Cluster algebras and Lie theory, I

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Séminaire Lotharingien de Combinatoire 69
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Coxeter frieze patterns (1971)

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local rule:

Coxeter frieze patterns (1971)

local rule:

$$\begin{array}{cc} & y \\ x & \\ & z \end{array}$$

Coxeter frieze patterns (1971)

local rule:

$$\begin{array}{ccc} & y & \\ x & & \dfrac{1+yz}{x} \\ & z & \end{array}$$

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1															
	1														
1															
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1	
1		2														
	1															
1																
	1		1		1		1		1		1		1		1	

Coxeter frieze patterns (1971)

1 1 1 1 1 1 1 1

1 2

1

1 2

1 1 1 1 1 1 1 1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2													
	1		5												
1		2													
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3											
	1		5												
1		2													
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3											
	1		5												
1		2		3											
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3											
	1		5		2										
1		2		3											
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		1									
	1		5		2										
1		2		3											
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		1									
	1		5		2										
1		2		3		1									
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		1									
	1		5		2		1								
1		2		3		1									
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		1		2		3		1		2	
	1		5		2		1		5		2		1		5
1		2		3		1		2		3		1		2	
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

1 1 1 1 1 1 1 1

1 2 3 1 2 3 1 2

1 5 2 1 5 2 1 5

1 2 3 1 2 3 1 2

1 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1

1

1

1

1 1 1 1 1 1 1 1

Coxeter frieze patterns (1971)

1 1 1 1 1 1 1 1
1 2 3 1 2 3 1 2
1 5 2 1 5 2 1 5
1 2 3 1 2 3 1 2
1 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1
1 2
1
1
1 1 1 1 1 1 1 1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		1		2		3		1		2	
	1		5		2		1		5		2		1		5
1		2		3		1		2		3		1		2	
	1		1		1		1		1		1		1		1

	1		1		1		1		1		1		1		1		1
1		2															
	1		3														
		1															
			1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		1		2		3		1		2	
	1		5		2		1		5		2		1		5
1		2		3		1		2		3		1		2	
	1		1		1		1		1		1		1		1

	1		1		1		1		1		1		1		1		1
1		2															
	1		3														
		1		4													
			1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

```
      1   1   1   1   1   1   1   1
    1   2   3   1   2   3   1   2
      1   5   2   1   5   2   1   5
    1   2   3   1   2   3   1   2
      1   1   1   1   1   1   1   1
```

```
   1   1   1   1   1   1   1   1   1
 1   2   2
   1   3
     1   4
       1   1   1   1   1   1   1   1
```

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		1		2		3		1		2	
	1		5		2		1		5		2		1		5
1		2		3		1		2		3		1		2	
	1		1		1		1		1		1		1		1

	1		1		1		1		1		1		1		1		1
1		2		2													
	1		3		3												
		1		4													
			1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

```
      1   1   1   1   1   1   1   1
    1   2   3   1   2   3   1   2
      1   5   2   1   5   2   1   5
    1   2   3   1   2   3   1   2
      1   1   1   1   1   1   1   1
```

```
  1   1   1   1   1   1   1   1   1
1   2   2   2
  1   3   3
    1   4
      1   1   1   1   1   1   1   1
```

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		1		2		3		1		2	
	1		5		2		1		5		2		1		5
1		2		3		1		2		3		1		2	
	1		1		1		1		1		1		1		1

	1		1		1		1		1		1		1		1		1
1		2		2		2											
	1		3		3												
		1		4		1											
			1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		1		2		3		1		2	
	1		5		2		1		5		2		1		5
1		2		3		1		2		3		1		2	
	1		1		1		1		1		1		1		1

	1		1		1		1		1		1		1		1		1
1		2		2		2											
	1		3		3		1										
		1		4		1											
			1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

1 1 1 1 1 1 1 1
1 2 3 1 2 3 1 2
1 5 2 1 5 2 1 5
1 2 3 1 2 3 1 2
1 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1
1 2 2 2 1
1 3 3 1
1 4 1
1 1 1 1 1 1 1 1

Coxeter frieze patterns (1971)

```
    1   1   1   1   1   1   1   1
  1   2   3   1   2   3   1   2
    1   5   2   1   5   2   1   5
  1   2   3   1   2   3   1   2
    1   1   1   1   1   1   1   1
```

```
  1   1   1   1   1   1   1   1   1
1   2   2   2   1   4   1
  1   3   3   1   3   3   1
    1   4   1   2   2   2   1
      1   1   1   1   1   1   1   1
```

Coxeter frieze patterns (1971)

1 1 1 1 1 1 1 1

1 2 3 1 2 3 1 2

1 5 2 1 5 2 1 5

1 2 3 1 2 3 1 2

1 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1

1 2 2 2 1 4 1 2 2 2

1 3 3 1 3 3 1 3 3

1 4 1 2 2 2 1 4

1 1 1 1 1 1 1 1

Coxeter frieze patterns (1971)

1 1 1 1 1 1 1 1

1

1

1

1

1

1 1 1 1 1 1 1 1

Coxeter frieze patterns (1971)

	1		1	1	1	1	1	1	1
1		2							
	1								
1									
	1								
1									
	1		1	1	1	1	1	1	1

Coxeter frieze patterns (1971)

	1		1	1	1	1	1	1	1
1		2							
	1								
1		2							
	1								
1		2							
	1		1	1	1	1	1	1	1

Coxeter frieze patterns (1971)

	1		1	1	1	1	1	1	1
1		2							
	1		5						
1		2							
	1		5						
1		2							
	1		1	1	1	1	1	1	1

Coxeter frieze patterns (1971)

	1		1		1	1	1	1	1	1
1		2		3						
	1		5							
1		2								
	1		5							
1		2		3						
	1		1		1	1	1	1	1	1

Coxeter frieze patterns (1971)

1 1 1 1 1 1 1 1

1 2 3

1 5

1 2 13

1 5

1 2 3

1 1 1 1 1 1 1 1

Coxeter frieze patterns (1971)

	1		1		1	1	1	1	1	1
1		2		3						
	1		5		8					
1		2		13						
	1		5		8					
1		2		3						
	1		1		1	1	1	1	1	1

Coxeter frieze patterns (1971)

	1		1		1		1	1	1	1	1
1		2		3		3					
	1		5		8						
1		2		13							
	1		5		8						
1		2		3		3					
	1		1		1		1	1	1	1	1

Coxeter frieze patterns (1971)

	1		1		1		1	1	1	1	1
1		2		3		3					
	1		5		8						
1		2		13		5					
	1		5		8						
1		2		3		3					
	1		1		1		1	1	1	1	1

Coxeter frieze patterns (1971)

	1		1		1		1	1	1	1	1
1		2		3		3					
	1		5		8		2				
1		2		13		5					
	1		5		8		2				
1		2		3		3					
	1		1		1		1	1	1	1	1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		3		1							
	1		5		8		2								
1		2		13		5		1							
	1		5		8		2								
1		2		3		3		1							
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

	1		1		1		1		1	1	1	1
1		2		3		3		1				
	1		5		8		2		1			
1		2		13		5		1				
	1		5		8		2		1			
1		2		3		3		1				
	1		1		1		1		1	1	1	1

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		3		1		2		3		3	
	1		5		8		2		1		5		8		2
1		2		13		5		1		2		13		5	
	1		5		8		2		1		5		8		2
1		2		3		3		1		2		3		3	
	1		1		1		1		1		1		1		1

Coxeter frieze patterns (1971)

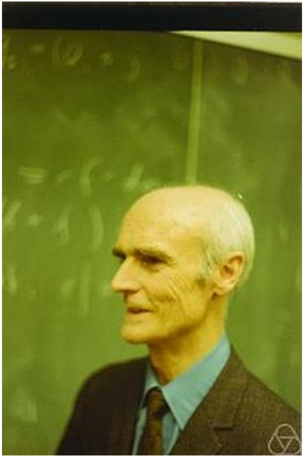
	1		1		1		1		1		1		1		1
1		2		3		3		1		2		3		3	
	1		5		8		2		1		5		8		2
1		2		13		5		1		2		13		5	
	1		5		8		2		1		5		8		2
1		2		3		3		1		2		3		3	
	1		1		1		1		1		1		1		1

- We get integer numbers !

Coxeter frieze patterns (1971)

	1		1		1		1		1		1		1		1
1		2		3		3		1		2		3		3	
	1		5		8		2		1		5		8		2
1		2		13		5		1		2		13		5	
	1		5		8		2		1		5		8		2
1		2		3		3		1		2		3		3	
	1		1		1		1		1		1		1		1

- We get integer numbers !
- It is periodic !



H. M. Coxeter 1907 – 2003

Somos sequence (1989)

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- Initial data : $x_1 = x_2 = x_3 = x_4 = 1$.

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- Recurrence :

$$x_k = \frac{x_{k-3}x_{k-1} + x_{k-2}^2}{x_{k-4}} \qquad (k \geq 5).$$

Somos sequence (1989)

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- Recurrence :

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k	5	6	7	8	9	10	11	12	13	$\cdots$
x_k	2	3	7	23	59	314	1529	8209	83313	$\cdots$

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Somos sequence (1989)

- Initial data : x_1, x_2, x_3, x_4.
- Recurrence :

$$x_k = \frac{x_{k-3}x_{k-1} + x_{k-2}^2}{x_{k-4}} \qquad (k \geq 5).$$

- $x_5 = \dfrac{x_2 x_4 + x_3^2}{x_1}$

Somos sequence (1989)

- Initial data : x_1, x_2, x_3, x_4.
- Recurrence :

$$x_k = \frac{x_{k-3}x_{k-1} + x_{k-2}^2}{x_{k-4}} \qquad (k \geq 5).$$

- $x_5 = \dfrac{x_2 x_4 + x_3^2}{x_1}$
- $x_6 = \dfrac{x_2 x_3 x_4 + x_3^3 + x_1 x_4^2}{x_1 x_2}$

Somos sequence (1989)

- Initial data : x_1, x_2, x_3, x_4.
- Recurrence :

$$x_k = \frac{x_{k-3}x_{k-1} + x_{k-2}^2}{x_{k-4}} \qquad (k \geq 5).$$

- $x_5 = \dfrac{x_2 x_4 + x_3^2}{x_1}$
- $x_6 = \dfrac{x_2 x_3 x_4 + x_3^3 + x_1 x_4^2}{x_1 x_2}$
- $x_7 = \dfrac{2x_2^2 x_3^2 x_4 + x_1 x_3^3 x_4 + x_1^2 x_4^3 + x_2^3 x_4^2 + x_2 x_3^4 + x_1 x_2 x_3 x_4^2}{x_1^2 x_2 x_3}$

Somos sequence (1989)

- $x_8 = (x_1^3 x_3 x_4^4 + 2x_1^2 x_3^4 x_4^2 + 3x_1^2 x_2 x_3^2 x_4^3 + x_1 x_3^7 + 3x_1 x_2 x_3^5 x_4 + 3x_1 x_2^2 x_3^3 x_4^2 + x_2^2 x_3^6 + 3x_2^3 x_3^4 x_4 + 3x_2^4 x_3^2 x_4^2 + x_2^5 x_4^3 + x_1^2 x_2^2 x_4^4 + x_1 x_2^3 x_3 x_4^3)/x_1^3 x_2^2 x_3 x_4$

Somos sequence (1989)

- $x_8 = (x_1^3 x_3 x_4^4 + 2x_1^2 x_3^4 x_4^2 + 3x_1^2 x_2 x_3^2 x_4^3 + x_1 x_3^7 + 3x_1 x_2 x_3^5 x_4 + 3x_1 x_2^2 x_3^3 x_4^2 + x_2^2 x_3^6 + 3x_2^3 x_3^4 x_4 + 3x_2^4 x_3^2 x_4^2 + x_2^5 x_4^3 + x_1^2 x_2^2 x_4^4 + x_1 x_2^3 x_3 x_4^3)/x_1^3 x_2^2 x_3 x_4$

- $x_9 = (x_1^4 x_4^6 + 2x_1^2 x_2^3 x_4^5 + 3x_1^3 x_2 x_3 x_4^5 + x_2^6 x_4^4 + 3x_1 x_2^4 x_3 x_4^4 + 5x_1^2 x_2^2 x_3^2 x_4^4 + 3x_1^3 x_3^3 x_4^4 + 4x_2^5 x_3^2 x_4^3 + 7x_1 x_2^3 x_3^3 x_4^3 + 6x_1^2 x_2 x_3^4 x_4^3 + 6x_2^4 x_3^4 x_4^2 + 6x_1 x_2^2 x_3^5 x_4^2 + 3x_1^2 x_3^6 x_4^2 + 4x_2^3 x_3^6 x_4 + 3x_1 x_2 x_3^7 x_4 + x_2^2 x_3^8 + x_1 x_3^9)/x_1^3 x_2^3 x_3^2 x_4$

Somos sequence (1989)

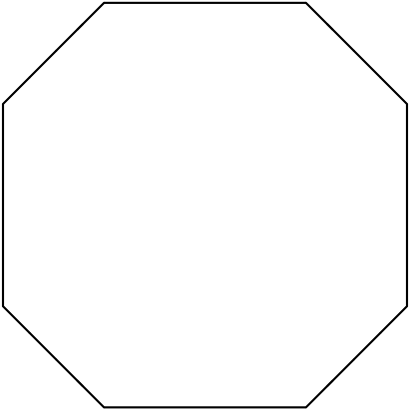
- $x_8 = (x_1^3 x_3 x_4^4 + 2x_1^2 x_3^4 x_4^2 + 3x_1^2 x_2 x_3^2 x_4^3 + x_1 x_3^7 + 3x_1 x_2 x_3^5 x_4 + 3x_1 x_2^2 x_3^3 x_4^2 + x_2^2 x_3^6 + 3x_2^3 x_3^4 x_4 + 3x_2^4 x_3^2 x_4^2 + x_2^5 x_4^3 + x_1^2 x_2^2 x_4^4 + x_1 x_2^3 x_3 x_4^3) / x_1^3 x_2^2 x_3 x_4$

- $x_9 = (x_1^4 x_4^6 + 2x_1^2 x_2^3 x_4^5 + 3x_1^3 x_2 x_3 x_4^5 + x_2^6 x_4^4 + 3x_1 x_2^4 x_3 x_4^4 + 5x_1^2 x_2^2 x_3^2 x_4^4 + 3x_1^3 x_3^3 x_4^4 + 4x_2^5 x_3^2 x_4^3 + 7x_1 x_2^3 x_3^3 x_4^3 + 6x_1^2 x_2 x_3^4 x_4^3 + 6x_2^4 x_3^4 x_4^2 + 6x_1 x_2^2 x_3^5 x_4^2 + 3x_1^2 x_3^6 x_4^2 + 4x_2^3 x_3^6 x_4 + 3x_1 x_2 x_3^7 x_4 + x_2^2 x_3^8 + x_1 x_3^9) / x_1^3 x_2^3 x_3^2 x_4$

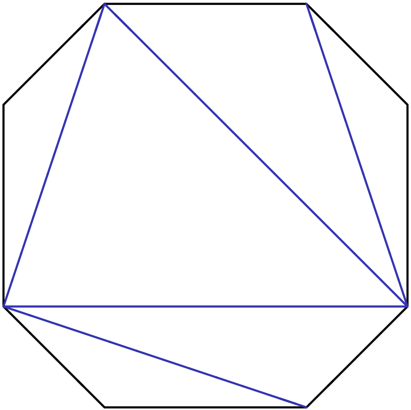
- We get Laurent polynomials with integer coefficients !

Triangulations

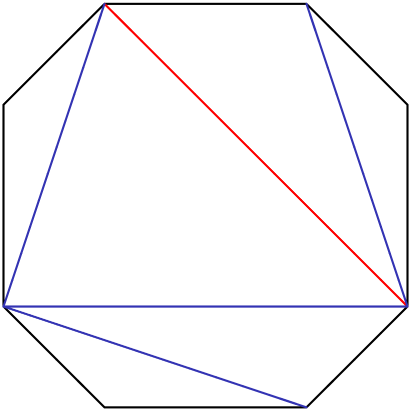
Triangulations



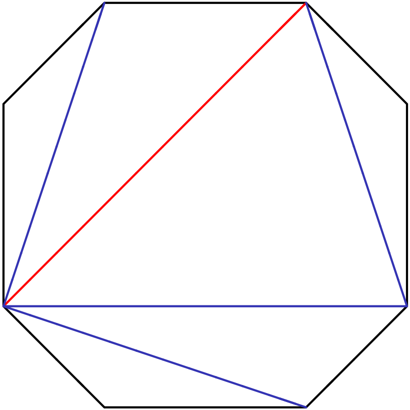
Triangulations



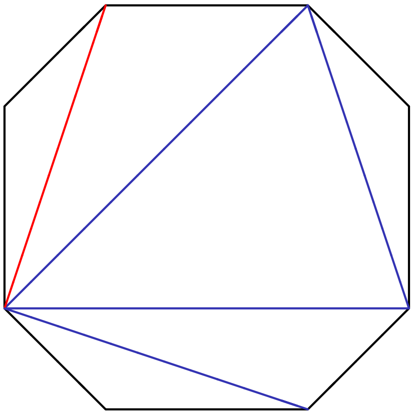
Triangulations



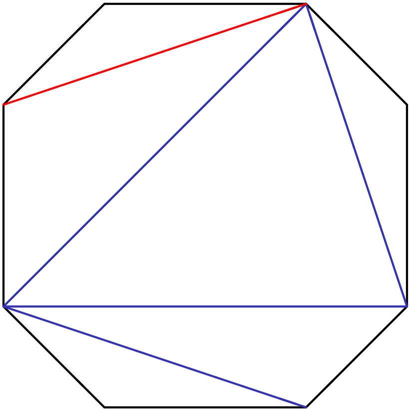
Triangulations



Triangulations



Triangulations



Triangulations

Triangulations

Let T be a triangulation, D a diagonal of T.

Triangulations

Let T be a triangulation, D a diagonal of T.

Definition

We call **mutation** of T with respect to D the triangulation $\mu_D(T)$ obtained by flipping D.

Triangulations

Let T be a triangulation, D a diagonal of T.

Definition

We call **mutation** of T with respect to D the triangulation $\mu_D(T)$ obtained by flipping D.

We can generate all triangulations by means of mutations:

Triangulations

Let T be a triangulation, D a diagonal of T.

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We call **mutation** of T with respect to D the triangulation $\mu_D(T)$ obtained by flipping D.

We can generate all triangulations by means of mutations:

- We fix an initial triangulation T_{init};

Triangulations

Let T be a triangulation, D a diagonal of T.

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We can generate all triangulations by means of mutations:

- We fix an initial triangulation T_{init};
- We mutate T_{init} with respect to each of its diagonals D;

Triangulations

Let T be a triangulation, D a diagonal of T.

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- We fix an initial triangulation T_{init};
- We mutate T_{init} with respect to each of its diagonals D;
- We mutate all the new triangulations;

Triangulations

Let T be a triangulation, D a diagonal of T.

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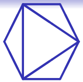
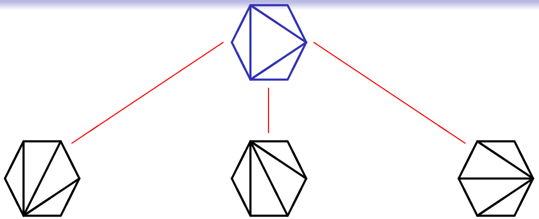
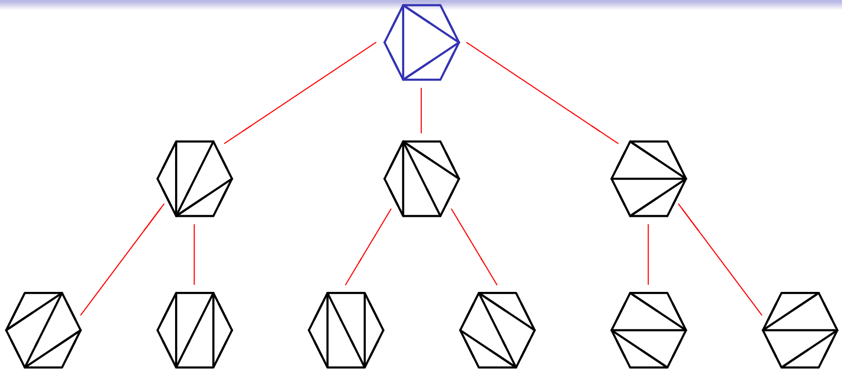
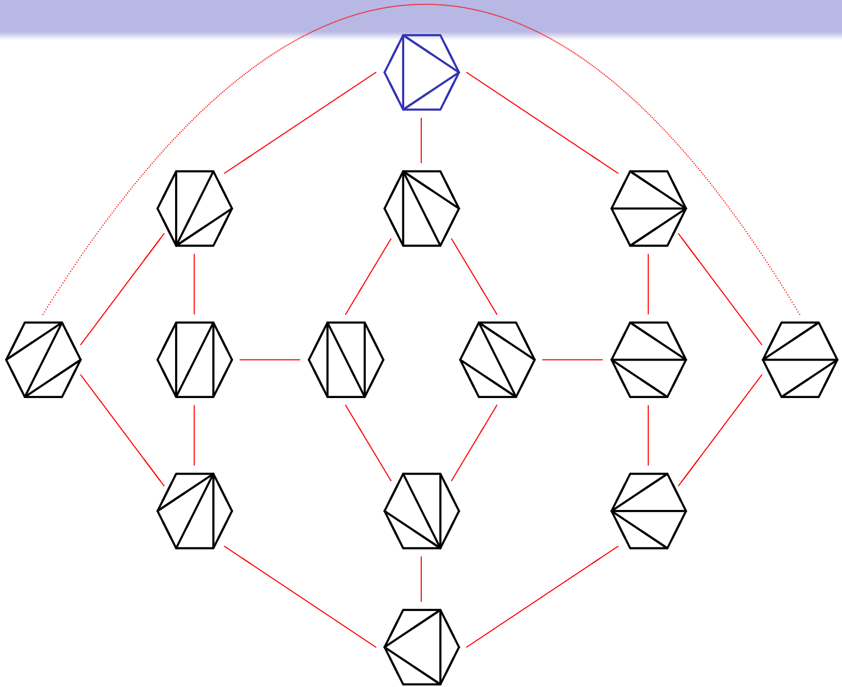
- *etc...*









Ptolemy’s theorem

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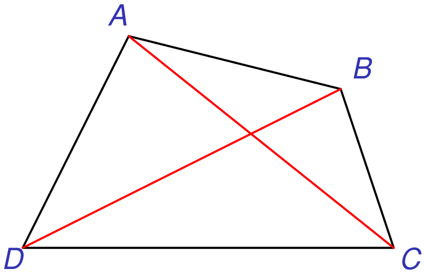

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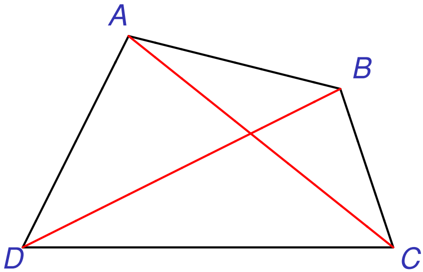
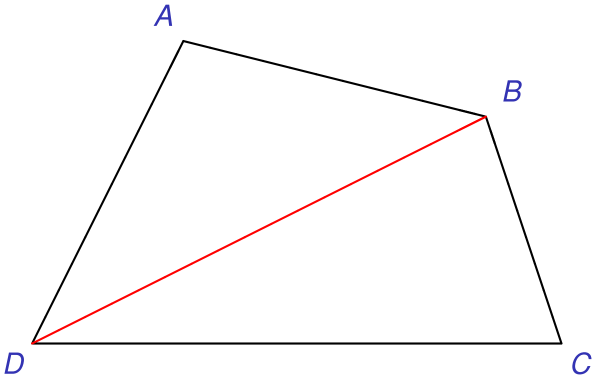


If A, B, C, D lie on a circle:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

Length mutation

Length mutation



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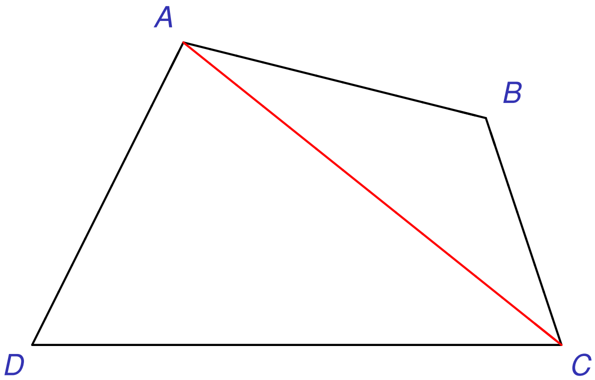

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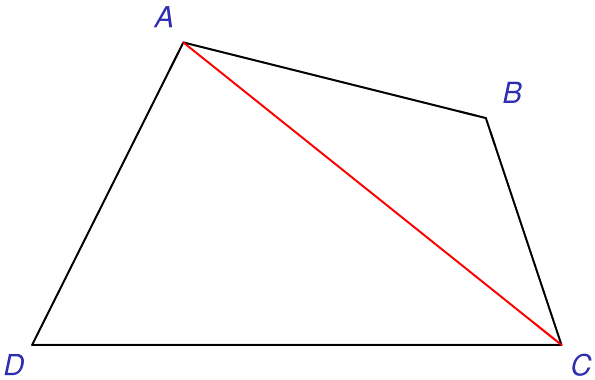
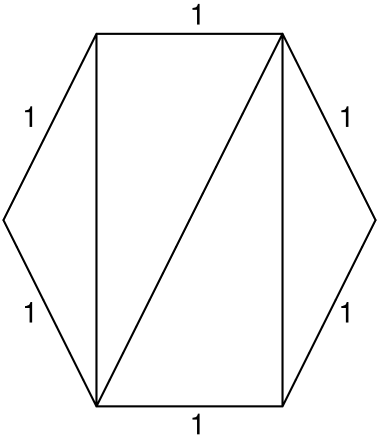


$$AC = \frac{AB \cdot CD + AD \cdot BC}{BD}$$

Length mutation



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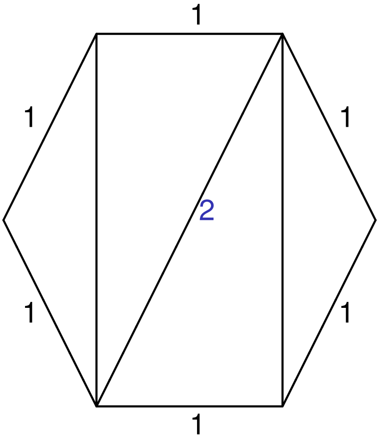

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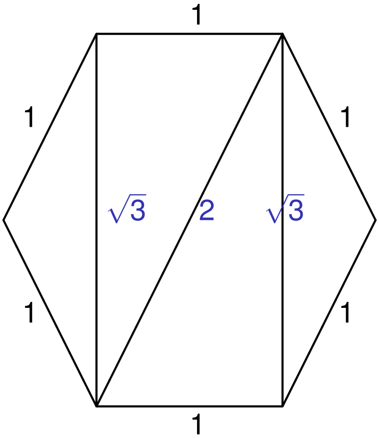

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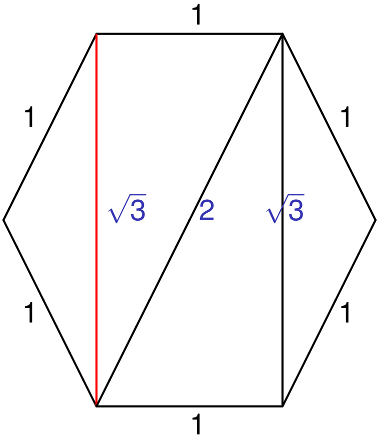

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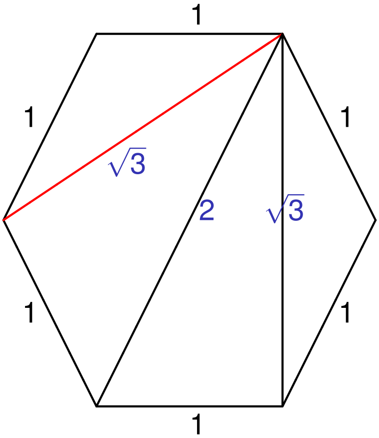
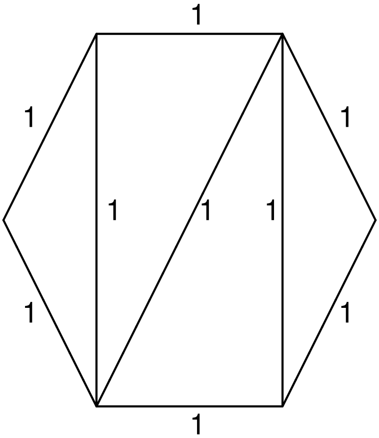


“Length” mutation



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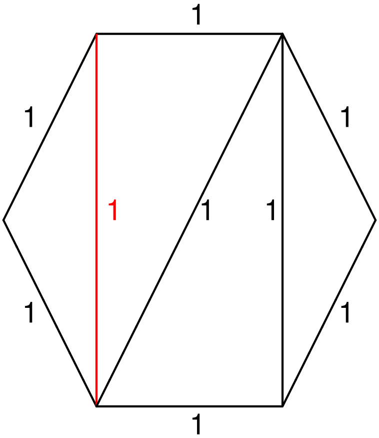

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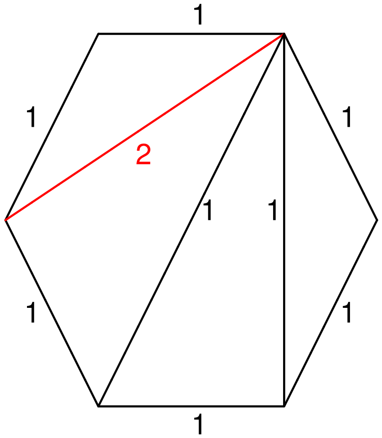

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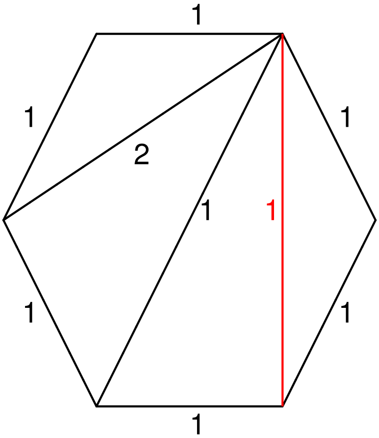

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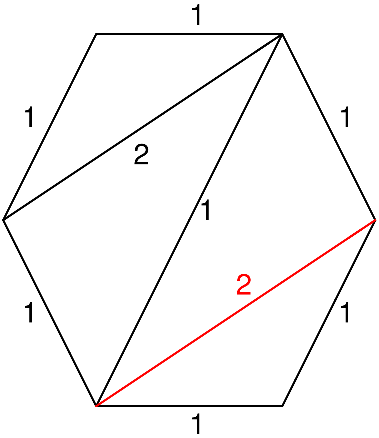

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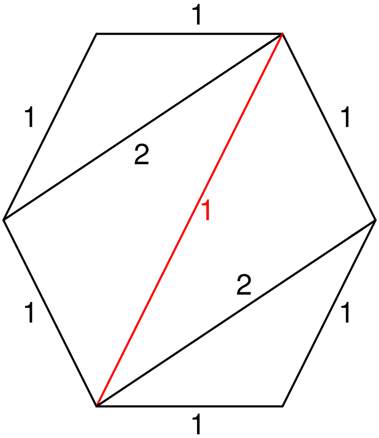

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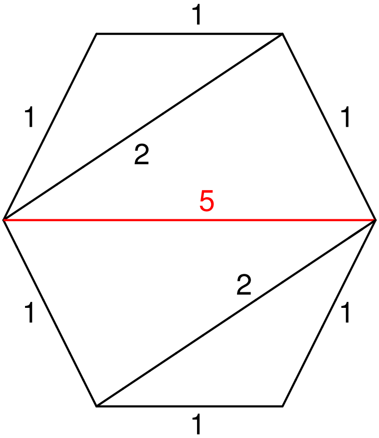

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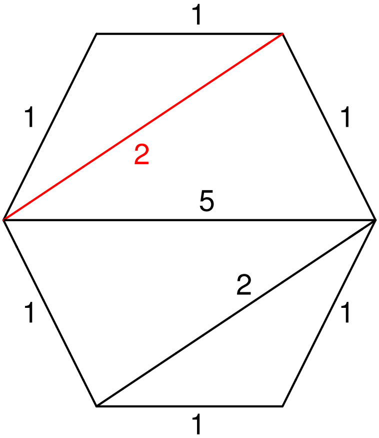

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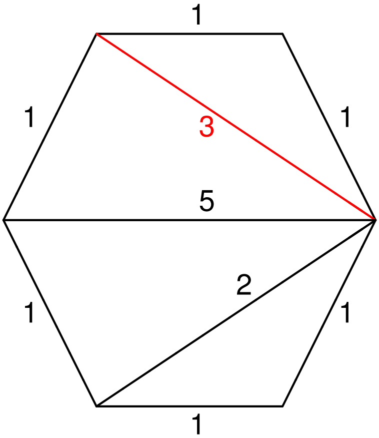
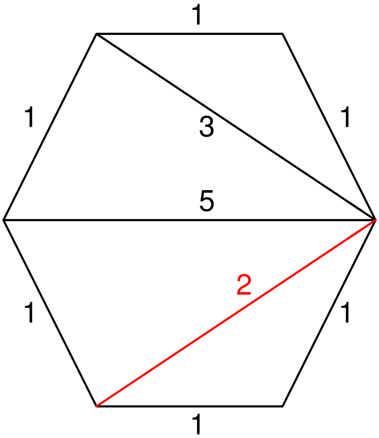


"Length" mutation



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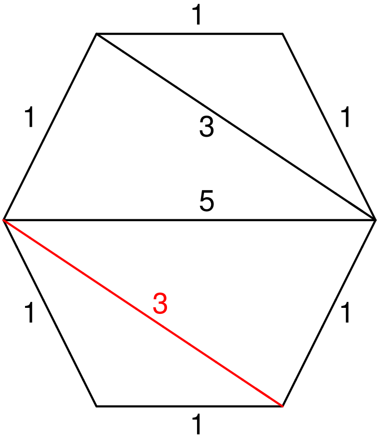

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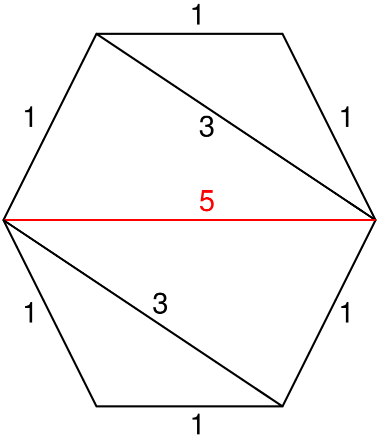

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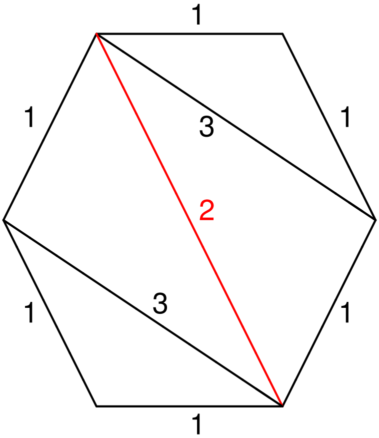

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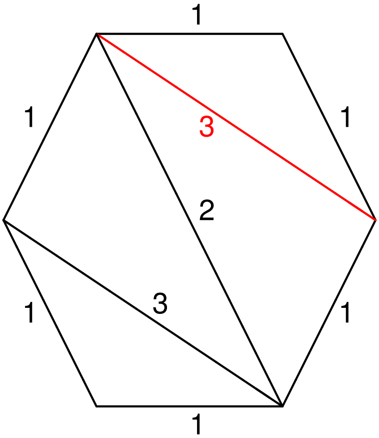

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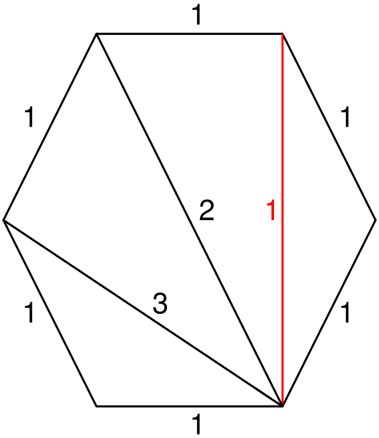

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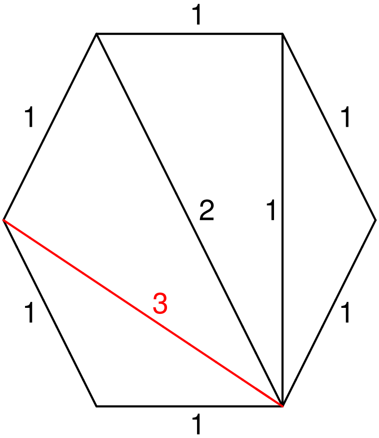

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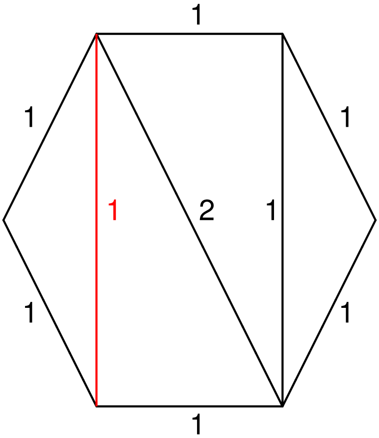

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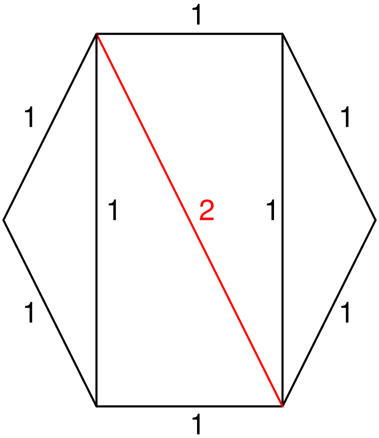

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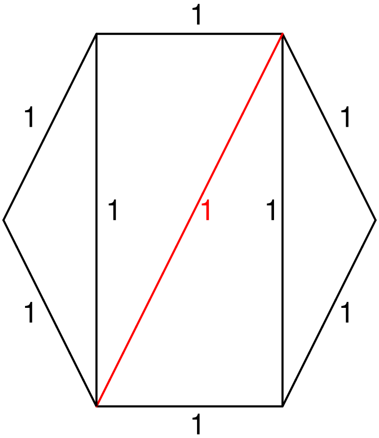


Back to Coxeter frieze patterns

1 1 1 1 1 1 1 1
1
1
1
1 1 1 1 1 1 1 1

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1	
1		2														
	1															
1																
	1		1		1		1		1		1		1		1	

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1
1		2													
	1														
1		2													
	1		1		1		1		1		1		1		1

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1
1		2													
	1		5												
1		2													
	1		1		1		1		1		1		1		1

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1	
1		2		3												
	1		5													
1		2														
	1		1		1		1		1		1		1		1	

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1	
1		2		3												
	1		5													
1		2		3												
	1		1		1		1		1		1		1		1	

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1
1		2		3											
	1		5		2										
1		2		3											
	1		1		1		1		1		1		1		1

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1
1		2		3		1									
	1		5		2										
1		2		3											
	1		1		1		1		1		1		1		1

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1
1		2		3		1									
	1		5		2										
1		2		3		1									
	1		1		1		1		1		1		1		1

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1
1		2		3		1									
	1		5		2		1								
1		2		3		1									
	1		1		1		1		1		1		1		1

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1
1		2		3		1		2		3		1		2	
	1		5		2		1		5		2		1		5
1		2		3		1		2		3		1		2	
	1		1		1		1		1		1		1		1

Back to Coxeter frieze patterns

	1		1		1		1		1		1		1		1
1		2		3		1		2		3		1		2	
	1		5		2		1		5		2		1		5
1		2		3		1		2		3		1		2	
	1		1		1		1		1		1		1		1

Exercise

- Obtain other friezes using mutations of triangulations.

Back to Coxeter frieze patterns

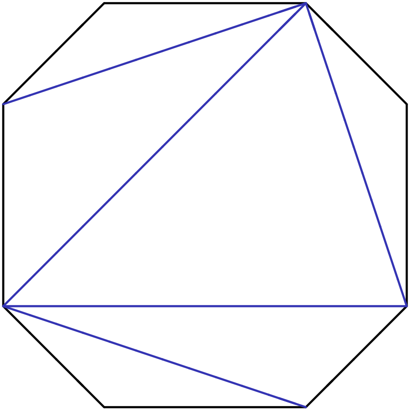
	1		1		1		1		1		1		1		1
1		2		3		1		2		3		1		2	
	1		5		2		1		5		2		1		5
1		2		3		1		2		3		1		2	
	1		1		1		1		1		1		1		1

Exercise

- Obtain other friezes using mutations of triangulations.
- Study the connections between friezes and triangulations.

Quiver of a triangulation

Quiver of a triangulation



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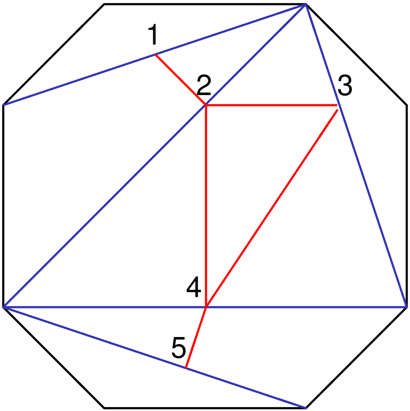

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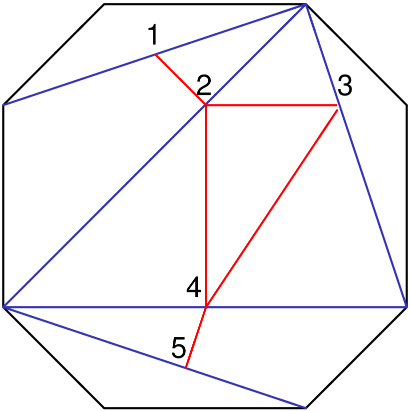
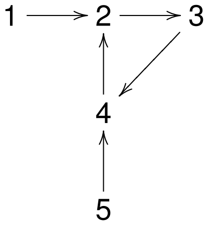
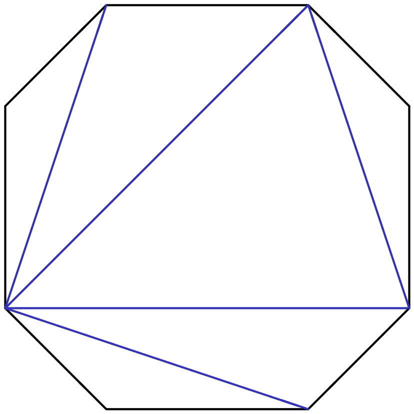




Quiver of a triangulation : mutation



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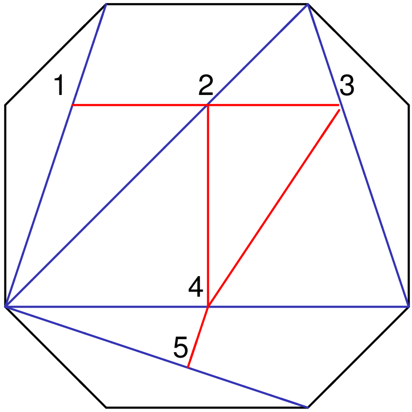

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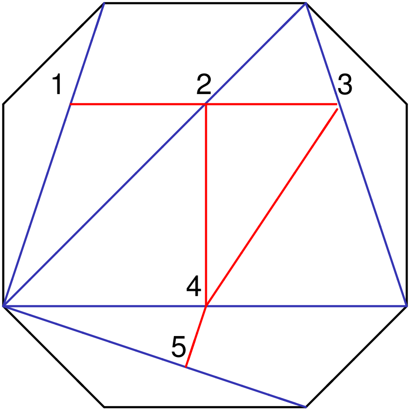

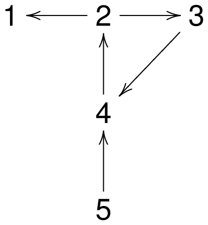

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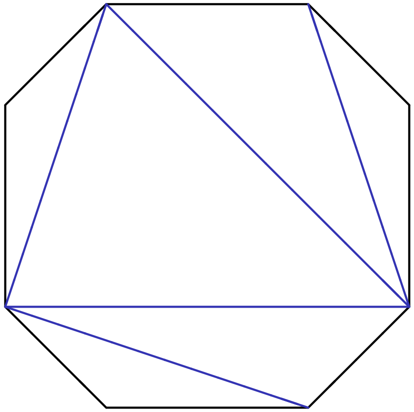

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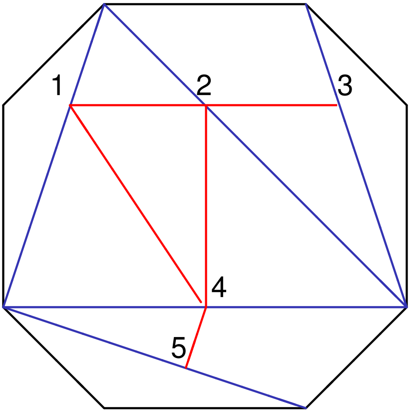

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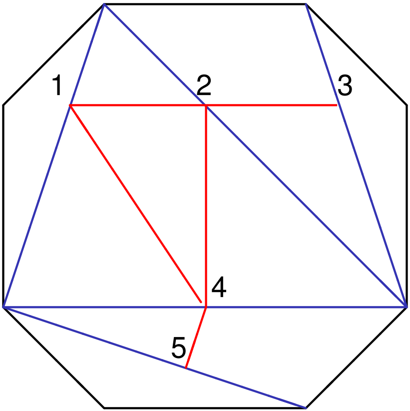

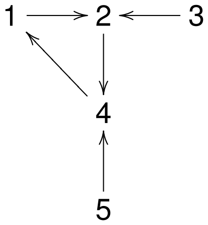

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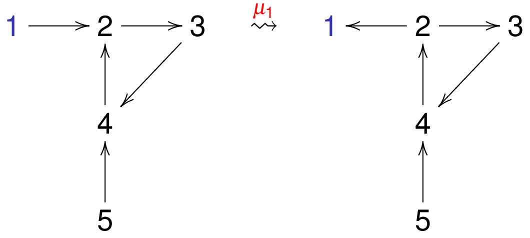

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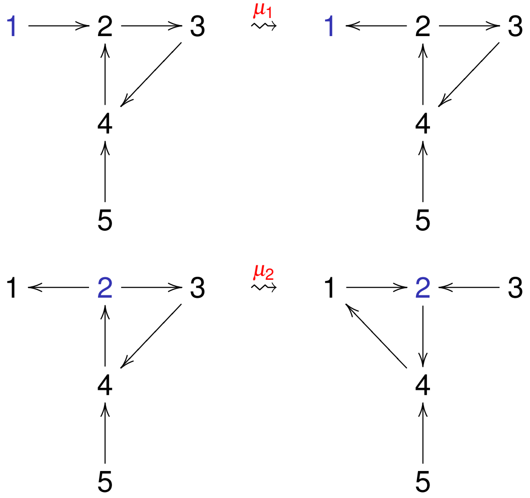

Mutation : general definition (2000)

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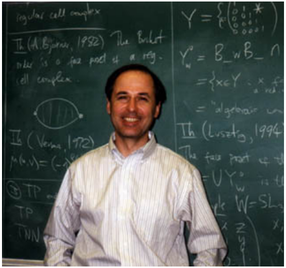
Andrei Zelevinsky



Sergey Fomin

Seed

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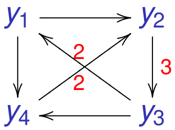

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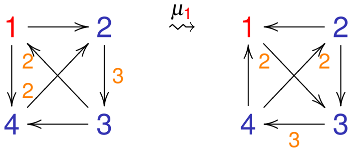

Mutation of variables

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Definition

For $1 \leq k \leq n$ and $1 \leq j \leq n$, define $\mu_k(y_j)$ by:

Mutation of variables

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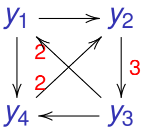

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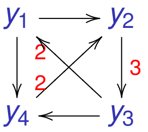

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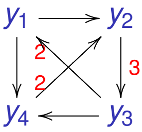

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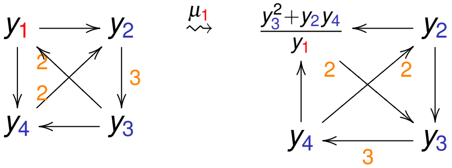

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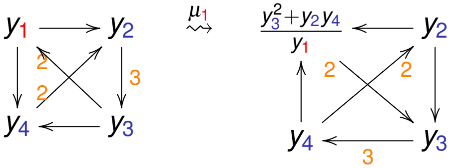

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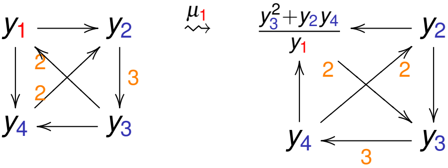

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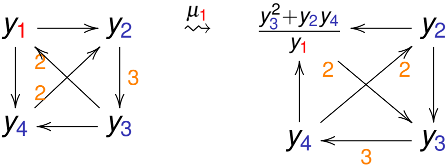

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⇝ We can iterate seed mutation.

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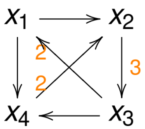

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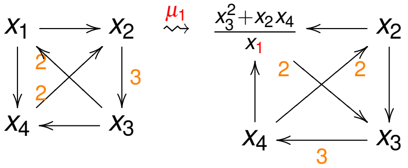

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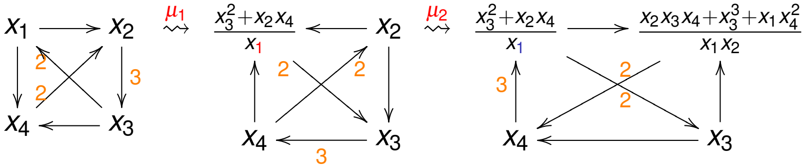

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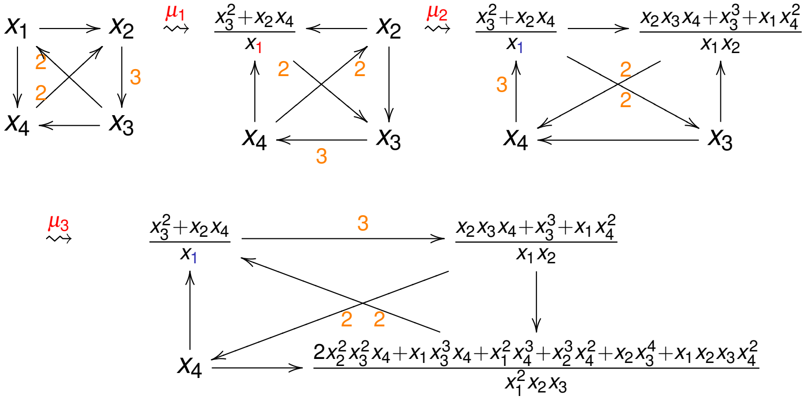

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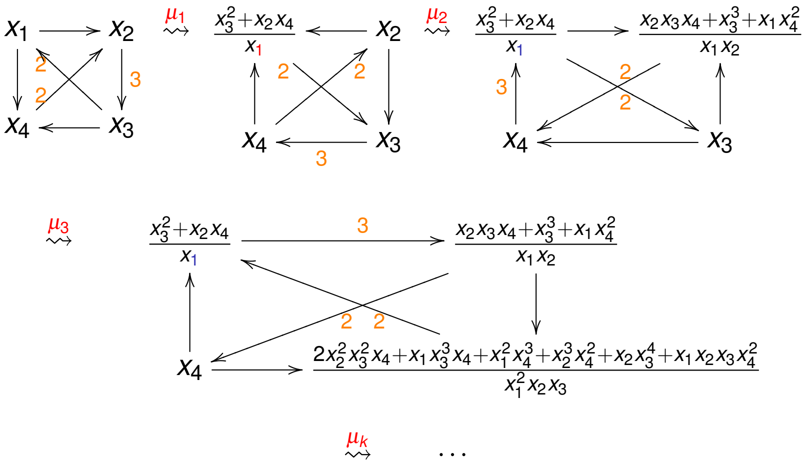

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Theorem (Fomin-Zelevinsky, “Laurent phenomenon”)

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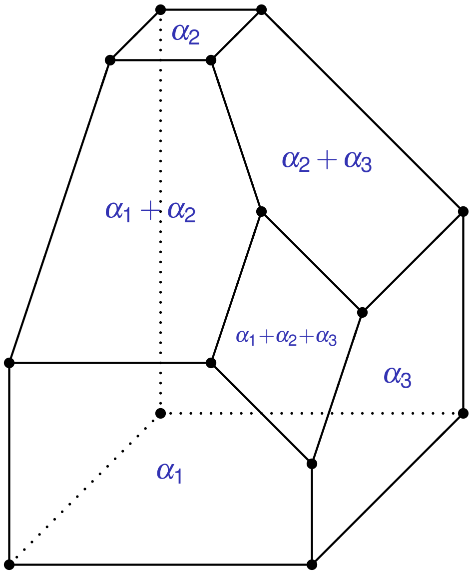

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Homework

Exercise

Show that the coordinate ring of the space of 2×3 matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

has a cluster algebra structure of type A_2 with

- 5 cluster variables : $a, b, e, f, g := af - cd$.
- 4 frozen variables : $A := ae - bd$, $B := bf - ec$, c, d.

References

- S. Fomin, A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529.
- S. Fomin, A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math. **154** (2003), 63–121.
- A. Zelevinsky, *What is a cluster algebra ?*, Notices of the AMS **54**, 11, (2007), 1494–1495.
- S. Fomin, *Cluster algebra portal*, `http://www.math.lsa.umich.edu/~fomin`

Cluster algebras and Lie theory, II

Bernard Leclerc,
Université de Caen

Séminaire Lotharingien de Combinatoire 69
Strobl, 11 septembre 2012

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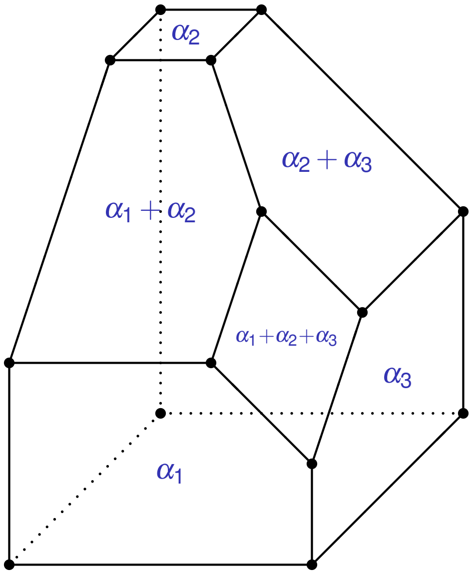

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Homework

Exercise

Show that the coordinate ring of the space of 2×3 matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

has a cluster algebra structure of type A_2 with

- 5 cluster variables : $a, b, e, f, g := af - cd$.
- 4 frozen variables : $A := ae - bd$, $B := bf - ec$, c, d.

Factorization problem

Factorization problem

The unipotent group

$$N = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix},\ a_{ij} \in \mathbb{C} \right\} \subset SL_4(\mathbb{C})$$

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is generated by the one-parameter subgroups

$$x_1(t_1) = \begin{pmatrix} 1 & t_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_2(t_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_3(t_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (t_i \in \mathbb{C}) \, .$$

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A generic element $x \in N$ has a unique factorization

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The map $(t_1, t_2, t_3, t_4, t_5, t_6) \mapsto x$ is a birational isomorphism

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Solved (for SL_n and any factorization pattern) by Berenstein, Fomin, Zelevinsky (1996).

Factorization problem

$$x_{(1,2,1,3,2,1)}(t_1,\ldots,t_6) = \begin{pmatrix} 1 & t_1+t_3+t_6 & t_1t_2+t_1t_5+t_3t_5 & t_1t_2t_4 \\ 0 & 1 & t_2+t_5 & t_2t_4 \\ 0 & 0 & 1 & t_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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$$D_{24} D_{134} = D_{14} D_{234} + D_{124} D_{34}$$

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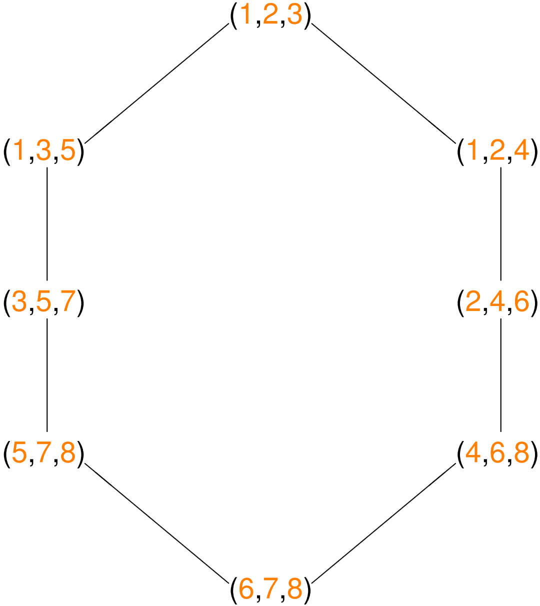
Put

$$1 := D_{124}, \quad 2 := D_{14}, \quad 3 := D_{134}, \quad 4 := D_{24}$$
$$5 := D_{13,34}, \quad 6 := D_{23}, \quad 7 := D_3, \quad 8 := D_2.$$



(1,2,3)
(1,3,5)
(1,2,4)
(3,5,7)
(2,4,6)
(5,7,8)
(4,6,8)
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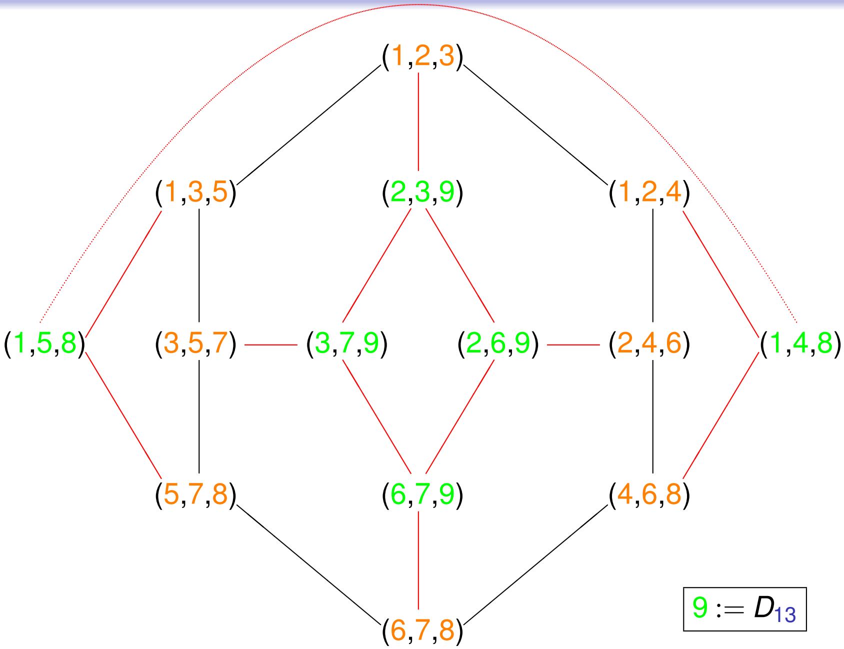

(1,2,3)
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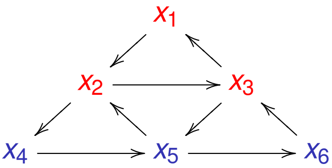
9 := D_{13}

Let $\mathscr{A}$ be the cluster algebra with initial seed :



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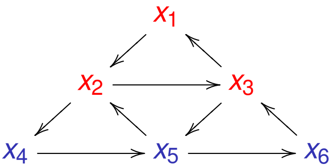

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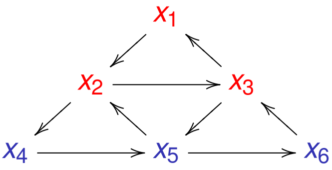

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Proposition

The assignment

$$x_1 \mapsto D_2,\ x_2 \mapsto D_3,\ x_3 \mapsto D_{23},\ x_4 \mapsto D_4,\ x_5 \mapsto D_{34},\ x_6 \mapsto D_{234}$$

extends to an isomorphism : $\mathscr{A} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}[N]$.

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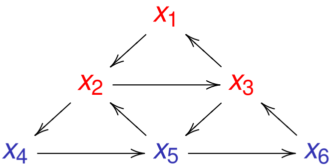

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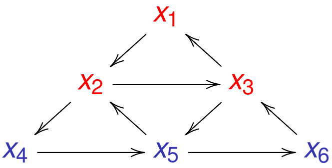

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- $\mathbb{C}[N]$ has finite cluster type A_3.
- The cluster monomials coincide with the elements of Lusztig's dual canonical basis of $\mathbb{C}[N]$ (Berenstein-Zelevinsky).

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- W the Weyl group, w_0 its longest element.

Theorem (Berenstein-Fomin-Zelevinsky)

$\mathbb{C}[N]$ is a cluster algebra, with explicit initial seeds labelled by reduced decompositions of w_0.

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$\rightsquigarrow$ preprojective algebra !!

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- Λ has finite representation type iff Q has type A_n $(n \leq 4)$!!

Type A_3

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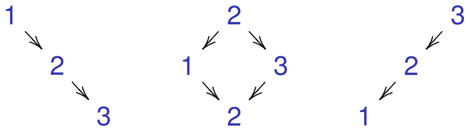
- Indecomposable projectives:



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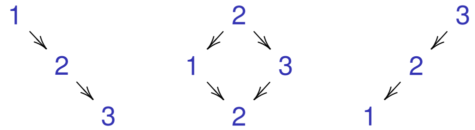

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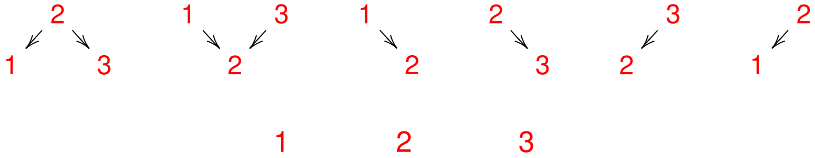

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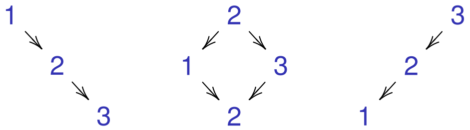

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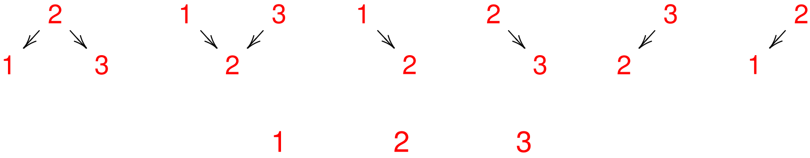

- Recall that $\mathbb{C}[N]$ has 3 frozen variables and 9 cluster variables !!

- Want a map : $\operatorname{mod} \Lambda \to \mathbb{C}[N]$ ...

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- Geiss-L-Schröer : Dualizing Lusztig's construction, get a nice map $M \mapsto \varphi_M$ from mod Λ to $\mathbb{C}[N]$.

The map $M \mapsto \varphi_M$

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- For $M \in \operatorname{mod} \Lambda$ and $\mathbf{i} = (i_1, \ldots, i_d)$ let $\mathscr{F}_{M,\mathbf{i}}$ be the variety of composition series of M of type $\mathbf{i}$:

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Theorem (Lusztig, Geiss-L-Schröer)

There exits a unique $\varphi_M \in \mathbb{C}[N]$ such that for all $\mathbf{j} = (j_1, \ldots, j_k)$

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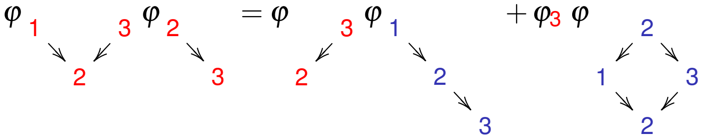

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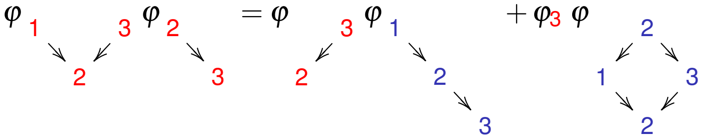

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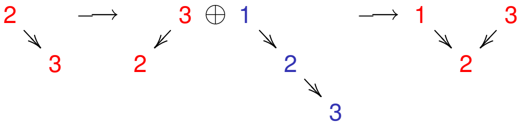

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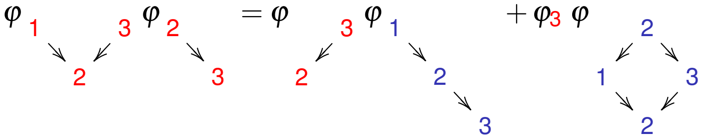

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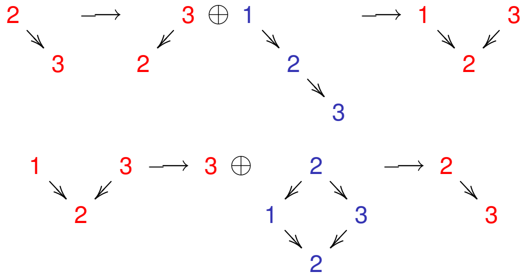

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Theorem (Geiss-L-Schröer)

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- if $\dim \operatorname{Ext}^1_\Lambda(M, L) = \dim \operatorname{Ext}^1_\Lambda(L, M) = 1$ then

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- if $\dim \operatorname{Ext}^1_\Lambda(M, L) > 1$, there is a more complicated formula involving all possible middle terms of non-split short exact sequences with end terms M and L.

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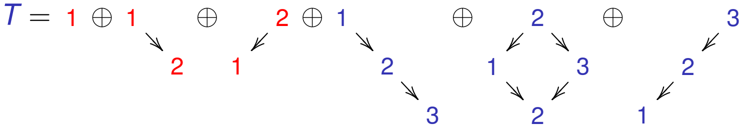

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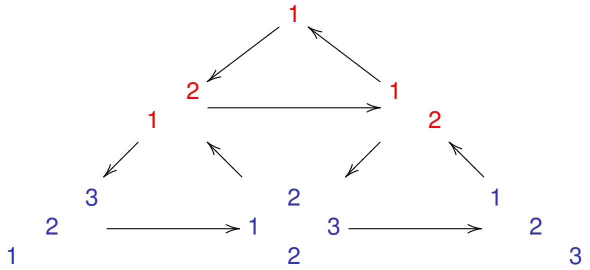

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  for $w \in W$. We get similar results where $\mathrm{mod}\,\Lambda$ is replaced by a certain full additive subcategory $\mathscr{C}_w$. The categories $\mathscr{C}_w$ were also introduced and studied independently by Buan, Iyama, Reiten, Scott.

Cluster algebras and Lie theory, III

Bernard Leclerc,
Université de Caen

Séminaire Lotharingien de Combinatoire 69
Strobl, 12 septembre 2012

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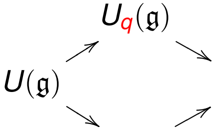

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Aim: Study the tensor category of finite-dimensional modules over $U_q(L\mathfrak{g})$.

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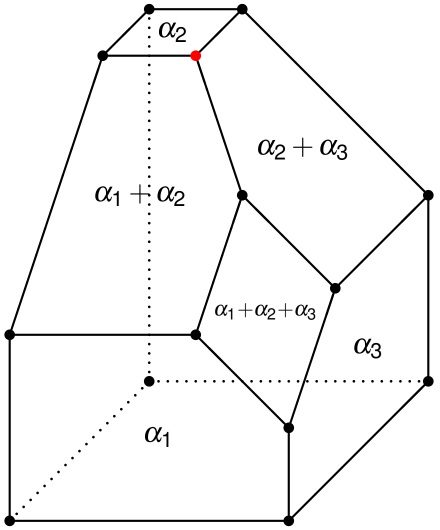
14 factorization patterns $\overset{1:1}{\longleftrightarrow}$ vertices of Stasheff associahedron


α_2
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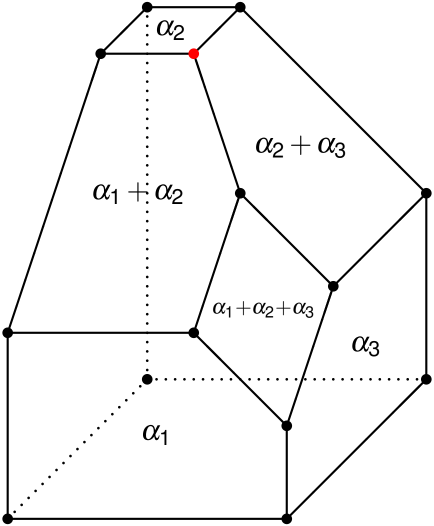

$$S(\alpha_2)^{\otimes a} \otimes S(\alpha_1+\alpha_2)^{\otimes b} \otimes S(\alpha_2+\alpha_3)^{\otimes c} \otimes F_1^{\otimes d} \otimes F_2^{\otimes e}$$

is simple for any $a, b, c, d, e \in \mathbb{N}$.

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Proposition

Simple objects S of $\mathscr{C}_1$ are characterized by their truncated q-character $\chi_q(S)_{\leq 2}$.

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Quiver grassmannians and F-polynomials

- M a representation of Q, $\gamma = \sum_i c_i \alpha_i$ a dimension vector,

$\rightsquigarrow$ $\mathrm{Gr}_\gamma(M)$:= Grassmannian of subrepresentations of M with dimension vector γ, a projective variety.

Definition

$F_M(v_1, \ldots, v_n) := \sum_\gamma \chi(\mathrm{Gr}_\gamma(M))\, \mathbf{v}^\gamma$ (χ Euler characteristic).

- β positive root $\rightsquigarrow$ indecomposable $M[\beta]$ with dimension β

$\rightsquigarrow$ $F_\beta := F_{M[\beta]}$

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Truncated q-characters and F-polynomials

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$\chi_q(S(\beta))_{\leq 2} = Y_{1,q^0} Y_{2,q^3} Y_{3,q^0}(1 + v_1 + v_3 + v_1 v_3 + v_1 v_2 v_3)$