Difference equations and Linearization of a product of orthogonal polynomials

Anisse Kasraoui
(Universität Wien)

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In the late 1960's Askey formulated several conjectures about the non-negativity of integrals of products of orthogonal polynomials.

In the 1970's it was realized that some of the integrals ($\equiv$ linearization coefficients) considered by Askey and his coauthors have combinatorial interpretations involving some kind of inhomogeneous partitions and permutations.

The simple Laguerre polynomials can be defined by one of the following equivalent conditions :

1. (Coefficients) $L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n}{k} x^k$.
2. (Generating function) $\sum_{n\geq 0} L_n(x)t^n = (1-t)^{-1}\exp\left(\frac{-xt}{1-t}\right)$.
3. (Orthogonality relation) $\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = \delta_{m,n}$.
4. (Recurrence relation) $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$.
5. (Moments) $\mu_n = \int_0^\infty e^{-x}\, dx = n!$.

Even and Gillis (1974) showed that

$$(-1)^{\sum_{k=1}^{m} n_k} \int_0^\infty e^{-x} \prod_{j=1}^{m} L_{n_j}(x) dx \tag{1}$$

is equal to the number of generalized derangements of sets of sizes $n_1, n_2, \ldots, n_m$.

The Hermite polynomials $\{H_n(x)\}_{n\geq 0}$ can be defined by one of the following five equivalent conditions :

1. (Coefficients) $H_n(x) = \sum_{0\leq 2k\leq n}(-1)^k \frac{n!}{2^k k!(n-2k)!}(2x)^{n-2k}$.
2. (Generating function) $\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp(2xt - t^2)$.
3. (Orthogonality relation) $\int_{\mathbb{R}} H_m(x)H_n(x)e^{-x^2}dx = 2^n n!\,\sqrt{\pi}\,\delta_{mn}$.
4. (Recurrence relation) $2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x)$, with $H_{-1}(x) = 0$, $H_0(x) = 1$.
5. (Moments) $\mu_{2n+1} = 0$, $\mu_{2n} = 1\cdot 3\cdots(2n-1)/2^n$.

Azor, Gillis, and Victor (1982) and independently Godsil (1982) showed that

$$2^{-(n_1+\cdots+n_m)/2}\int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}}\prod_{j=1}^{m} H_{n_j}(x)dx,$$

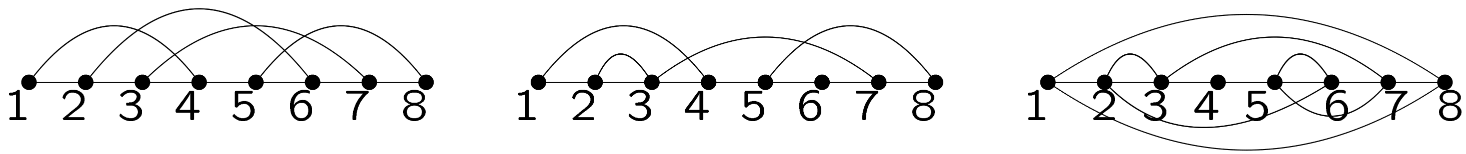
where the $H_n(x)$'s are the Hermite polynomials, is the number of perfect inhomogeneous matchings of sets of sizes $n_1, n_2, \ldots, n_m$.

We denote by Π_n and $\mathfrak{S}_n$ the set of partitions and the set of permutations, respectively, of $[n] := \{1, 2, \ldots, n\}$.

A perfect matching of $[n]$ is just a set partition of $[n]$ the blocks of which have exactly two elements.

Let $\mathcal{M}_n$ denote the set of perfect matchings of $[n]$.

We represent pictorially matchings, set partitions and permutations of $[n]$.



Diagrams of, from left to right, the matching $M = 1\,4/2\,6/3\,7/5\,8$, the partition $\pi = 1\,4/2\,3\,7/5\,8/6$ and the permutation $\sigma = 8\,3\,7\,4\,6\,2\,5\,1$

Inhomogeneous partitions and permutations.

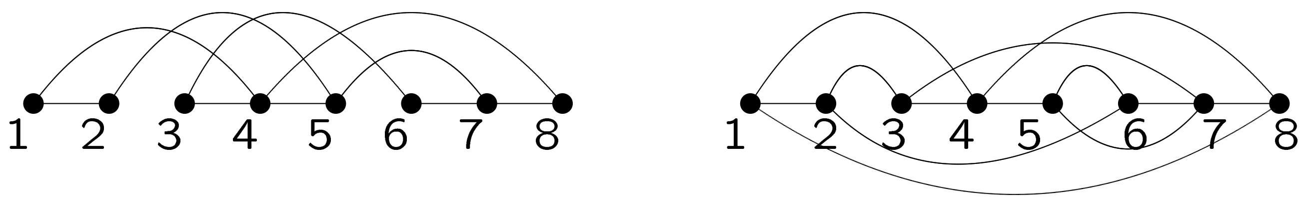
$\boldsymbol{n} = (n_1, \ldots, n_m)$ m-tuple of positive integers, $n := n_1 + \cdots + n_m$.

Set $S_j = \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j\}$ for $j = 1, \ldots, m$.

A partition π of $[n]$ is said to be *inhomogeneous* if each block of π contains at least two elements and no two elements in the same block belong to the same set S_i $(1 \leq i \leq m)$.

We denote by $\mathcal{P}(\boldsymbol{n})$ and $\mathcal{K}(\boldsymbol{n})$ the set of inhomogeneous partitions and the set of inhomogeonous perfect matchings, respectively, of $[\boldsymbol{n}]$.

Similarly, a permutation σ of $[\boldsymbol{n}]$ is an *inhomogeneous derangement* if $\sigma(S_i) \cap S_i = \emptyset$ for all $i \in [m]$.



Diagrams of, from left to right, a partition in $\mathcal{P}(2,3,3)$ and permutation in $\mathcal{D}(2,3,3)$.

Note that a set partition (resp., permutation) is inhomogeneous if and only in its diagram, there is no isolated vertex and no arc connecting two elements in the same S_i $(1 \leq i \leq m)$.

The Tchebycheff polynomials of the second kind are the (unique) polynomials $U_n(x)$ defined by $\frac{sin(n+1)\theta}{sin\theta} = U_n(cos\theta)$. De Sainte Catherine and Viennot (1975) proved that

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\prod_{j=1}^{m}U_{n_m}(x)dx,$$

where the $U_n(x)$'s are the Tchebycheff polynomials of the second kind, is the number of perfect inhomogeneous noncrossing matchings of sets of sizes $n_1, n_2, \ldots, n_m$.

The Charlier polynomials $C_n^{(a)}(x)$ can be defined by one of the following five equivalent conditions :

1. (Explicit formula) $C_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} k!(-a)^{n-k}$.
2. (Generating function) $\sum_{n=0}^{\infty} C_n^{(a)}(x)\frac{w^n}{n!} = e^{-aw}(1+w)^x$,
3. (Orthogonality) $\int_0^\infty C_n^{(a)}(x)C_m^{(a)}(x)d\psi^{(a)}(x) = a^n n!\delta_{mn}$, where $\psi^{(a)}$ is the step function of which the jumps at the points $x = 0, 1, \ldots$ are $\psi^{(a)}(x) = \frac{e^{-a}a^x}{x!}$.
4. (Recursion relation) $C_{n+1}^{(a)}(x) = (x-n-a)C_n^{(a)}(x) - anC_{n-1}^{(a)}(x)$.
5. (Moments) $\mu_n = \sum_{k=1}^{n} S(n,k)a^k$, where $S(n,k)$ are the Stirling numbers of the second kind.

Zeng (1988) and Gessel (1989) showed that

$$\int_0^\infty C_{n_1}^{(a)}(x)\cdots C_{n_m}^{(a)}(x)d\psi^{(a)}(x) = \sum_{\pi\in\mathcal{P}(\boldsymbol{n})} a^{\mathsf{bl}(\pi)},$$

where $\mathcal{P}(\boldsymbol{n})$ is the set of inhomogeneous partitions of type $\boldsymbol{n}$ and $\mathsf{bl}(\pi) =$ number of blocks in π.

The (generalized) Laguerre polynomials can be defined by

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!(\alpha+1)_k} x^k, \tag{2}$$

and satisfy the orthogonality relation

$$\int_0^\infty \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\, dx = \frac{(\alpha+1)_n}{n!}\, \delta_{m,n}. \tag{3}$$

Foata and Zeilberger (1988) showed that

$$(-1)^{\sum_{k=1}^{m} n_m} n_1! \cdots n_m! \int_0^\infty \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} \prod_{j=1}^{m} L_{n_j}^{(\alpha)}(x) dx = \sum_{\sigma \in \mathcal{D}(\boldsymbol{n})} (\alpha+1)^{\mathrm{cyc}(\pi)},$$

$\mathrm{cyc}(\sigma)$ = number of cycles of σ and $\mathcal{D}(\boldsymbol{n})$ is the set of generalized derangements of type $\boldsymbol{n}$.

Zeng (1992) extended this study to Meixner and Meixner-Pollaczek.

Kim and Zeng (2001) found a common generalization of all these combinatorial interpretations.

There are essentially three methods to establish these combinatorial interpretations.

The generating function approach. Combining the β-extension of MacMahon's Master theorem (Foata and Zeilberger, 1988) and the exponential formula, all the known combinatorial interpretations of the linearization coefficients of the orthogonal Sheffer polynomials can be deduced by computing their generating functions.

The direct combinatorial approach. (De Sainte Catherine et Viennot) Using the combinatorial interpretations of the orthogonal polynomials (Hermite and Laguerre) as matching polynomials of certain graphs and the combinatorial interpretations to obtain a messy sum, and then using a killing involution to reduce it to some nicer models.

A more classical approach. Find a recurrence for the linearization coefficients and show that the desired combinatorial interpretations satisfy the same recurrence.

The 3 methods can be used to prove the combinatorial interpretations of orthogonal Sheffer polynomials.

However, the generating function approach seems to fail when one tries to extend the previous results to their q-analogues, even though a conjecture for the combinatorial interpretation is formulated.

Rogers q-Hermite polynomials :

$$xH_n(x|q) = H_{n+1}(x|q) + [n]_q H_{n-1}(x|q).$$

Here q is (say) in $(-1, 1)$, and

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}.$$

Orthogonal with respect to

$$d\mu_q(x) = \frac{1}{\pi}\sqrt{1-q}\,\sin(\theta)(q;q)_\infty \left|(qe^{2i\theta;q})_\infty\right|^2 dx,$$

for $x = \frac{2}{\sqrt{1-q}}\cos(\theta)$, $\theta \in [0, \pi]$, and

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j).$$

Moments

$$\mu_n = \int x^n d\mu_q(x) = \sum_{\pi} q^{\mathrm{cros}_2(\pi)},$$

where the summation is over all perfect matchings of $\{1, 2, \ldots, n\}$.

Ismail, Stanton and Viennot (1987), using the pure combinatorial approach, proved that

$$\int H_{n_1}(x|q) \ldots H_{n_k}(x|q) d\mu_q(x) = \sum_{\pi \in \mathcal{K}(n_1, n_2, \ldots, n_k)} q^{\mathrm{cros}_2 \pi}.$$

For $k = 4$, it gives a remarkable combinatorial evaluation of the Askey-Wilson integral

$$\frac{(q;\, q)_\infty}{2\pi} \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta};\, q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta};\, q)_\infty} d\theta = \frac{(t_1 t_2 t_3 t_4;\, q)_\infty}{\prod_{1 \le j < k \le 4} (t_j t_k;\, q)_\infty}.$$

The Al-Salam-Chihara q-Charlier polynomials

Recurrence

$$xC_{q,n}(x,a) = C_{q,n+1}(x,a) + (a + [n]_q)C_{q,n}(x,a) + a[n]_q C_{q,n-1}(x,a).$$

Moments (Biane 1997)

$$\mu_n(a,q) = \sum_{\pi \in \Pi_n} q^{\mathsf{cros}_2(\pi)} a^{|\pi|}.$$

Linearization coefficients, using stochastic process, Anshelevich (2005)

$$\mathcal{L}_q\left(C_{q,n_1}(x,a) \ldots C_{q,n_k}(x,a)\right) = \sum_{\pi \in \Pi(n_1,n_2,\ldots,n_k)} q^{\mathsf{cros}_2 \pi} a^{|\pi|}.$$

The Al-Salam-Chihara q-Laguerre polynomials

Recurrence :

$$L_{n+1}(x;\, q) = (x - y[n+1]_q - [n]_q)L_n(x;\, q) - y[n]_q^2 L_{n-1}(x;\, q).$$

Explicit formulas :

$$L_n(x;\, q) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!_q}{k!_q} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} y^{n-k} \prod_{j=0}^{k-1} \left(x - (1 - yq^{-j})[j]_q\right).$$

Moments (Randrianarivony (1994), Corteel (2006))

$$\mu_n^{(\ell)}(y, q) := \sum_{\sigma \in S_n} y^{wex(\sigma)} q^{\mathsf{cros}(\sigma)}.$$

Linearization coefficients. Using recursive approach, K., Stanton and Zeng (2009) proved

$$\mathcal{L}_q(L_{n_1}(x;\, q) \ldots L_{n_k}(x;\, q)) = \sum_{\sigma \in \mathcal{D}(n_1, \ldots, n_k)} y^{wex(\sigma)} q^{\mathsf{cros}(\sigma)}.$$

The q-Laguerre polynomials have the following interpretation :

$$L_n(x;\, q) = \sum_{A \subset [n], f : A \to [n]} (-1)^{|A|} x^{n-|A|} y^{\alpha(A,f)} q^{w(A,f)},$$

where f is injective.

Difficult to use for a combinatorial proof of the linearization formula

$$\mathcal{L}_q(L_{n_1}(x;\, q) \dots L_{n_k}(x;\, q)) = \sum_{\sigma \in \mathcal{D}(n_1, \dots, n_k)} y^{wex(\sigma)} q^{\mathsf{cros}(\sigma)}.$$

Characterization of linearization coeffcients with difference equations.

For simplicity (and without loss of generality), we only consider sequence of monic orthogonal polynomials. Let $\{p_n(x)\}$ be a sequence of monic orthogonal polynomials

$$\int_{\mathbb{R}} p_m(x)p_n(x)d\mu(x) = \zeta_n\delta_{m,n}, \quad \zeta_0 = 1. \tag{4}$$

Linearization coefficients in the expansion of $\prod_{j=1}^{m-1} p_{n_j}(x)$ in $\{p_n(x)\}$.

$$I(\boldsymbol{n}) := \int_{\mathbb{R}} \prod_{j=1}^{m} p_{n_j}(x)\ d\mu(x), \tag{5}$$

where $\boldsymbol{n} = (n_1, \ldots, n_m)$, n_j is a nonnegative integer for $1 \le j \le m$. We shall use the following notation :

$$I_j^{\pm}(\boldsymbol{n}) = I(n_1, \ldots, n_{j-1}, n_j \pm 1, n_{j+1}, \ldots, n_m).$$

By (Favard's Theorem) the polynomials $\{p_n(x)\}$ must satisfy a three term recurrence relation of the form

$$p_{n+1}(x) = (x - b_n)p_n(x) - c_n p_{n-1}(x), \quad n > 0, \tag{6}$$

with $p_{-1}(x) := 0$, $p_0(x) = 1$.

Remark. The numbers $I(\boldsymbol{n})$ satisfy the system of difference equations

$$I_j^+(\boldsymbol{n}) - I_k^+(\boldsymbol{n}) = (b_{n_k} - b_{n_j})I(\boldsymbol{n}) - c_{n_j}I_j^-(\boldsymbol{n}) + c_{n_k}I_k^-(\boldsymbol{n}). \tag{7}$$

Proof. Suppose $1 \leq t \leq m$. By Favard's recurrence,

$$I_t^+(\boldsymbol{n}) = \int_{\mathbb{R}} \big((x - b_{n_t})p_{n_t}(x) - c_{n_t}p_{n_j-1}(x)\big) \prod_{r \neq t} p_{n_r}(x)d\mu(x),$$

whence, by linearity of the integral,

$$I_t^+(\boldsymbol{n}) = \int_{\mathbb{R}} x \prod_{r=1}^{m} p_{n_t}(x)d\mu(x) - b_{n_t}I(\boldsymbol{n}) - c_{n_t}I_t^-(\boldsymbol{n}).$$

Theorem 1 *The system of difference equations*

$$(1 \le j \le m) \quad Y_j^+(\boldsymbol{n}) - Y_{j+1}^+(\boldsymbol{n}) = (b_{n_{j+1}} - b_{n_j})I(\boldsymbol{n}) - c_{n_j}Y_j^-(\boldsymbol{n}) + c_{n_{j+1}}Y_{j+1}^-(\boldsymbol{n}).$$

and the boundary conditions
(i) $Y(0,\ldots,0) = 1$,
(ii) $Y(n_1, n_2, \cdots, n_m) = 0$ *if* $\sum_{j=1}^{m-1} n_j < n_m$,
(ii) $Y_j^+(0,\ldots,0,1) = c_1$ *if* $1 \le j \le m-1$.
have a unique solution which is given by (5).

Proof.

The numbers $I(\boldsymbol{n})$ satisfy the system of difference equations

$$I_j^+(\boldsymbol{n}) - I_k^+(\boldsymbol{n}) = (b_{n_k} - b_{n_j})I(\boldsymbol{n}) - c_{n_j}I_j^-(\boldsymbol{n}) + c_{n_k}I_k^-(\boldsymbol{n}). \tag{8}$$

The boundary condition leads easily to the uniqueness.

Linearization coefficients of Charlier polynomials.

Set $C(\boldsymbol{n},a)=\sum_{\pi\in\mathcal{P}(\boldsymbol{n})} a^{\mathsf{bl}(\pi)}$.

Lemma 2 *For* $1\le k,j\le m$, $k\neq j$, *the polynomials* $C(\boldsymbol{n},a)$ *satisfy*

$$C_j^+(\boldsymbol{n},a)-C_k^+(\boldsymbol{n},a)=(n_k-n_j)C(\boldsymbol{n},a)+an_kC_k^-(\boldsymbol{n},a)-an_jC_j^-(\boldsymbol{n},a). \tag{9}$$

Proof. Let $N_j=n_1+\cdots+n_j$. The partitions of $\mathcal{P}_j^+(\boldsymbol{n})$ can be divided into three categories :

– N_j+1 and one element of S_k form a block of two elements, the corresponding generating function is $an_kC_k^-(\boldsymbol{n},a)$;
– N_j+1 and one element of S_k belong to a block containing at least one another element, the corresponding generating function is $\sum_{\pi\in\mathcal{P}(\boldsymbol{n})}(n_k-n_{k,j}(\pi))a^{\mathsf{bl}(\pi)}$, where $n_{k,j}(\pi)$ is the number of blocks in π containing both elements of S_j and S_k (Clearly $n_{k,j}(\pi)=n_{j,k}(\pi)$) ;
– N_j+1 is in a block without any element of S_k, let $R_{k,j}(\boldsymbol{n},a)$ be the corresponding generating function.

Thus we have

$$C_j^+(\boldsymbol{n},a)=\sum_{\pi\in\mathcal{P}(\boldsymbol{n})}(n_k-n_{k,j}(\pi))a^{\mathsf{bl}(\pi)}+an_kC_k^-(\boldsymbol{n},a)+R_{k,j}(\boldsymbol{n},a).$$

Clearly, we have $R_{k,j}(\boldsymbol{n},a)=R_{j,k}(\boldsymbol{n},a)$.

Rogers q-Hermite polynomials.

$$K(\boldsymbol{n}|q) = \sum_{M \in \mathcal{K}(\boldsymbol{n})} q^{\mathsf{cros(M)}} \tag{10}$$

Lemma 3 *For* $1 \le j \le m-1$*, the polynomials* $K(\boldsymbol{n};q)$ *satisfy*

$$K_j^+(\boldsymbol{n}|q) - K_{j+1}^+(\boldsymbol{n}|q) = [n_{j+1}]_q K_{j+1}^-(\boldsymbol{n}|q) - [n_j]_q K_j^-(\boldsymbol{n}|q). \tag{11}$$

Proof. Let $u = n_1 + \cdots + n_j + 1$.

(left) the blocks S_j and S_{j+1} in $\mathcal{K}_j^+(\boldsymbol{n})$, (right) the blocks S_j and S_{j+1} in $\mathcal{K}_{j+1}^+(\boldsymbol{n})$

More formally, Let $u = n_1 + \cdots + n_j + 1$. The matchings in $\mathcal{K}_j^+(\boldsymbol{n})$ (resp. $\mathcal{K}_{j+1}^+(\boldsymbol{n})$) can be divided into two categories :

- the integer $u \in S_j$ (resp, $u \in S_{j+1}$) is matched with the ℓth element $u + \ell$ in S_{j+1} (resp. $u - \ell$ in S_j), from left (resp., right), with $\ell \in [n_{j+1}]$ (resp. $\ell \in [n_j]$), then the corresponding edge will cross each of the $\ell - 1$ edges of which one vertex is $u + t$ (resp. $u - t$) with $1 \leq t \leq \ell - 1$. An illustration is given below. Hence the generating function of such matchings is

$$(1 + q + \cdots + q^{n_{j+1}-1})K_{j+1}^-(\boldsymbol{n}|q) \quad \text{(resp.} \quad (1 + q + \cdots + q^{n_j-1})K_j^-(\ n|q));$$

- the integer u is matched with an element not in $S_j \cup S_{j+1}$, let $R_u(\boldsymbol{n}|q)$ be the generating polynomial of such matchings.

It follows that $K_j^+(\boldsymbol{n}|q) = [n_{j+1}]_q K_{j+1}^-(\boldsymbol{n}|q) + R_u(\boldsymbol{n}|q)$ and $K_{j+1}^+(\boldsymbol{n}|q) = [n_j]_q K_j^-(\boldsymbol{n}|q) + R_u(\boldsymbol{n}|q)$.

We can obtain the linearization coefficients of the Al-Salam Chihara q-Charlier and q-Laguerre polynomials with similar methods but the proofs are more technical. See

– Ismail, K. and Zeng, Separation of variables and combinatorics of linearization coefficients of orthogonal polynomials, 2011.

– K., Stanton and Jiang Zeng, The combinatorics of Al–Salam–Chihara q-Laguerre polynomials, 2009.