(B))|^2 \prod_{j \in B} x_j,$$

  where $A(B)$ is the $r \times r$ submatrix with columns indexed by B.
- Thm. (B., D'Leon): No projective geometry is WHPP. A binary matroid is WHPP iff it is regular.

- Let $\mathcal{B}$ be the collection of all subsets of size 4 of $\{1, \ldots, 8\}$ such that the corresponding vertices do not lie in an affine plane in the following figure



- $\mathcal{B}$ is the set of bases of the Vámos cube, V_8.
- Let further

$$V(\mathbf{x}) = \sum_{B \in \mathcal{B}} \prod_{j \in B} x_j = x_1x_2x_3x_5 + x_1x_2x_3x_6 + \cdots .$$

- Theorem (Wagner-Wei, 2009)

  $V(\mathbf{x})$ is stable, hence V_8 is WHPP.

Generalized Lax Conjecture

- The above questions can be thought of as discrete versions of questions considered in convex optimization.
- A polynomial $P(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is a real zero polynomial (RZ) if
  - $P(0) \neq 0$, and
  - for all $\mathbf{x} \in \mathbb{R}^n$, the polynomial $t \mapsto P(t\mathbf{x})$ is real-rooted.
- If $A_1, \ldots, A_n$ are hermitian matrices, then

  $$\det(I + x_1A_1 + \cdots + x_nA_n)$$

  is a RZ polynomial.
- If $P \in \mathbb{R}[\mathbf{x}]$ and $P(0) \neq 0$ let $\mathcal{C}_P$ be the connected component of

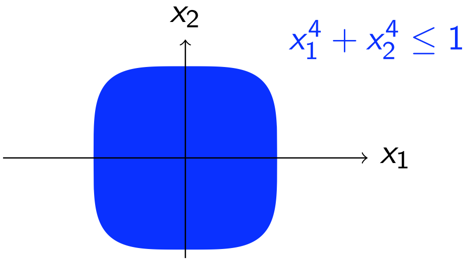
  $$\{\mathbf{x} \in \mathbb{R}^n : P(\mathbf{x}) \neq 0\}$$

  containing the origin.

- Theorem (Gårding)

  If P is a RZ polynomial, then $\mathcal{C}_P$ is a convex set, called rigidly convex.

- The ball $\{x_1^2 + \cdots + x_n^2 \leq 1\}$ is rigidly convex since $1 - x_1^2 - \cdots - x_n^2$ is an RZ polynomial.
- The ancient tv-screen is not rigidly convex:



- If $P(\mathbf{x}) = \det(I + x_1 A_1 + \cdots + x_n A_n)$ where $A_1, \ldots, A_n$ are hermitian, then

  $$\mathcal{C}_P = \{\mathbf{x} \in \mathbb{R}^n : I + x_1 A_1 + \cdots + x_n A_n \text{ is positive semidefinite}\}.$$

- Such sets are called spectrahedral, and are the feasible sets for semidefinite optimization.
- Methods generalizing semidefinite optimization have been developed for rigidly convex sets.

- Generalized Lax conjecture

  {rigidly convex sets} = {spectrahedral sets}.

- Conjecture (P. Lax)

  If $P(x, y)$ is a RZ polynomial of degree d, then there are symmetric $d \times d$ matrices such that

  $$P(x, y) = \det(I + xA + yB).$$

  - Proved by Helton and Vinnikov.

- The exact analog of the Lax conjecture fails in more than three variables by a count of parameters.
- Helton and Vinnikov proposed the following two conjectures.
- Conjecture 1. If $P \in \mathbb{R}[x_1, \ldots, x_n]$ is a RZ polynomial, then there exists symmetric matrices $A_1, \ldots, A_n$ such that $P(\mathbf{x}) = \det(I + x_1 A_1 + \cdots + x_n A_n)$.
- Conjecture 2. If $P \in \mathbb{R}[x_1, \ldots, x_n]$ is a RZ polynomial, then there exists symmetric matrices $A_1, \ldots, A_n$ and a positive integer N such that $P(\mathbf{x})^N = \det(I + x_1 A_1 + \cdots + x_n A_n)$.
- Suppose that $H(\mathbf{x})$ is a homogeneous and stable polynomial. Then $P(\mathbf{x}) = H(x_1 + 1, \ldots, x_n + 1)$ is an RZ polynomial.

- Theorem (B.)

  There is no power N such that

  $$V(x_1 + 1, \ldots, x_8 + 1)^N$$

  is a determinantal polynomial.

  - The idea of the proof is that V_8 is a WHPP matroid which is not representable over $\mathbb{C}$.