Nichols algebras

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Let (V, c) be a braided vector space. That is: V is a vector space and $c \in \mathrm{Aut}(V \otimes V)$ is a solution of the braid equation in $\mathrm{Aut}(V \otimes V \otimes V)$:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$

Examples:

- $V = \langle x_1, x_2, \ldots, x_n \rangle$, $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for $q_{ij} \in \mathbb{C}^\times$;
- G a group, $V = \mathbb{C}G$, $c(g \otimes h) = ghg^{-1} \otimes g$.

A braided vector space V gives a special type of algebra called the Nichols algebra $\mathfrak{B}(V)$.

To define Nichols algebras we need Artin's braid group $\mathbb{B}_n$. This is the quotient of the free group in $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ by the relations:

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad && 1 \leq i \leq n-2 \\ \sigma_i \sigma_j &= \sigma_j \sigma_i && |i-j| \geq 2 \end{aligned}$$

Recall that $\mathbb{S}_n$ is the quotient of the free group in $\tau_1, \cdots, \tau_{n-1}$ by the relations:

$$\begin{aligned} \tau_i \tau_{i+1} \tau_i &= \tau_{i+1} \tau_i \tau_{i+1} \qquad && 1 \leq i \leq n-2 \\ \tau_i \tau_j &= \tau_j \tau_i && |i-j| \geq 2 \\ \tau_i^2 &= 1 && 1 \leq i \leq n-1 \end{aligned}$$

Some remarks:

- There exists a surjection $\mathbb{B}_n \to \mathbb{S}_n$ defined by $\sigma_i \mapsto \tau_i$.
- (Matsumoto) There exists a section of sets

$$\mu : \mathbb{S}_n \to \mathbb{B}_n$$
$$\tau_i \mapsto \sigma_i$$

  such that $\mu(xy) = \mu(x)\mu(y)$ for any $s, t \in \mathbb{S}_n$ with $\mathrm{length}(xy) = \mathrm{length}(x) + \mathrm{length}(y)$.
- Let (V, c) a braided vector space and let

$$c_i = c_{i,i+1} = \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}} \in \mathrm{Aut}(V^{\otimes n}).$$

  Then $c_1, \cdots, c_{n-1}$ satisfy the relations of the Braid group and hence $\rho_n : \mathbb{B}_n \to \mathrm{Aut}(V^{\otimes n})$, defined by $\rho(\sigma_i) = c_i$, is a representation.

Let (V, c) be a braided vector space. We construct the Nichols algebra of V as

$$\mathfrak{B}(V) = \bigoplus_n \mathfrak{B}^n(V) = \bigoplus_n T^n(V)/(\ker \mathfrak{S}_n).$$

The map $\mathfrak{S}_n$ is the quantum symmetrizer:

$$\mathfrak{S}_n = \sum_{\sigma \in \mathbb{S}_n} \rho_n(\mu(\sigma)),$$

where ρ_n is the representation of $\mathbb{B}_n$ induced by c and μ is the Matsumoto section.

Examples:

- $\mathfrak{S}_2 = 1 + c$,
- $\mathfrak{S}_3 = 1 + c_{12} + c_{23} + c_{12}c_{23} + c_{23}c_{12} + c_{12}c_{23}c_{12}$.

Some well-known examples of Nichols algebras:

- (V, flip) gives the symmetric algebra;
- $(V, -\text{flip})$ gives the exterior algebra.

Problem

Classify finite-dimensional Nichols algebras

Nichols algebras appear, for example, in:

- Quantum groups (Drinfeld; Jimbo; Lusztig; Rosso);
- Differential calculus in quantum groups (Woronowicz);
- Quantized Lie superalgebras (Khoroshkin, Tolstoy);
- Deformations of Lie (super)algebras (Hodges);
- Pointed Hopf algebras (Nichols; Andruskiewitsch, Schneider);
- Cohomology rings of flag manifolds (Fomin, Kirillov; Postnikov; Bazlov);
- Mathematical-physics (Majid; Semikhatov).

Definition:
A Nichols algebra is of diagonal type if there exists a basis $\{v_1, \cdots, v_n\}$ such that

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad q_{ij} \in \mathbb{K}^{\times}.$$

Nichols algebras of diagonal type have many interesting properties and applications.

Heckenberger classified finite-dimensional Nichols algebras of diagonal type in terms of generalized Dynkin diagrams. The key: the Weyl groupoid.

How to construct (non-diagonal) braided vector spaces?

In 1992 Drinfeld proposed to study a set theoretical version of the braid equation.

Let X be a set. A bijective function $c : X \times X \to X \times X$ is a solution of the set-theoretical braid equation if

$$(c \times \mathrm{id})(\mathrm{id} \times c)(c \times \mathrm{id}) = (\mathrm{id} \times c)(c \times \mathrm{id})(\mathrm{id} \times c).$$

Definition:
A rack is a pair $(X, \triangleright)$, where X is a finite set and $\triangleright : X \times X \to X$ is a map such that:

- $\varphi_i : x \mapsto i \triangleright x$ is bijective for all $i \in X$.
- $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$.

Remark:
The map $c(x, y) = (x \triangleright y, x)$ is a solution of the set-theoretical braid equation if and only if $(X, \triangleright)$ is a rack.

Example (Important!):
A conjugacy class X with the conjugation $x \triangleright y = xyx^{-1}$ is a rack.

A rack $(X, \triangleright)$ is faithful if the map $i \mapsto \varphi_i$ is injective for all $i \in X$.

Remark:
Let $(X, \triangleright)$ be a rack. Let $V = \mathbb{C}X$ and define $c \in \mathrm{GL}(V \otimes V)$ by

$$c(x \otimes y) = (x \triangleright y) \otimes x.$$

Then (V, c) is a braided vector space.

Questions:

- Is it possible to construct more "braidings" from $(X, \triangleright)$?
- Let $q : X \times X \to \mathbb{C}^\times$ be a map. When is the map

  $$c(x \otimes y) = q(x, y)(x \triangleright y) \otimes x$$

  a solution of the braid equation?

Remark: The map

$$c(x \otimes y) = q(x, y)(x \triangleright y) \otimes x$$

is a solution of the braid equation if and only if q is an abelian rack 2-cocycle:

$$q(x, y \triangleright z)q(y, z) = q(x \triangleright y, x \triangleright z)q(x, z).$$

Notation:
$\mathfrak{B}(X, q)$ is the Nichols algebra $\mathfrak{B}(V, c)$ where $V = \mathbb{C}X$ and the braiding is $c(x \otimes y) = q(x, y)(x \triangleright y) \otimes x$.

Examples of racks, 2-cocycles and Nichols algebras

3-cycles in $\mathbb{A}_4$:
Let $X = (123)^{\mathbb{A}_4}$ be the rack associated to the conjugacy class of (123) in $\mathbb{A}_4$:

	(243)	(123)	(134)	(142)
(243)	(243)	(134)	(142)	(123)
(123)	(142)	(123)	(243)	(134)
(134)	(123)	(142)	(134)	(243)
(142)	(134)	(243)	(123)	(142)

For example:

$$\begin{aligned}(243) \triangleright (123) &= (243)(123)(243)^{-1} \\ &= (243)(123)(234) \\ &= (243)(13)(24) \\ &= (134)\end{aligned}$$

Remark:
It is possible to prove that

$$H^2((123)^{\mathbb{A}_4}, \mathbb{C}^\times) = \mathbb{C}^\times \times \langle \eta \rangle,$$

where η is the 2-cocycle defined by

	(243)	(123)	(134)	(142)
(243)	1	1	1	1
(123)	1	1	-1	-1
(134)	1	-1	1	-1
(142)	1	-1	-1	1

Affine racks:
Let $\mathbb{F}_q$ be the field of q elements and let $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$. Define

$$x \triangleright y = (1 - \alpha)(x) + \alpha(y).$$

Then (A, α) is a rack (called affine rack), and it will be denoted by $\mathrm{Aff}(q, \alpha)$.

Transpositions in $\mathbb{S}_n$:
Let X_n be the conjugacy class of transpositions in $\mathbb{S}_n$ and let χ be the 2-cocycle:

$$\chi(\sigma,\tau) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ -1 & \text{otherwise,} \end{cases}$$

where $\tau = (i\ j)$ with $i < j$.

Remarks:

- These algebras appear in the work of Fomin & Kirillov.
- χ is cohomologous to the trivial 2-cocycle for $n = 3$.
- $H^2(X_n, \mathbb{C}^\times) = \mathbb{C}^\times \times \langle\chi\rangle$ for $n \in \{4, \dots, 10\}$.

Conjeture

$H^2(X_n, \mathbb{C}^\times) = \mathbb{C}^\times \times \langle\chi\rangle$ for all $n \geq 4$.

As an example we work out the case $n = 3$.

Let X_3 be the conjugacy class of transpositions in $\mathbb{S}_3$:

$\triangleright$	$(1\ 2)$	$(2\ 3)$	$(1\ 3)$
$(1\ 2)$	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$
$(2\ 3)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2)$
$(1\ 3)$	$(2\ 3)$	$(1\ 2)$	$(1\ 3)$

The algebra $\mathfrak{B}(X_3, \chi)$ is generated by $x_{(12)}, x_{(23)}, x_{(13)}$ in degree 1 and has relations:

$$
\begin{gathered}
x_{(12)}^2 = x_{(23)}^2 = x_{(13)}^2 = 0 \\
x_{(12)}x_{(23)} + x_{(23)}x_{(13)} = x_{(12)}x_{(13)}, \\
x_{(23)}x_{(12)} + x_{(13)}x_{(23)} = x_{(13)}x_{(12)}.
\end{gathered}
$$

It is a graded algebra of dimension 12. The Hilbert series is

$$
\mathcal{H}(t) = 1 + 3t + 4t^2 + 3t^3 + t^4.
$$

Remark:
$\mathfrak{B}(X_n, \chi)$ is finite-dimensional for $n \in \{3, 4, 5\}$:

n	rank	top	dimension	Hilbert series
3	3	4	12	$(2)_t^2(3)_t$
4	6	12	576	$(2)_t^2(3)_t^2(4)_t^2$
5	10	40	8294400	$(4)_t^4(5)_t^2(6)_t^4$

where $(k)_t = 1 + t + t^2 + \cdots + t^{k-1}$.

Conjetures

- $\dim \mathfrak{B}(X_n, \chi) = \infty$ for $n \geq 6$.
- $\mathfrak{B}(X_n, \chi)$ is quadratic.

Definition:
A rack X is decomposable if there exists a subset $\emptyset \neq Y \subseteq X$ such that $X = Y \sqcup (X \backslash Y)$ and $X \triangleright Y \subseteq Y$.

The problem:
For a given rack X classify all 2-cocycles of X such that $\dim \mathfrak{B}(X, q) < \infty$.

Definition:
A rack X is decomposable if there exists a subset $\emptyset \neq Y \subseteq X$ such that $X = Y \sqcup (X \backslash Y)$ and $X \triangleright Y \subseteq Y$.

The problem:
For a given indecomposable rack X classify all 2-cocycles of X such that $\dim \mathfrak{B}(X, q) < \infty$.

Only a few examples of non-diagonal finite-dimensional Nichols algebras over indecomposable racks are known

Examples of Nichols algebras over $\mathbb{C}$

	$\dim V$	$\dim \mathfrak{B}(V)$
Milinski, Schneider (1996)[1]	3	12
Graña (2000)	4	72
Andruskiewitsch, Graña (2002)	5	1280
Andruskiewitsch, Graña (2002)	5	1280
Milinski, Schneider[1] (2002)	6	576
Andruskiewitsch, Graña (2002)	6	576
Andruskiewitsch, Graña (2002)	6	576
Graña (2002)	7	326592
Graña (2002)	7	326592
Graña (2002)	10	8294400
Graña[1] (2002)	10	8294400

[1] Based on the work of Fomin & Kirillov (1995)

X	q	$\lvert X\rvert$	$\dim \mathfrak{B}(X,q)$	Hilbert series
$(12)^{\mathbb{S}_3}$	χ	3	12	$(2)_t^2(3)_t$
$(123)^{\mathbb{A}_4}$	-1	4	72	$(2)_t^2(3)_t(6)_t$
$\mathrm{Aff}(5,2)$	-1	5	1280	$(4)_t^4(5)_t$
$\mathrm{Aff}(5,3)$	-1	5	1280	$(4)_t^4(5)_t$
$(1234)^{\mathbb{S}_4}$	-1	6	576	$(2)_t^2(3)_t^2(4)_t^2$
$(12)^{\mathbb{S}_4}$	-1	6	576	$(2)_t^2(3)_t^2(4)_t^2$
$(12)^{\mathbb{S}_4}$	χ	6	576	$(2)_t^2(3)_t^2(4)_t^2$
$\mathrm{Aff}(7,3)$	-1	7	326592	$(6)_t^6(7)_t$
$\mathrm{Aff}(7,5)$	-1	7	326592	$(6)_t^6(7)_t$
$(12)^{\mathbb{S}_5}$	-1	10	8294400	$(4)_t^4(5)_t^2(6)_t^4$
$(12)^{\mathbb{S}_5}$	χ	10	8294400	$(4)_t^4(5)_t^2(6)_t^4$

$$(k)_t = 1 + t + t^2 + \cdots + t^{k-1}$$

Theorem (with Graña & Heckenberger)

Let X be a non-trivial indecomposable faithful rack of size d and let q be a 2-cocycle of X. The following are equivalent:

1. $\dim \mathfrak{B}_2(X,q) \leq \frac{d(d+1)}{2}$.
2. X is one of the racks

$$(12)^{\mathbb{S}_n} \text{ for } n \in \{3,4,5\},$$
$$(1234)^{\mathbb{S}_4}, (123)^{\mathbb{A}_4},$$
$$\mathrm{Aff}(p,\alpha) \text{ for } (p,\alpha) \in \{(5,2),(5,3),(7,3),(7,5)\},$$

   and the Nichols algebra is one of the algebras listed before.
3. There exist $n_1, n_2, \dots, n_d \in \mathbb{N}$ such that the Hilbert series of $\mathfrak{B}(X,q)$ factorizes as

$$\mathcal{H}(t) = (n_1)_t (n_2)_t \cdots (n_d)_t.$$

A new example (with Heckenberger & Lochmann):

Recall that the rack $X = (123)^{\mathbb{A}_4}$ can be presented as

	a	b	c	d
a	a	c	d	b
b	d	b	a	c
c	b	d	c	a
d	c	a	b	d

Consider the 2-cocycle q:

	a	b	c	d
a	ω	ω	ω	ω
b	ω	ω	$-\omega$	$-\omega$
c	ω	$-\omega$	ω	$-\omega$
d	ω	$-\omega$	$-\omega$	ω

where ω is a cubic root of 1.

Then $\mathfrak{B}(X, q)$ has dimension 5184.

- Generators: a, b, c, d,
- Relations: four relations in degree 2, four in degree 3 and one in degree 6,
- Hilbert series: $\mathcal{H}(t) = (6)_t^4 (2)_{t^2}^2$.