Integer partition models for extended Catalan arrangements and generalized cluster complexes

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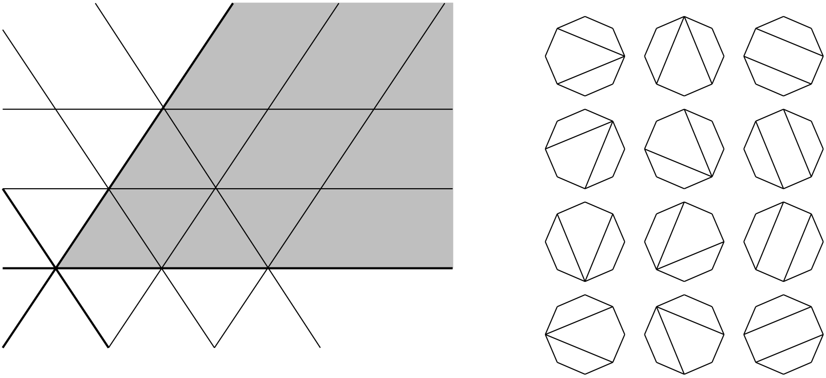
An example

12 regions of the hyperplane arrangement $\mathrm{Cat}^2(A_2)$ and 12 dissections of an 8-gon, corresponding to the facets of $\Delta^2(A_2)$.



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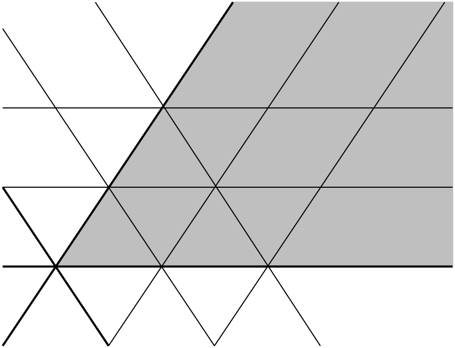

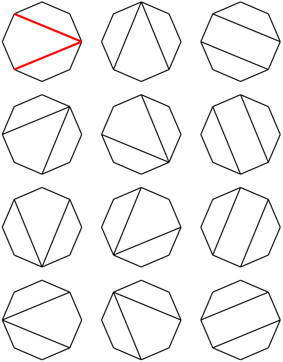

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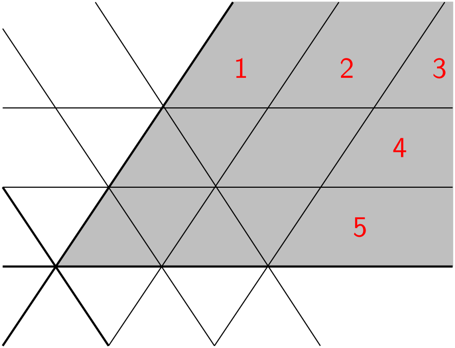

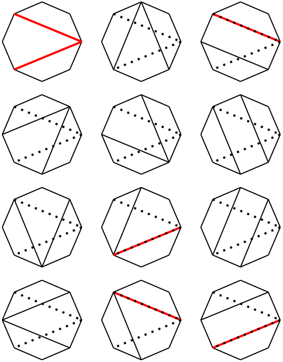

Outline

1. Introduction
    - Extended Catalan arrangement of type A
    - Generalized cluster complex of type A
2. Problem setup
3. Integer partition models
    - A bijection between the set of "gray" regions and a set of partitions
    - A bijection between the set of polygon dissections and a set of partitions
4. Results

Simple roots and positive roots

- Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}\}$ be the standard basis and $\langle \cdot, \cdot \rangle$ the standard inner product of $\mathbb{R}^{n+1}$.
- For $1 \leq i \leq n$ we set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.
- For $1 \leq i \leq j \leq n$ we set $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$.

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- $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of *simple roots* of type A_n.

- $\Phi_{>0} = \{\alpha_{ij}, 1 \leq i \leq j \leq n\}$ is the set of *positive roots* of type A_n.

Extended Catalan arrangement

- $H_{\alpha,k} = \{v \in \mathbb{R}^{n+1} : \langle v, \alpha \rangle = k\}$.
- $H^+_{\alpha,k} = \{v \in \mathbb{R}^{n+1} : \langle v, \alpha \rangle \geq k\}$.
- The *dominant chamber* is the intersection

$$\bigcap_{\alpha \in \Phi_{>0}} H^+_{\alpha,0}.$$

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Definition

The m-(extended) Catalan arrangement $\mathrm{Cat}^m(A_n)$ is the set of hyperplanes $\{H_{\alpha,k} : \alpha \in \Phi_{>0}, 0 \leq k \leq m\}$.

Some more definitions

- The *regions* of a hyperplane arrangement are the connected components of the complement of the arrangement.
- Each hyperplane which supports a facet of a region R is called a *wall* of the region R.
- A wall H of a region is a *separating wall* if the origin and the region lie in different half-spaces relative to H.
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Notation:

- $\mathcal{R}^m(A_n) := \{R,\ R$ dominant region of $\mathrm{Cat}^m(A_n)\}$.
- $\mathcal{R}^m_+(A_n) := \{R,\ R$ bounded dominant region of $\mathrm{Cat}^m(A_n)\}$.

The hyperplane arrangement $\mathrm{Cat}^2(A_2)$

- Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ be the standard basis of $\mathbb{R}^3$.
- $\Pi = \{\alpha_1, \alpha_2\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.
- $\Phi_{>0} = \{\alpha_1, \alpha_2, \alpha_{12}\}$, where $\alpha_{12} = \alpha_1 + \alpha_2$.

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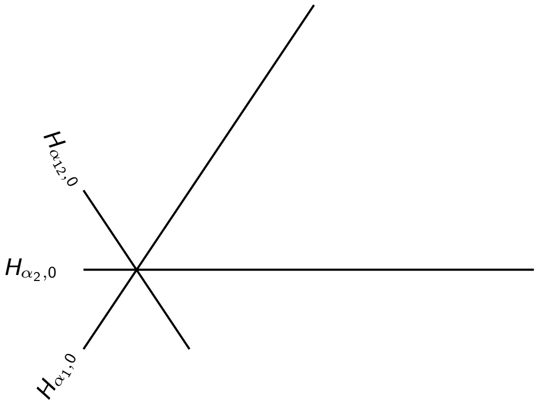

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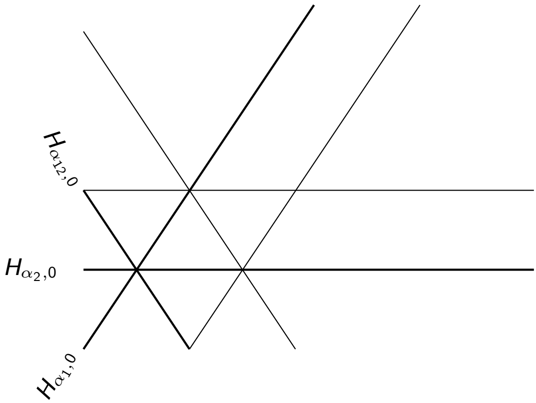

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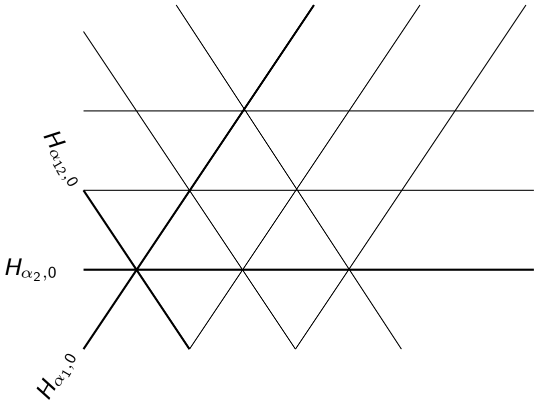

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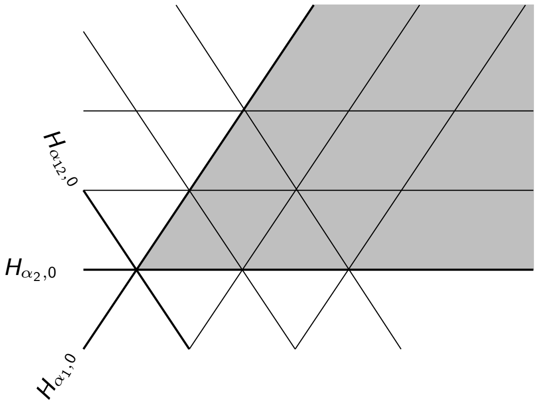

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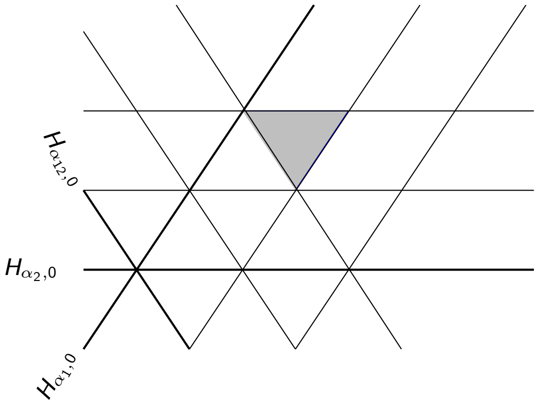

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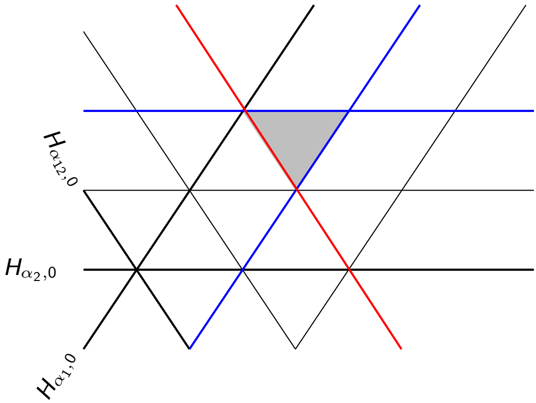

Generalized cluster complex

- Let $\Phi_{>0}$ be set of positive roots and Π the set of simple roots of type A_n.
- $\Phi^m_{\geq -1}$: set of *colored almost positive roots*.
  $\Phi^m_{\geq -1} = \{\alpha^k : \alpha \in \Phi_{>0}, k \in \{1, 2, \ldots, m\}\} \cup (-\Pi)$.

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- *The m-generalized cluster complex $\Delta^m(A_n)$* is a pure simplicial complex of dimension $n-1$ on the ground set of colored almost positive roots.
- *The positive part $\Delta^m_+(A_n)$* is a subcomplex of $\Delta^m(A_n)$ whose facets contain no negative simple roots.

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We will use the following combinatorial model realizing the complex $\Delta^m(A_n)$. [Fomin, Reading, Tzanaki '05]

Construction of $\Delta^m(A_n)$

Consider an $(m(n+1)+2)$-gon P.

- Vertices: diagonals which dissect the polygon into two subpolygons with number of vertices $2 \bmod m$. We call these diagonals m-diagonals.
- k-faces: dissections having $k+1$ many m-diagonals.
- Facets: dissections having n many m-diagonals.

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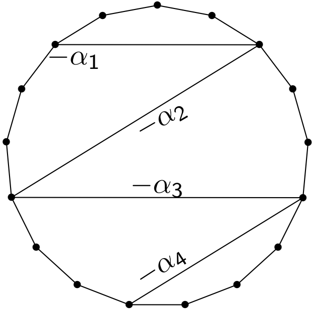
Recall: The ground set of $\Delta^m(A_n)$ is the set $\Phi^m_{\geq -1}$.

Q: Which diagonals correspond to negative simple roots and which to colored positive roots?

Polygon dissections of type A

$n = 4, m = 3$



- Negative simple roots: n consecutive m-diagonals ("snake").

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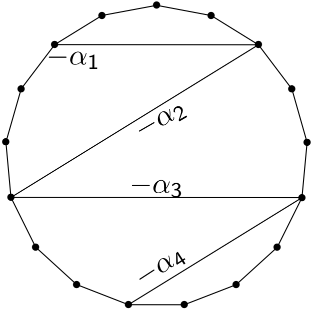

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- The m-colored copies of each positive root are determined from the snake:
  - for each positive root α_{ij} there exist m many m-diagonals, which "intersect" with the roots $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$.These are the m colored copies of α_{ij}.

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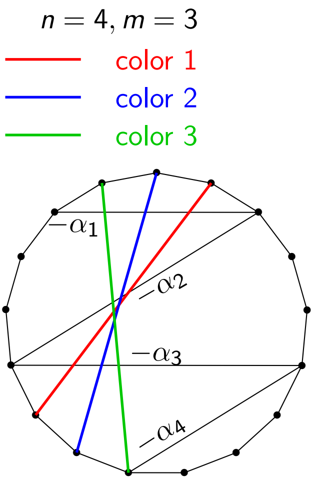

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- α_{13}, α_{13}, α_{13}

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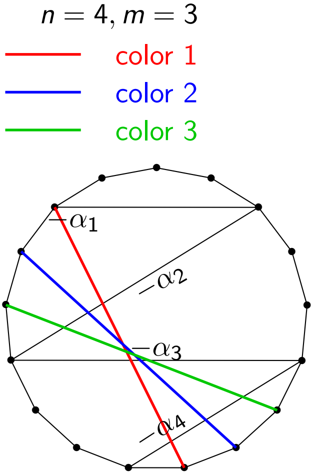

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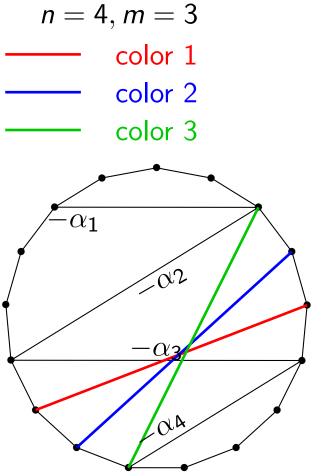

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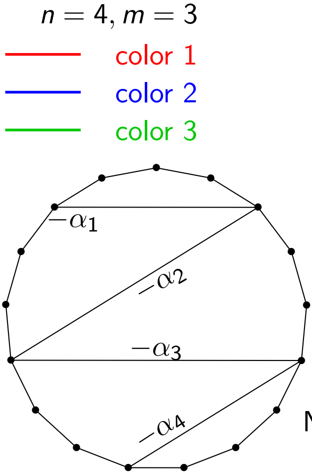
- α_3, α_3, α_3

Polygon dissections of type A



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Notation:

- $\mathcal{D}^m(A_n) = \{F, F \text{ facet of } \Delta^m(A_n)\}$.
- $\mathcal{D}^m_+(A_n) = \{F, F \text{ facet of } \Delta^m_+(A_n)\}$.

Motivation (formally)

The following hold.

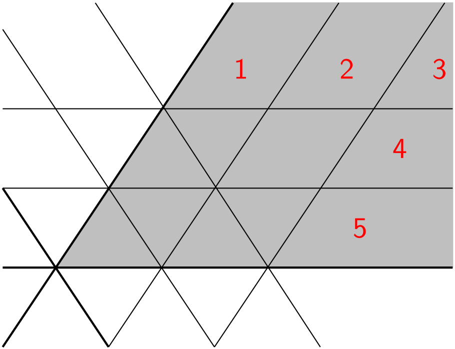
- $\#\mathcal{R}^m(A_n) = \#\mathcal{D}^m(A_n) = \frac{1}{n+1}\binom{(m+1)(n+1)}{n}$
- $\#\mathcal{R}^m_+(A_n) = \#\mathcal{D}^m_+(A_n) = \frac{1}{n+1}\binom{m(n+1)+n-1}{n}$

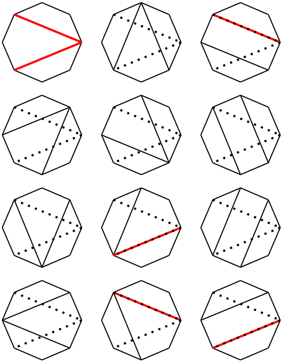
Property

For any $J \subseteq \{1, \ldots, n\}$, the number of facets of $\Delta^m(A_n)$ containing exactly the negative simple roots $-\alpha_i$ with $i \in J$, is equal to the number of dominant regions in $\mathrm{Cat}^m(A_n)$ with separating walls $H_{\alpha_i,m}$ with $i \in J$.

back to our example

- $\#\mathcal{R}^2(A_2) = \#\mathcal{D}^2(A_2) = 12$
- $\#\mathcal{R}^2_+(A_2) = \#\mathcal{D}^2_+(A_2) = 7$



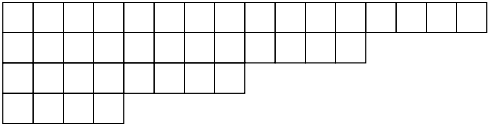


Goal: Find a bijection between $\mathcal{R}^m(A_n)$ and $\mathcal{D}^m(A_n)$ that preserves our property.

Tool: Integer partitions.

We consider the set of *(n, m)-dilated* partitions:

$$\mathcal{DL}^m(n) := \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \,|\, 0 \leq \lambda_i \leq m(n-i+1)\}.$$



Idea: Encode both objects in terms of (n, m)-dilated partitions.

Shi tableaux A_n

Proposition

There is a bijection between the dominant regions in $Cat^m(A_n)$ and the so-called m-Shi tableaux of type A. [Athanasiadis '05, Fishel, Tzanaki, Vazirani '11]

k_{14}	k_{13}	k_{12}	k_{11}
k_{24}	k_{23}	k_{22}	
k_{34}	k_{33}		
k_{44}			

- An m-Shi tableau is an n-staircase diagram with entries being positive integers between 0 and m that satisfy certain conditions. It can be considered as the "coordinates" of a given region.

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α_{44}			

$H_{\alpha_{ij},k}$

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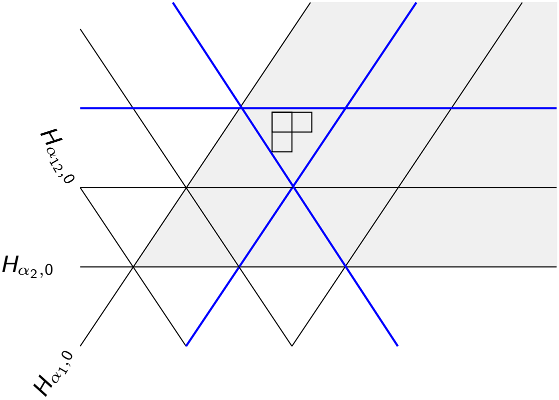
Example: Regions in $\mathrm{Cat}^2(A_2)$ and their tableaux



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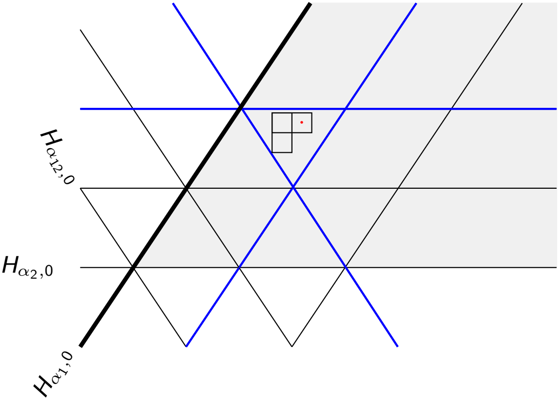

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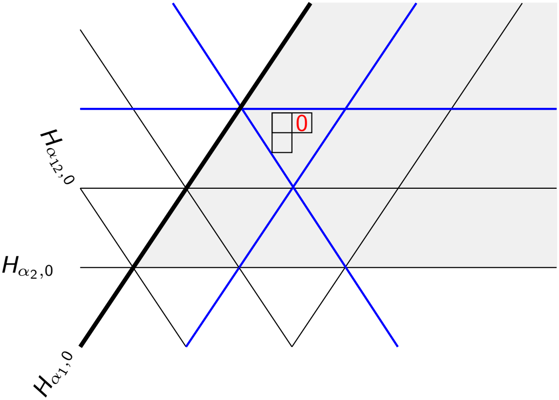

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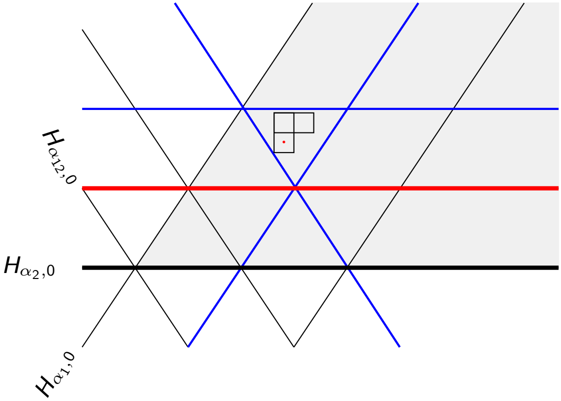

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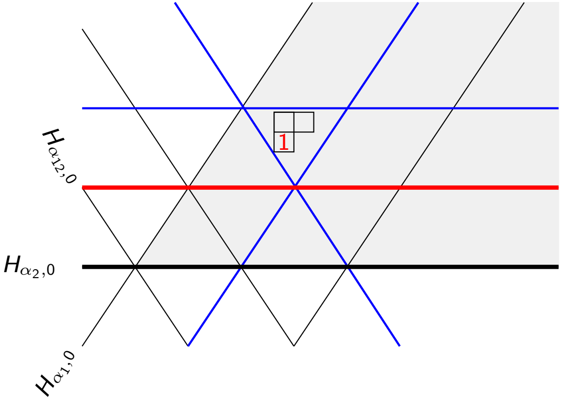

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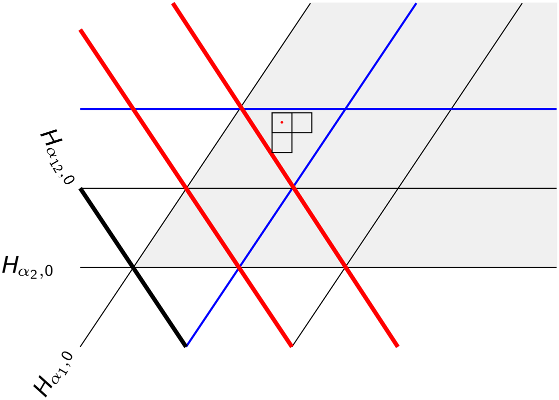

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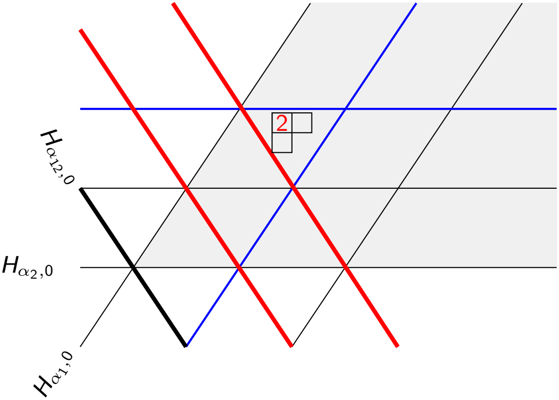

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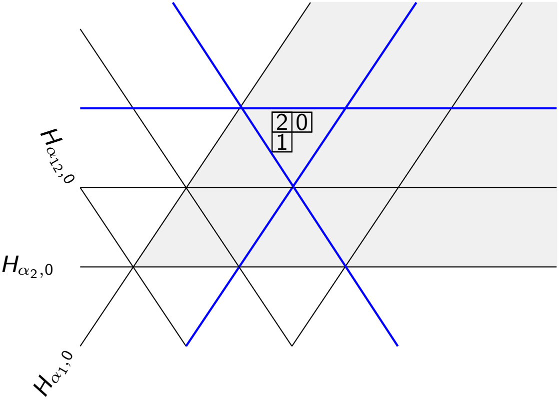

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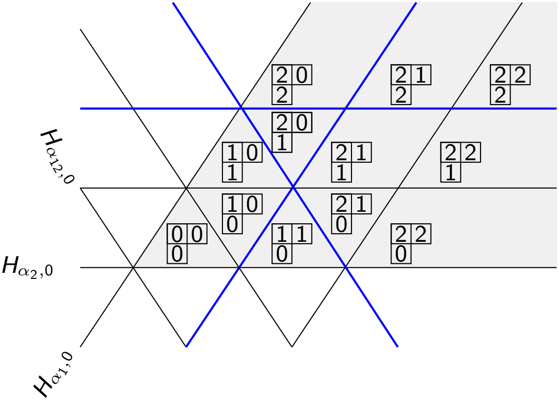

The bijection between $\mathcal{R}^m(A_n)$ and $\mathcal{DL}^m(n)$

Let $\phi : \mathcal{R}^m(A_n) \to \mathcal{DL}^m(n)$ be the map which sends each m-Shi tableau to the partition whose parts are the sum of the entries of each row.

Example: $n = 3, m = 4$

4	4	2
4	4	
1		

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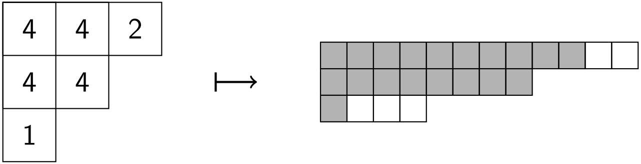

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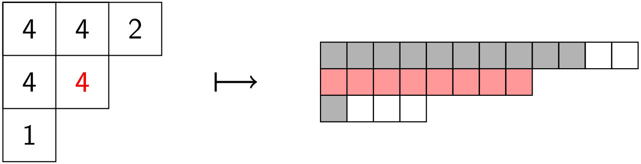
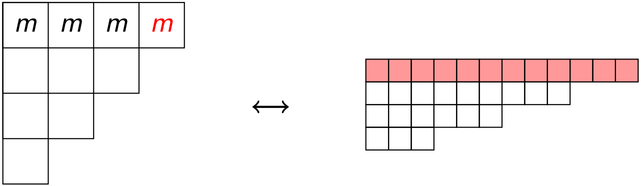


- Separating wall: the hyperplane $H_{\alpha_2,4}$.

First result

Theorem (FKT '11)

The map ϕ is a bijection. In particular, the hyperplane $H_{\alpha_i,m}$ (where $\alpha_i \in \Pi$) is a separating wall of the region R if and only if the partition $\phi(R) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfies $\lambda_i = (n-i+1)m$.



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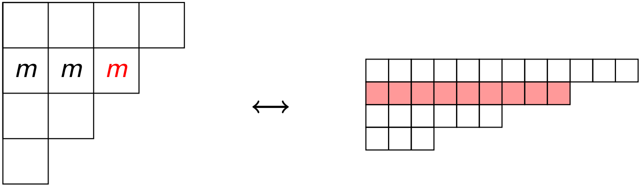

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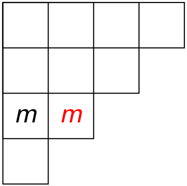
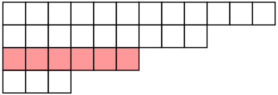


$\longleftrightarrow$



First result

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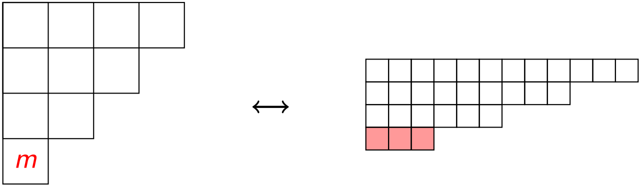
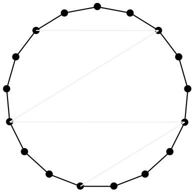
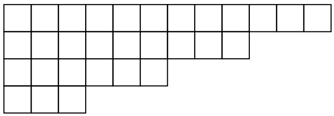
The map ϕ is a bijection. In particular, the hyperplane $H_{\alpha_i,m}$ (where $\alpha_i \in \Pi$) is a separating wall of the region R if and only if the partition $\phi(R) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfies $\lambda_i = (n - i + 1)m$.



The bijection between $\mathcal{D}^m(A_n)$ and $\mathcal{DL}^m(n)$

In view of the map ϕ and our property, we need a bijection such that
the dissection $D \in \mathcal{D}^m(A_n)$ contains the negative simple root $-\alpha_i$ if and only if its image $(\lambda_1, \ldots, \lambda_n)$ satisfies $\lambda_i = (n - i + 1)m$.





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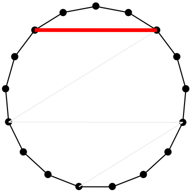

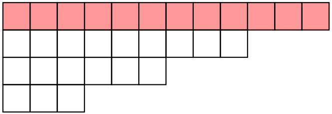

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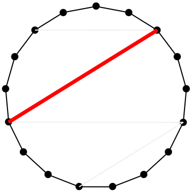

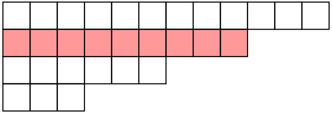

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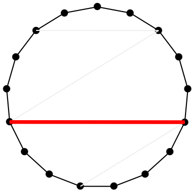

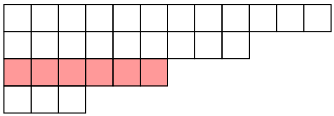

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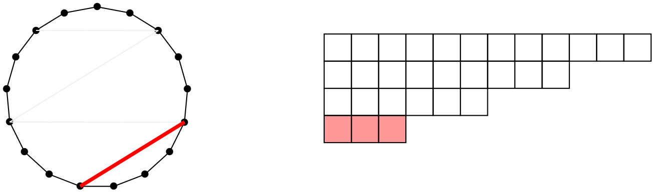


The bijection between $\mathcal{D}^m(A_n)$ and $\mathcal{DL}^m(n)$

Idea: We label the vertices of the $(m(n+1)+2)$-gon P. For $1 \leq i \leq n$, let i_a, i_b with $i_a < i_b$ be a pair of labels corresponding to each diagonal. The point i_a is called *initial* point of the diagonal $\{i_a, i_b\}$. We map the dissection D to the partition defined by the initial points $1_a, 2_a, \ldots, n_a$.

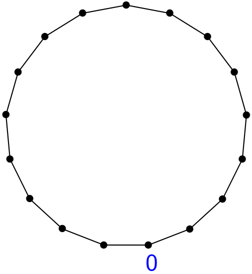
Problem: How do we label the vertices so that the property is satisfied? For instance, the "natural" labeling does not work.

Solution: Use a labeling which we call *the alternating labeling*.

The bijection between $\mathcal{D}^m(A_n)$ and $\mathcal{DL}^m(n)$

The alternating labeling

$n = 4, m = 3$



- Let P be a $(m(n+1)+2)$-gon.
- Fix some vertex 0 of P and label its vertices from 1 to $m(n+1)+1$ as follows:
  - the vertices on the right of 0 are labeled in increasing order with those $k \in \{0, 1, \ldots, m(n+1)+1\}$ for which $\lfloor \frac{k}{m} \rfloor$ is odd.
  - the vertices on the left of 0 are labeled in increasing order with those $k \in \{0, 1, \ldots, m(n+1)+1\}$ for which $\lfloor \frac{k}{m} \rfloor$ is even.

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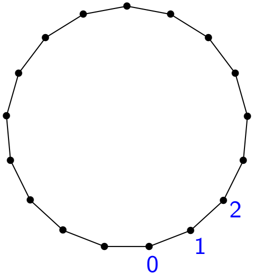

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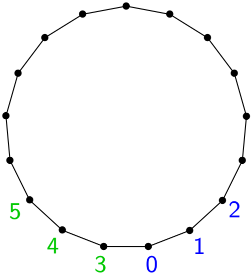

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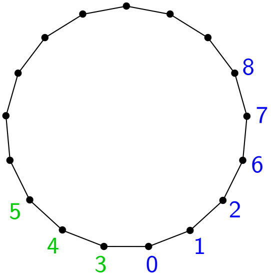

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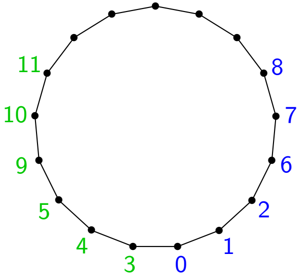

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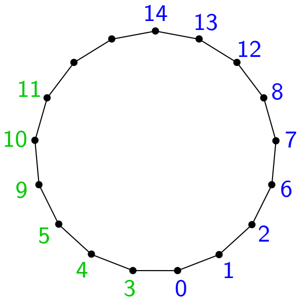

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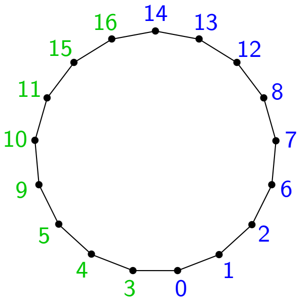

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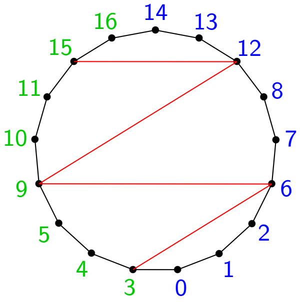

- The m-diagonals $\{im, (i+1)m\}$, for $1 \leq i \leq n$, form a "snake".

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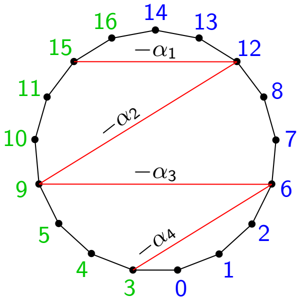
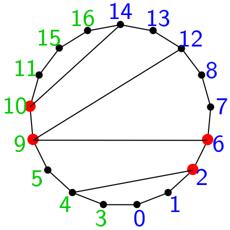


- The m-diagonals $\{im, (i+1)m\}$, for $1 \leq i \leq n$, form a "snake".
- We set $-\alpha_i$ to be the diagonal with endpoints $(n-i+1)m, (n-i+2)m$.

The bijection between $\mathcal{D}^m(A_n)$ and $\mathcal{DL}^m(n)$

Let $\psi : \mathcal{D}^m(A_n) \to \mathcal{DL}^m(n)$ be the map which sends each dissection to the partition whose parts are the initial points w.r.t the alternating labeling.

Example: $n = 4, m = 3$



- Initial points:
  $\lambda_1 = 10, \lambda_2 = 9, \lambda_3 = 6, \lambda_4 = 2$
- $\lambda = (10, 9, 6, 2)$.

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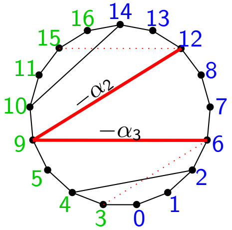

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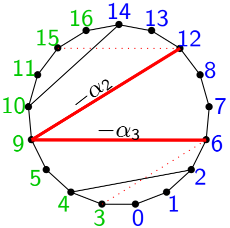
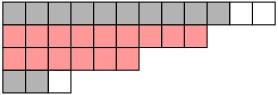


$\longmapsto$

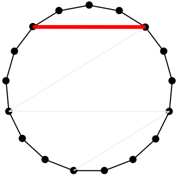


- negative simple roots $-\alpha_2, -\alpha_3$

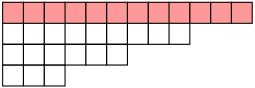
Second result

Theorem (FKT'11)

The map ψ is a bijection. In particular, the negative simple root $-\alpha_i$ is contained in D if and only if the partition $\psi(D) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfies $\lambda_i = (n - i + 1)m$.



$\longleftrightarrow$



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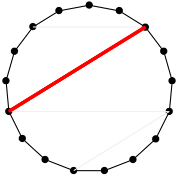

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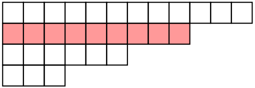

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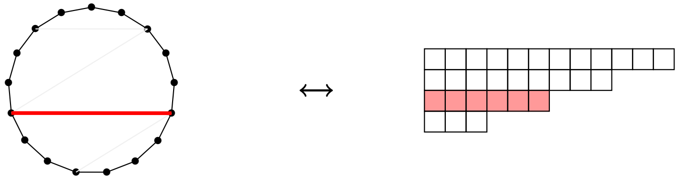

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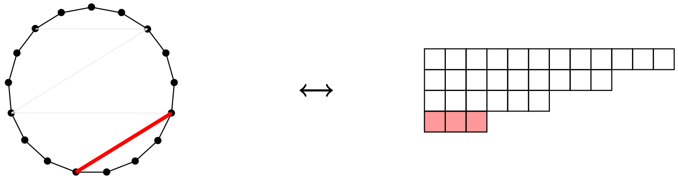

Composing ψ^{-1} and ϕ

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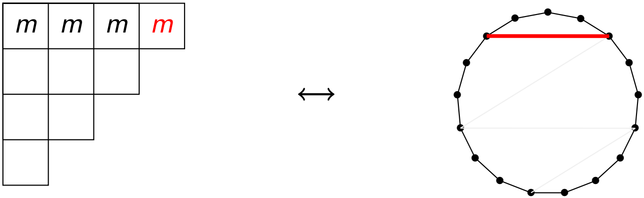
The map $\psi^{-1} \circ \phi$ is a bijection. In particular, the hyperplane $H_{\alpha_i,m}$ (where $\alpha_i \in \Pi$) is a separating wall of R if and only if $\psi^{-1}(\phi(R))$ contains the negative simple root $-\alpha_i$.



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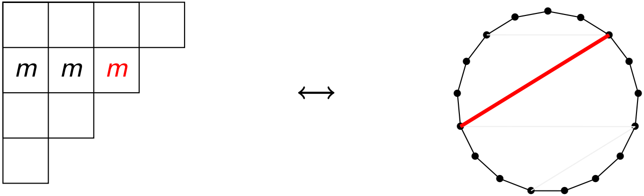

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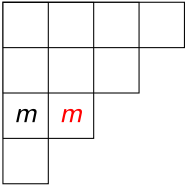

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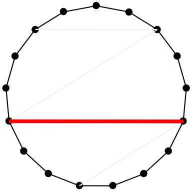

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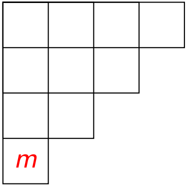

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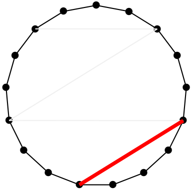
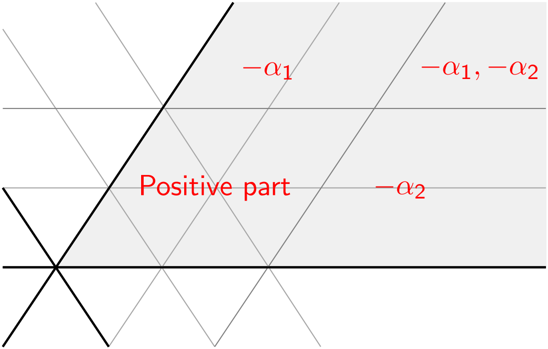


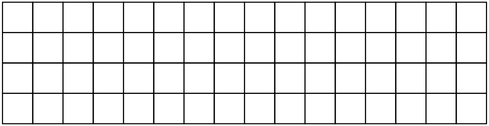
Composing ψ^{-1} and ϕ



Types B and C

We employ the set of *(n, m)-bounded* partitions:

$$\mathcal{B}^m(n) := \{(\lambda_1, \lambda_2, \dots, \lambda_n) | 0 \leq \lambda_i \leq mn\}.$$

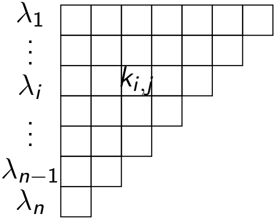


- We give a bijection between the sets $\mathcal{D}^m(B_n), \mathcal{D}^m(C_n)$ and the set $\mathcal{B}^m(n)$ and characterize the facets which contain the negative simple root $-\alpha_i$, $1 \leq i \leq n$.
- We give a bijection between a subset of $\mathcal{R}^m(B_n), \mathcal{R}^m(C_n)$ and a subset of $\mathcal{B}^m(n)$ and characterize the dominant regions which are separated from the origin by certain hyperplanes of the form $H_{\alpha_i, m}$.

Danke schön!

The map $\phi^{-1} : \mathcal{DL}^m(n) \to \mathcal{R}^m(n)$

Let $(\lambda_1, \ldots, \lambda_n) \in \mathcal{DL}^m(n)$. For each $1 \leq i \leq j \leq n$ we define recursively:



$$k_{i,j} = \min\left\{ m, \left\lceil \frac{\lambda_i - \qquad + \qquad}{j-i+1} \right\rceil \right\}.$$

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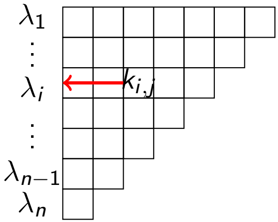

$$k_{i,j} = \min\left\{ m, \left\lceil \frac{\lambda_i - \sum_{\ell=j+1}^{n} k_{i,\ell} +}{j-i+1} \right\rceil \right\}.$$

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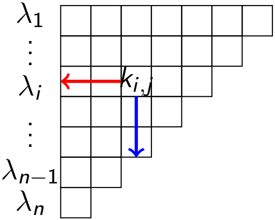

$$k_{i,j} = \min\left\{ m, \left\lceil \frac{\lambda_i - \sum_{\ell=j+1}^{n} k_{i,\ell} + \sum_{\ell=i+1}^{j} k_{\ell,j}}{j-i+1} \right\rceil \right\}.$$

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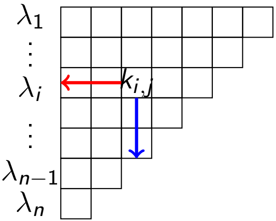

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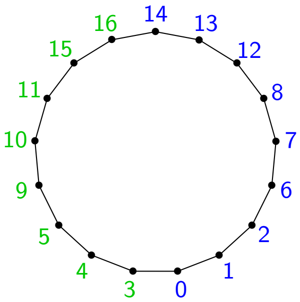

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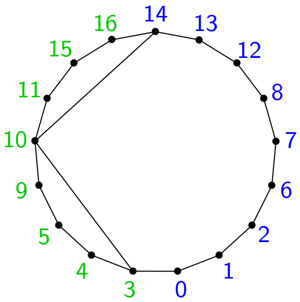

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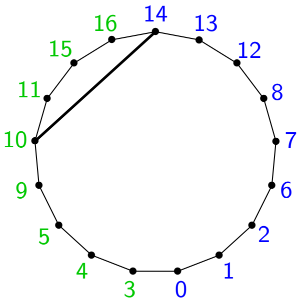

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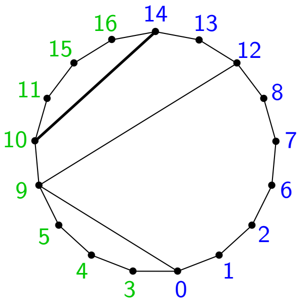

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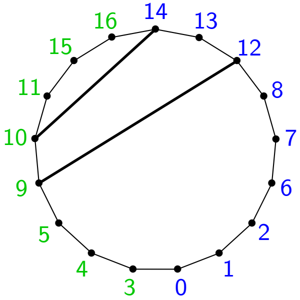

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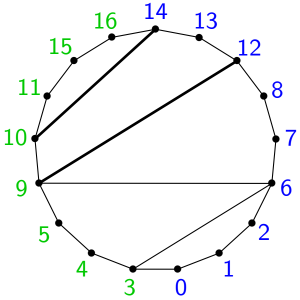

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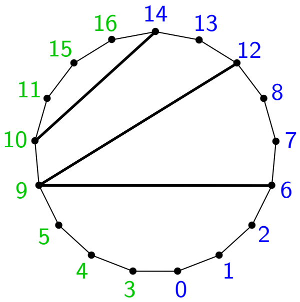

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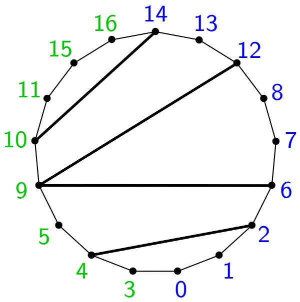

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Conditions for a Shi tableau

The filling of an n-staircase diagram is called an *m-Shi tableau* if:

k_{14}	k_{13}	k_{12}	k_{11}
k_{24}	k_{23}	k_{22}	
k_{34}	k_{33}		
k_{44}			

- for each $k_{ij} < m$ the sum of the values of the endpoints of each hook on k_{ij} of length $j - i + 2$ sum up to k_{ij} or $k_{ij} - 1$.
- for each $k_{ij} = m$ the sum of the values of the endpoints of each hook on k_{ij} of length $j - i + 2$ sum up to a value $\geq m - 1$.

Example of a Shi tableau

$n = 5, m = 5$

5	4	3	2	2
5	2	1	0	
4	1	1		
3	0			
2				

- for each $k_{ij} < 5$ we check if the sum of the values of the endpoints of each hook on k_{ij} of length $j - i + 2$ sum up to k_{ij} or $k_{ij} - 1$.
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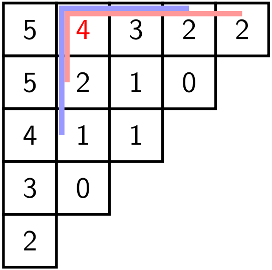

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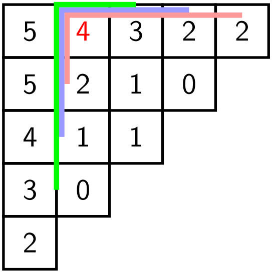

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