A Coxeter theoretic interpretation of Euler numbers

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SLC 69, Strobl, September 2012.

Introduction

The Euler numbers T_n are defined via the generating function

$$\tan(z) + \sec(z) = \sum_{n \geq 0} T_n \frac{z^n}{n!}$$

Désiré André showed that T_n is the number of alternating permutations in S_n, i.e. those w such that $w(1) > w(2) < w(3) > \dots$

Since then, a lot of interest has been given to these numbers and these permutations [Stanley, a survey of alternating permutations (2010)]

Outline

Springer showed that, considering alternating permutations as the largest *descent class* in S_n, there is an analogue of T_n for other finite irreducible Coxeter groups (he also computed the value in each case of the ABDE... classification).

There is another way to relate T_n with the symmetric group as a Coxeter group, relying on a result of Stanley about orbits of maximal chains in the set partition lattice. We present a method to compute the value in each case of the classification.

Let $\mathcal{P}(n)$ be the lattice of set partitions of $\{1, \ldots, n\}$. It is ordered by refinement: $\mu \leq \pi$ if every block of μ is contained in a block of π.
For example: $\{\{1,2,4\},\{3,6\},\{5\}\} \leq \{\{1,2,4\},\{3,5,6\}\}$.

A *maximal chain* in $\mathcal{P}(n)$ is a sequence $\pi_1 < \cdots < \pi_n$ where

- $\pi_1 = \{\{1\},\{2\},\ldots,\{n\}\}$,
- $\pi_n = \{\{1,\ldots,n\}\}$,
- π_{i+1} is obtained by joining two blocks of π_i.

For example: $\{\{1\},\{2\},\{3\},\{4\}\} < \{\{1,3\},\{2\},\{4\}\} < \{\{1,3\},\{2,4\}\} < \{\{1,2,3,4\}\}$.

S_n acts on these maximal chains in an "obvious way", and Stanley proved that the number of orbits is T_{n-1}.

In this talk, we see a finite Coxeter group W as a real reflection group, i.e. we have a *defining representation* $W \subset GL(V)$ for some Euclidian space V, such that W is generated by orthogonal reflections.

H is a *reflecting hyperplane* if the orthogonal reflection through it is in W.

Definition

The set partition lattice $\mathcal{P}(W)$ is the set of all subspaces

$$H_1 \cap \cdots \cap H_r$$

where each H_i is a reflecting hyperplane.
It is ordered by reverse inclusion.

In type A_{n-1}, $W = S_n$ acts on $V = \{v \in \mathbb{R}^n : \sum v_i = 0\}$ by permuting coordinates.
$w \in S_n$ is a reflection if it permutes v_i and v_j for some $i < j$, i.e. is the orthogonal reflection through the subspace $H_{ij} = \{v \in V : v_i = v_j\}$.

Then the set of all subspaces

$$H_{i_1 j_1} \cap \cdots \cap H_{i_k j_k}$$

is in bijection with set partitions. For example if $n = 7$:

$$H_{1,7} \cap H_{2,4} \cap H_{4,5} = \{v \in V : v_1 = v_7, \; v_2 = v_4 = v_5\}$$

$$\leftrightarrow \{\{1,7\},\{2,4,5\},\{3\},\{6\}\}.$$

And the refinement order on set partitions corresponds to reverse inclusion in subspaces of V.

Let $\mathcal{M}(W)$ the set of maximal chains of the set partition lattice $\mathcal{P}(W)$.
There is an action of W on $\mathcal{M}(W)$, we consider the number of orbits $K(W) = \#(\mathcal{M}(W)/W)$.

So Stanley's result is $K(A_n) = T_n$.
(Remark: In Springer's problem of the largest descent class, T_n is the number associated to S_n i.e. type A_{n-1} and not A_n.)

What is $K(W)$ for the other cases of the classification ?

The general method

W acts on lines (=coatoms) in $\mathcal{P}(W)$. Let $L_1, \ldots, L_k$ be some orbit representatives. For each line L_i, let

$$\mathrm{Fix}(L_i) = \{w \in W \,:\, w(x) = x, \forall x \in L_i\},$$
$$\mathrm{Stab}(L_i) = \{w \in W \,:\, w(L_i) = L_i\}.$$

Proposition

$\mathrm{Fix}(L_i)$ *is itself a Coxeter group, so* $\mathcal{P}(\mathrm{Fix}(L_i))$ *and* $\mathcal{M}(\mathrm{Fix}(L_i))$ *are defined, they are acted on by* $\mathrm{Stab}(L_i)$ *, and*

$$K(W) = \sum_{i=1}^{k} \#\big(\mathcal{M}(\mathrm{Fix}(L_i))/\mathrm{Stab}(L_i)\big).$$

Remark

If

- $\mathrm{Stab}(L_i) = \mathrm{Fix}(L_i)$, or
- $\mathrm{Stab}(L_i) = \mathrm{Fix}(L_i) \rtimes \{\pm Id\}$,

then we have

$$\#\big(\mathcal{M}(\mathrm{Fix}(L_i))/\mathrm{Stab}(L_i)\big) = K(\mathrm{Fix}(L_i)),$$

which we assume we already know by induction.
(note that $\{\pm Id\}$ acts trivially on $\mathcal{P}(W)$)

Proof.

From each orbit of maximal chains, we can extract an orbit of lines. It suffices to show that the number of orbits of maximal chains associated to the orbit of L_i is $\#\big(\mathcal{M}(\mathrm{Fix}(L_i))/\mathrm{Stab}(L_i)\big)$.

$\mathrm{Fix}(L_i)$ is seen as a Coxeter group with defining representation acting on $L_i^\perp$. The interval $[\hat{0}, L_i]$ in $\mathcal{P}(W)$ is identified with $\mathcal{P}(\mathrm{Fix}(L_i))$ (we use the bijection: subspaces containing $L_i \leftrightarrow$ subspaces in $L_i^\perp$).

The number of W-orbits of chains $\hat{0} < \cdots < w(L_i) < \hat{1}$ for some $w \in W$, is also the number of $\mathrm{Stab}(L_i)$-orbits of chains $\hat{0} < \cdots < L_i < \hat{1}$. Hence it is $\#\big(\mathcal{M}(\mathrm{Fix}(L_i))/\mathrm{Stab}(L_i)\big)$.

□

Another useful result:

Proposition

If W_1, and W_2 have ranks m and n, we have:

$$K(W_1 \times W_2) = \binom{m+n}{m} K(W_1)K(W_2).$$

Proof.

A maximal chain in $\mathcal{P}(W_1 \times W_2)$ is obtained by "shuffling" two maximal chains in $\mathcal{P}(W_1)$ and $\mathcal{P}(W_2)$. For example from $x_0 < \cdots < x_m$ and $y_0 < \cdots < y_n$ we can form $(x_0, y_0) < (x_0, y_1) < (x_1, y_1) < (x_2, y_1) < \ldots$.

The number of shuffles is the binomial coefficient.
This operation is still well-defined for the orbits of maximal chains.

□

Case of the symmetric group S_n (type A_{n-1})

Let $V = \{v \in \mathbb{R}^n : \sum v_i = 0\}$. The coatoms are the 2-block set partitions and a set of orbit representatives is:

$$L_i = \{v \in V : v_1 = \cdots = v_i,\ v_{i+1} = \cdots = v_n\}$$

with $1 \leq i \leq \frac{n}{2}$.

- If $i < \frac{n}{2}$, $\mathrm{Fix}(L_i) = \mathrm{Stab}(L_i) = S_i \times S_{n-i}$.
- If $i = \frac{n}{2}$, $\mathrm{Fix}(L_i) = S_i \times S_i$ and $\mathrm{Stab}(L_i) = (S_i \times S_i) \rtimes S_2$ where S_2 permutes the two factors in $S_i \times S_i$.

In the second case, we have

$$\mathcal{M}(\mathrm{Fix}(L_i))/\mathrm{Stab}(L_i) = \mathcal{M}(S_i \times S_i)/(S_i \times S_i)/S_2$$

where the S_2-action has no fixed point, so

$$\#(\mathcal{M}(\mathrm{Fix}(L_i))/\mathrm{Stab}(L_i)) = \frac{1}{2}K(S_i \times S_i).$$

Case of the symmetric group S_n (type A_{n-1})

Let $a_n = K(A_n)$, we obtain:

$$a_{n-1} = \sum_{1 \le i < \frac{n}{2}} \binom{n-2}{i-1} a_{i-1} a_{n-i-1} + \chi[n \text{ even}] \frac{1}{2} \binom{n-2}{n/2-1} a_{n/2-1}^2.$$

This is equivalent to

$$a_{n-1} = \frac{1}{2} \sum_{i=1}^{n-1} \binom{n-2}{i-1} a_{i-1} a_{n-i-1}.$$

So $A(z) = \sum_{n \ge 0} a_n \frac{z^n}{n!}$ satisfies $A'(z) = \frac{1}{2}(1 + A(z)^2)$ with $A(0) = 1$.

The solution is $A(z) = \tan(z) + \sec(z)$.

So $a_n = T_n$ (number of alternating permutations in S_n).

Case of B_n

Let $V = \mathbb{R}^n$. In the case B_n, the reflecting hyperplanes are: $\{v \in V : v_i = 0\}$, $\{v \in V : v_i = v_j\}$, and $\{v \in V : v_i = -v_j\}$ $(i < j)$.

As orbit representatives of the lines in $\mathcal{P}(B_n)$, we can take:

$$L_i = \{v \in V : v_1 = \cdots = v_i = 0, \quad v_{i+1} = \cdots = v_n\},$$

with $0 \leq i \leq n-1$. We have

$$\mathrm{Fix}(L_i) = B_i \times A_{n-i-1}, \text{ and } \mathrm{Stab}(L_i) = (B_i \times A_{n-i-1}) \rtimes \{\pm Id\}.$$

So

$$b_n = \sum_{i=0}^{n-1} \binom{n-1}{i} b_i a_{n-i-1}.$$

Case of B_n

Let $B(z) = \sum_{n \geq 0} b_n \frac{z^n}{n!}$. The recursion

$$b_n = \sum_{i=0}^{n-1} \binom{n-1}{i} b_i a_{n-i-1}, \qquad b_0 = 1.$$

is equivalent to $B'(z) = B(z)A(z)$ and $B(0) = 1$.

The solution is $B(z) = A'(z) = \frac{1}{1-\sin(z)}$.

So $b_n = T_{n+1}$ (number of alternating permutations in S_{n+1}).

Case of D_n

Let $V = \mathbb{R}^n$. In type D_n, the reflecting hyperplanes are:
$\{v \in V \,:\, v_i = v_j\}$ and $\{v \in V \,:\, v_i = -v_j\}$ $(i < j)$.

As orbit representatives of the lines in $\mathcal{P}(D_n)$, we can take:

- $L_i = \{v \in V \,:\, v_1 = \cdots = v_i = 0, \quad v_{i+1} = \cdots = v_n\}$, with $i = 0$ or $2 \le i \le n-1$. We have $\mathrm{Fix}(L_i) = D_i \times A_{n-i-1}$,

$$\mathrm{Stab}(L_i) = \begin{cases} (D_i \times A_{n-i-1}) \rtimes \{\pm Id\} \text{ if } n \text{ is even,} \\ (D_i \times A_{n-i-1}) \rtimes \{Id, (1,-1,\ldots,-1)\} \text{ if } n \text{ is odd and } i > 0. \\ (D_i \times A_{n-i-1}) \text{ if } n \text{ is odd and } i = 0. \end{cases}$$

- And if n is even, we also include $L_0' = \{v \in V \,:\, -v_1 = v_2 = \cdots = v_n\}$. We have $\mathrm{Fix}(L_0') = \mathrm{Stab}(L_0') = A_{n-1}$.

Case of D_n

Let $d_n = K(D_n)$ and $\bar{d}_n = \#(\mathcal{M}(D_n)/B_n)$. The recursion for d_n is:

$$d_n = 2a_{n-1} + \sum_{i=2}^{n-1} \binom{n-1}{i} d_i a_{n-1-i}$$

if n is even, and

$$d_n = a_{n-1} + \sum_{\substack{2 \le i \le n-1 \\ n-i \text{ even}}} \binom{n-1}{i} d_i a_{n-1-i} + \sum_{\substack{2 \le i \le n-1 \\ n-i \text{ odd}}} \binom{n-1}{i} \bar{d}_i a_{n-1-i}$$

if n is odd.

We need to compute $\bar{d}_n = \#(\mathcal{M}(D_n)/B_n)$.

If n is odd, $B_n = D_n \rtimes \{\pm Id\}$ so $\bar{d}_n = d_n$.

The scheme for computing $K(W)$ works for $\bar{d}_n$ too and gives:

$$\bar{d}_n = a_{n-1} + \sum_{i=2}^{n-1} \binom{n-1}{i} \bar{d}_i a_{n-1-i}.$$

Let $\bar{D}(z) = \sum_{n\geq 0} \bar{d}_n \frac{z^n}{n!}$, the recursion is equivalent to:

$$\bar{D}'(z) = (\bar{D}(z) - z)A(z), \qquad D(0) = 1.$$

This is solved by

$$\bar{D}(z) = \frac{2 - \cos(z) - z\sin(z)}{1 - \sin(z)}.$$

It follows $\bar{d}_n = 2T_{n+1} - (n+1)T_n$ if $n \geq 2$.

So for odd $n \geq 2$, we have $d_n = 2T_{n+1} - (n+1)T_n$.

From the recursions for $\bar{d}_n$ and d_n, we have for even n:

$$(d_n - \bar{d}_n) = a_{n-1} + \sum_{i=2}^{n-1} (d_n - \bar{d}_n) a_{n-i-1}.$$

Let $U(z) = 1 + \sum_{n \geq 2} (d_n - \bar{d}_n) \frac{z^n}{n!}$, the recursion is equivalent to:

$$U'(z) = U(z) \tan(z), \qquad U(0) = 1.$$

This is solved by $U(z) = \sec(z)$.
So for even $n \geq 2$ we have $d_n = (d_n - \bar{d}_n) + d_n = 2T_{n+1} - nT_n$.

Remaining cases

Dihedral groups:

$$K(I_2(m)) = \begin{cases} 1 \text{ if } n \text{ is odd,} \\ 2 \text{ if } n \text{ is even.} \end{cases}$$

Exceptional groups: one method is to see them as symmetry groups of some semiregular polytopes, and use the geometry of these polytopes.

$$K(H_3) = 4, \quad K(H_4) = 12, \quad K(F_4) = 16,$$

$$K(E_6) = 82, \quad K(E_7) = 768, \quad K(E_8) = 4056.$$

In all cases except E_6, the polytope is centrally symmetric and this ensures we have $\mathrm{Stab}(L_i) = \mathrm{Fix}(L_i) \rtimes \{\pm Id\}$.

The general method to find orbit representatives of lines in $\mathcal{P}(W)$ is the following.
Let $H_1, \ldots, H_n$ so that the orthogonal reflections are simple generators and

$$L_i = \bigcap_{j \neq i} H_j.$$

Then $L_1, \ldots, L_n$ are representatives, except that if $L_i = w_0(L_j)$ we only take one of L_i and L_j (w_0 is the longest element).

thanks for your attention