Tilings of half a hexagon

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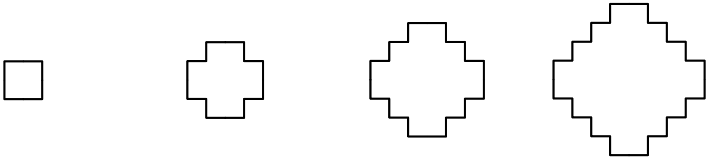
The Aztec Diamond

Aztec diamonds of orders 1, 2, 3 and 4.



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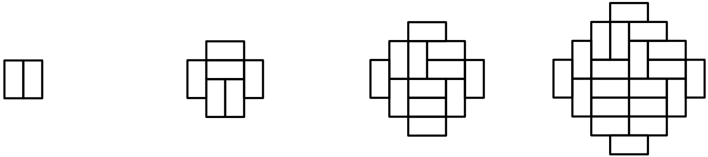
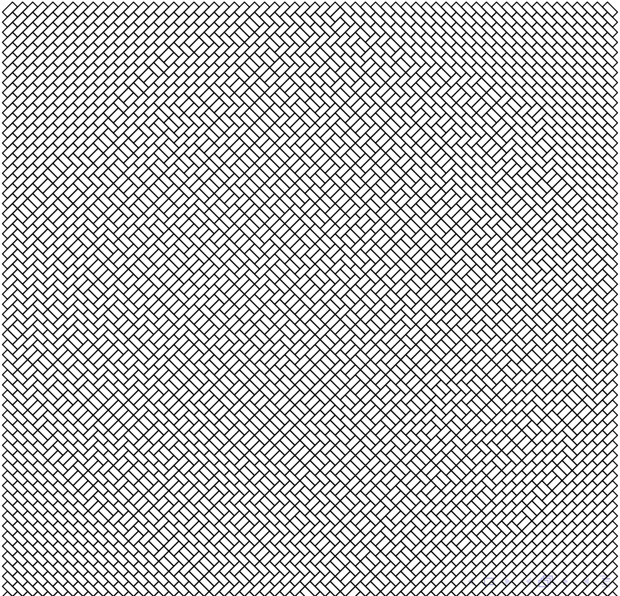


The diamond of order n can be tiled in $2^{n(n+1)/2}$ ways.
Elkies, Kuperberg, Larsen & Propp 1992

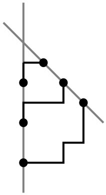
The Aztec Diamond

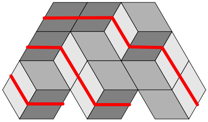


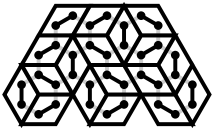
The number of tilings of an order n Aztec diamond is $2^{\binom{n+1}{2}}$. Jonathan Novak observed that

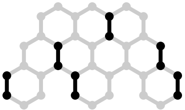
$$\det\left[\binom{2i}{j}\right]_{i,j=1}^{n} = 2^{\binom{n+1}{2}}.$$

Novak half-hexagon



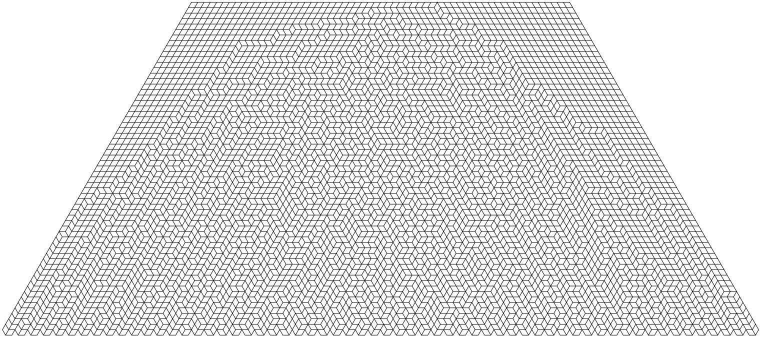




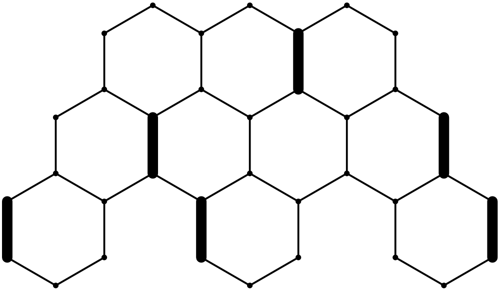


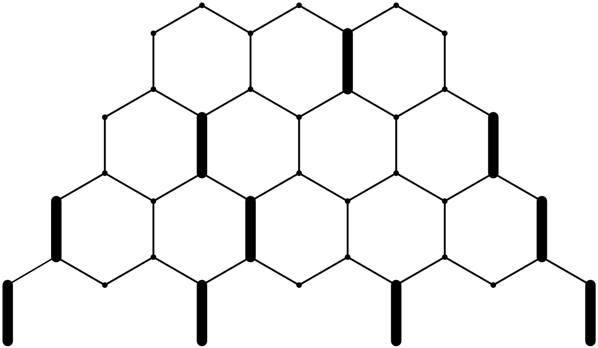
3
2 5
1 3 6
1 3 5 7

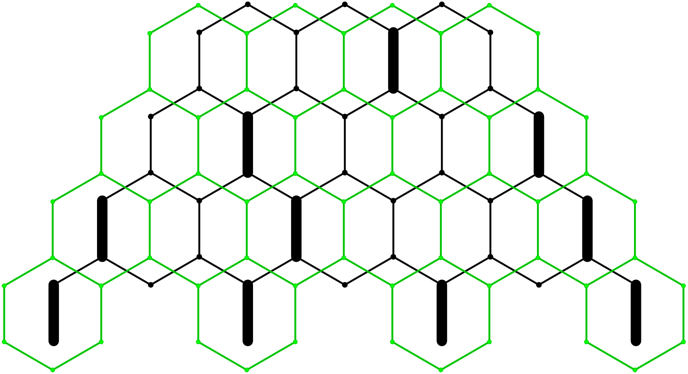
Limit shape

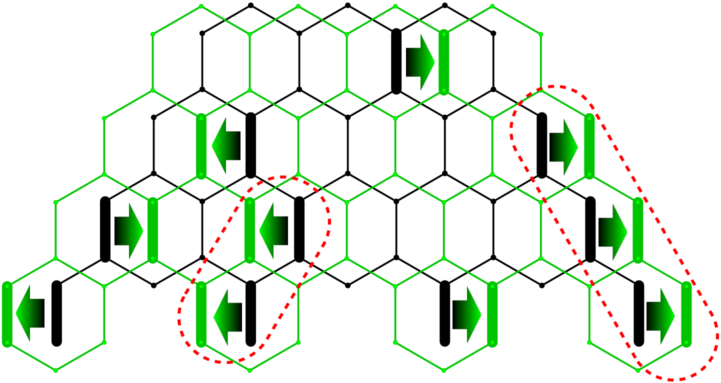


- The shuffling algorithm
- The Arctic Parabola Theorem.
- Correlation kernel

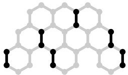








Introduce a coordinate system: $X_j^i(t)$ is the position of the jth particle on level i at time t.



3			
2	5		
1	3	6	
1	3	5	7

Note that

$$X_j^i(t) \leq X_j^{i-1}(t) < X_{j+1}^i(t)$$

Recursion equations

$$
\begin{aligned}
X_1^1(t) &= X_1^1(t-1) + \beta_1^1(t) \\
X_1^j(t) &= X_1^j(t-1) + \beta_1^j(t) \\
&\quad - \mathbf{1}\{X_1^j(t-1) + \beta_1^j(t) = X_1^{j-1}(t) + 1\} \quad \text{for } j \geq 2 \\
X_j^j(t) &= X_j^j(t-1) + \beta_j^j(t) \\
&\quad + \mathbf{1}\{X_j^j(t-1) + \beta_j^j(t) = X_{j-1}^{j-1}(t)\} \quad \text{for } j \geq 2 \\
X_i^j(t) &= X_i^j(t-1) + \beta_i^j(t) \\
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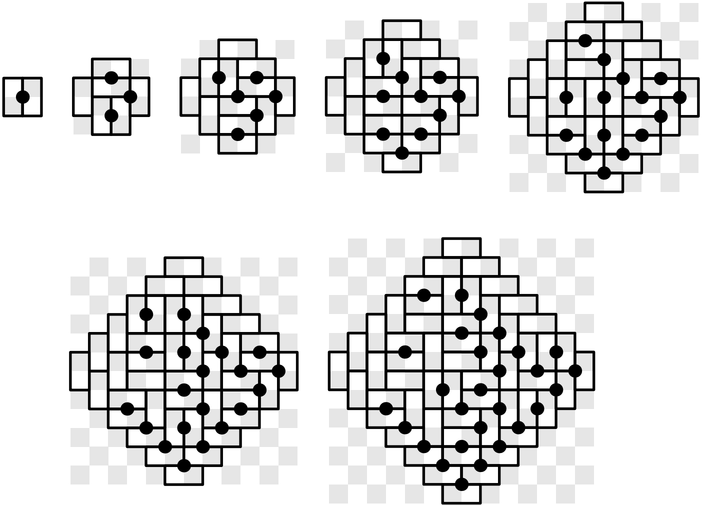
&\quad - \mathbf{1}\{X_i^j(t-1) + \beta_i^j(t) = X_i^{j-1}(t) + 1\} \quad \text{for } j > i > 1
\end{aligned}
$$

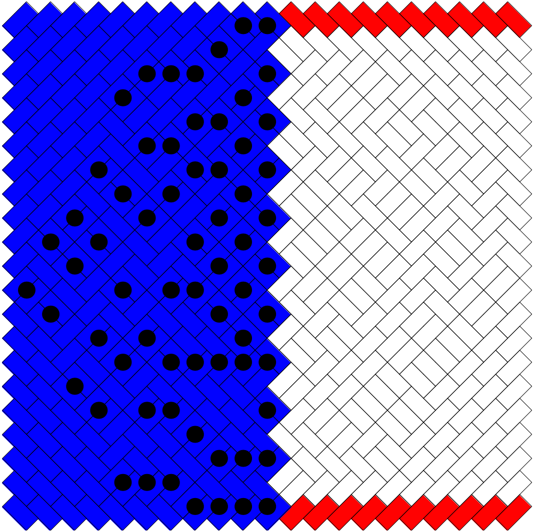
where all $\beta_j^i(t)$ are independent coin flips.

Particle dynamics



Time shift: $x_i^j(t) = X_i^j(t-j)$

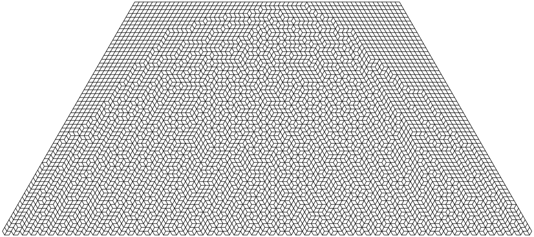
Half-Aztec diamond



The Arctic Parabola Theorem

Theorem

Consider uniform measure on tilings of the Novak half-hexagon. The region in which the density of particles (i.e. vertical lozenges) is assymptotically non-zero is bounded by a parabola.

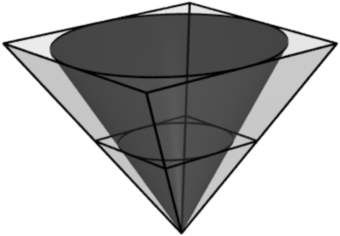


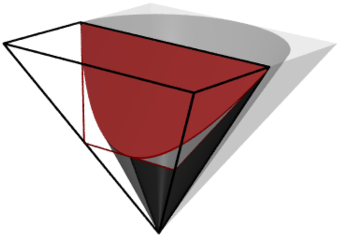
Proposition (N & Y 2011)

The limit shape in the Half-Aztec diamond is the semi-circle.

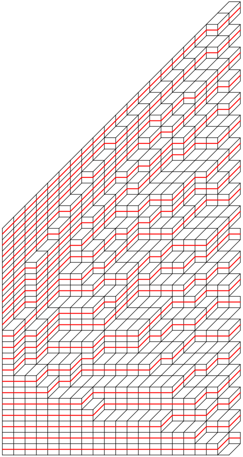
Jockusch, Propp & Shor (1998)
Cohn, Kenyon & Propp (2001)

The Arctic Parabola Theorem





Correlations



Correlations

Consider n Bernoulli walkers started at $1, 2, \ldots, n$, and conditioned to end up at positions $y_1, \ldots, y_n$, at time N conditioned never to intersect.

The number of such configurations is given by the Lindström-Gessel-Viennot Theorem as the determinant of

$$M = \left[\binom{N}{y_j - i} \right]_{i,j=1}^{n}$$

Eynard-Mehta Theorem

Theorem (Eynard & Mehta (1998), Borodin & Rains (2005))

The probability that there is a walker at each of $(x_1, t_1), \ldots, (x_k, t_k)$ *is*

$$\det[K(t_i, x_i; t_j, x_j)]_{i,j=1}^{k}$$

where

$$K(r, x; s, y) = -\mathbf{1}\{s > r\}\binom{s-r}{y-x} + \sum_{i,j=1}^{n} \binom{N-r}{y_i - x}[M^{-1}]_{i,j}\binom{s}{y-j}$$

$$\det\left[\binom{N}{y_i - j - \mathbf{1}\{j \geq s\}}\right]_{i,j=1}^{n-1} =$$

$$\left(\prod_{i=1}^{n-1} \frac{N!}{(y_i-1)!(N-y_i+n)!}\right) \times \det[f(i,j,s)]_{i,j=1}^{n-1}$$

where

$$f(i,j,s) =$$

$$\begin{cases} (y_i - j + 1)\cdots(y_i - 1)(N - y_i + j + 1)\cdots(N - y_i + n), & j < s, \\ (y_i - j)\cdots(y_i - 1)(N - y_i + j + 2)\cdots(N - y_i + n), & j \geq s. \end{cases}$$

Let Δ mean taking the Vandermonde determinant in the variables. For $s = 1$, sage gave us

$$P_{n,1}(N, y) = \Delta(y) \left(\prod_{i=1}^{n-2} (N+i)^{n-1-i} \right) \left(\prod_{j=1}^{n-1} (y_j - 1) \right)$$

Let Δ mean taking the Vandermonde determinant in the variables. For $s = 2$, sage gave us

$$
\begin{aligned}
P_{3,2}(N,y) =& (N+1)\Delta(y)(-2e_2(y) + (N+4)e_1(y) - (3N+8)) \\
P_{4,2}(N,y) =& (N+1)^2(N+2)\Delta(y)(-3e_3(y) + (N+6)e_2(y) \\
& - (3N+12)e_1(y) + (7N+24)) \\
P_{5,2}(N,y) =& (N+1)^3(N+2)^2(N+3)\Delta(y)(-4e_4(y) \\
& + (N+8)e_3(y) - (3N+16)e_2(y) \\
& + (7N+32)e_1(y) - (15N+64)) \\
P_{6,2}(N,y) =& (N+1)^4(N+2)^3(N+3)^2(N+4)\Delta(y)(-5e_5(y) \\
& + (N+10)e_4(y) - (3N+20)e_3(y) + (7N+40)e_2(y) \\
& - (15N+80)e_1(y) + (31N+160))
\end{aligned}
$$

$$P_{n,s}(N,y) = \Delta(y) \prod_{r=1}^{n-2} (N+r)^{n-1-r} \times$$
$$\times \sum_{l=0}^{n-1} \sum_{k=0}^{s-1} \sum_{j=1}^{s} \sum_{i=0}^{j} (-1)^{n+s+l+j} \frac{N^k j^l e_{n-1-l}(y)}{i!(s-1)!} \mathbf{s}(s-1-j, k-i) \times$$
$$\times \left(\left(\frac{d}{dn} \right)^i (n-1) \cdots (n-j) \right) \binom{s-1}{j} \quad (1)$$

where $\mathbf{s}(n,k)$ are the Stirling numbers of the first kind.

Matrix Inverse

Theorem

Let

$$M = \left[\binom{N}{y_i - j} \right]_{i,j=1}^{n}.$$

Then

$$[M^{-1}]_{i,j} = \sum_{k=1}^{j} \frac{\binom{N+n-1}{k-1}\binom{N-1+j-k}{j-k}}{\binom{N+n-1}{y_i-1}} (-1)^{k+j} \prod_{l=1,\, l\neq i}^{n} \frac{k - y_l}{y_i - y_l}.$$

Proof

$$[MM^{-1}]_{\alpha,\gamma} = \sum_{\beta=1}^{n} [M]_{\alpha,\beta} [M^{-1}]_{\beta,\gamma}$$

$$= \sum_{\beta=1}^{n} \sum_{k=1}^{\gamma} (-1)^{k+\gamma} \binom{N+n-1}{y_\beta - 1}^{-1} \binom{N+n-1}{k-1} \times$$

$$\binom{N-1+\gamma-k}{\gamma-k} \binom{N}{y_\beta - \alpha} \prod_{i=1,\, i\neq\beta}^{n} \frac{k-y_i}{y_\beta - y_i}. \quad (2)$$

Lagrange interpolation

Let $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$ and let

$$p_k(x) = \prod_{i=1, i \neq k}^{n} \frac{x - x_i}{x_k - x_i}.$$

Then

$$f(x) = \sum_{k=1}^{n} y_k p_k(x)$$

has the property that $f(x_i) = y_i$ for $i = 1, \ldots, n$.

Proof

$$[MM^{-1}]_{\alpha,\gamma} = \sum_{k=1}^{\gamma} (-1)^{\beta+j} \binom{N-1+\beta-k}{\beta-k} \binom{N}{k-\gamma} = \binom{0}{\alpha-\gamma} = \delta_{\alpha,\gamma} \quad (3)$$

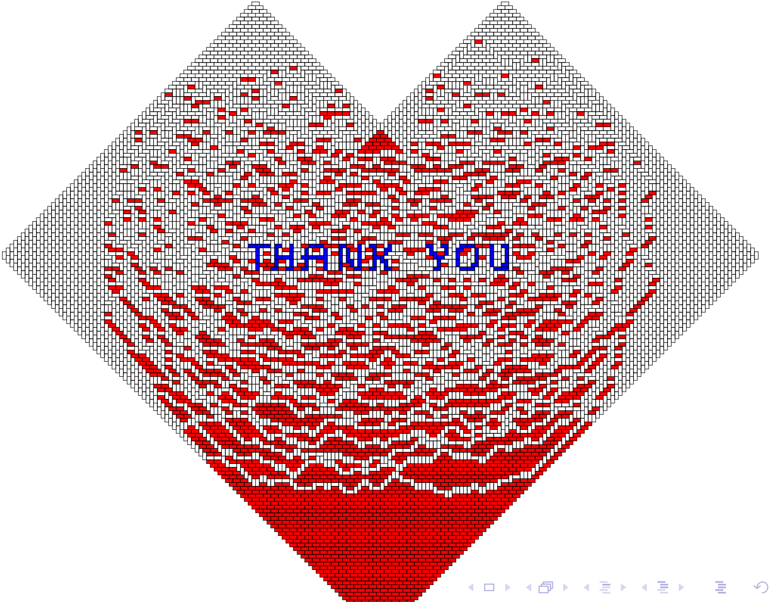
Corollary

The correlation functions for the Novak half-hexagon are determinental, with kernel given by

$$K(r,x;s,y) = -\,\phi_{r,s}(x,y) + \sum_{i,j=1}^{n} \frac{\binom{n+1-r}{2i-x}\binom{s}{y-j}}{\binom{2n}{2i-1}} \sum_{k=1}^{j} \binom{2n}{k-1}\binom{n+j-k}{j-k} \times \frac{(-1)^{k+j+i+n}}{(i-1)!(n-i)!} \prod_{l=1,\, l\neq i}^{n} (k-2l)$$

where for $r \geq s$, $\phi \equiv 0$ *and for* $r < s$,

$$\phi_{r,s}(x,y) = \binom{s-r}{y-x}.$$


THANK YOU

Thank you four your attention

- Nordenstam, Young, *Domino shuffling on Novak half-hexagons and Aztec half-diamonds*, Electron. J. of Combin. 18 (2011), no. 1.
- Nordenstam, Young, *Correlations for the Novak Process*, FPSAC 2012 proceedings, arXiv:1201.4138.

q-analog

Theorem

$$N = \left[\begin{bmatrix} A \\ B_j - i \end{bmatrix}_q q^{\binom{B_j - i}{2}} \right]_{i,j=1}^{n},$$

has inverse

$$[N^{-1}]_{i,j} = \frac{q^{nB_i - \binom{B_i}{2}}}{\begin{bmatrix} A+n-1 \\ B_i - 1 \end{bmatrix}_q} \left(\prod_{k=1,\, k \neq i}^{n} \frac{1}{q^{B_i} - q^{B_k}} \right) \times$$

$$\sum_{a=0}^{j-1} \sum_{b=0}^{n-1} \begin{bmatrix} b \\ j-1-a \end{bmatrix}_q \begin{bmatrix} n-b-1 \\ a \end{bmatrix}_q q^{\binom{j-1}{2} + (a+b)(a-j-1) - b - 1 + aA} \times$$

$$(-1)^b e_b(q^{B_1}, \dots, \widehat{q^{B_i}}, \dots, q^{B_n}).$$