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COMBINATORIAL HOPF ALGEBRAIC DESCRIPTION OF THE MULTI-SCALE RENORMALIZATION IN QUANTUM FIELD THEORY

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Abstract. We define combinatorial Hopf algebras on assigned Feynman graphs and on Gallavotti–Nicolò trees, which we then prove to underly the multi-scale renormalization in quantum field theory. Moreover, homomorphisms between these Hopf algebras and the Connes–Kreimer Hopf algebras on rooted trees and on Feynman graphs are given. Finally, we show how this formalism can be used to investigate some algebraic properties of the effective expansion in multi-scale renormalization.

Contents

1. Introduction and motivation

The interplay between *combinatorics* and *physics* has been very fruitful for both recently, and it leads to the new and emerging interdisciplinary field of *combinatorial physics*. For instance, combinatorial tools have been used successfully for a better understanding of the algebraic structures underlying quantum mechanics (see [3], [14] and references therein), and the interplay between combinatorics and statistical physics or integrable

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systems has been extensively studied both by combinatorialists and theoretical physicists.

In quantum field theory (QFT), a similar success story is the elegant description of the combinatorial backbone of perturbative renormalization via the combinatorial Connes–Kreimer Hopf algebra on Feynman graphs (see the original paper [8] as well as Section 1.6 of the book [11])[1]. The Connes–Kreimer Hopf algebra allows one to recover the analytic expressions of a renormalized Feynman amplitude and the usual forest structure of the subtraction operators, e.g., in the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) renormalization, by using the recursive computation of the antipode, which automatically generates all Zimmermann forests with their correct weight.

This elegant point of view and its relationship with other mathematical problems such as the Riemann–Hilbert problem [9, 10] has made renormalization a popular subject of mathematics. But there is a drawback: it has become so famous among mathematicians that it may have obscured for some of them the true physical meaning of renormalization. Indeed, the key physical notion in renormalization, namely the notion of scale, is absent or hidden in the Connes–Kreimer formalism.

It is the goal of this paper to attract the attention of the mathematics community to this point, and to propose a possible compromise, by supplementing the Connes–Kreimer algebra with discrete scale assignments. The corresponding algebra is generated by *assigned graphs*, which are ordinary Feynman graphs supplemented with the assignment of an integer to each edge. This integer physically represents the resolution scale of that edge or propagator.

As a matter of fact, the modern version of renormalization, namely the renormalization group discovered by Wilson and followers [37], tells us that the main purpose of renormalization is not to remove divergences from Feynman amplitudes, nor to hide them into unobservable infinite bare parameters[2]. Renormalization is much more general and powerful. It explains why and how, for physical systems with many coupled degrees of freedom, the *laws of nature change with the observation scale* [38]. This fundamental aspect of renormalization is captured mathematically by a *multi-scale analysis*.

For general systems, the Wilsonian slicing into scales can be implemented technically in many ways (block-spins, wavelet analysis, etc.). The most convenient technique in the context of QFT *slices* the propagator of the theory according to a geometric sequence of cutoffs. Each slice represents a particular energy scale, and has a particular spatial resolution power; it has an ultraviolet and an infrared cutoff with constant ratio between both. The renormalization group then performs many times the same step, namely a functional integral over a slice or fluctuation field and the computation of the resulting effective action for the remaining sum of fields of lower slices, called the background field [38].

The need for such a discrete multi-scale analysis of QFT was quite independently also discovered by mathematical physicists such as J. Glimm, A. Jaffe, and their followers in the constructive field theory program [35]. They called it the *phase space expansion*. Over

[1]Let us also mention here that a Hopf algebraic description was also used to describe the combinatorics of perturbative renormalization on noncommutative Moyal space scalar QFT (where graphs are replaced by ribbon graphs, or combinatorial maps) [34], [33]; moreover, Connes–Kreimer-like Hopf algebras have been defined for quantum gravity spin-foam models [25], [32].

[2]This is particularly clear in the case of asymptotically free theories, such as quantum chromodynamics, the theory of strong interactions, for which the bare coupling tends to *zero* at a high ultraviolet scale.

the years this constructive program, in which perturbative QFT is summed, effectively merged completely with the Wilsonian renormalization group approach and became its mathematically rigorous version[3]. Constructive analysis comes at a high price: many elegant perturbative tools in QFT such as dimensional regularization, dimensional renormalization, and differential (rather than finite-difference) renormalization group equations, had to be discarded by the constructive community in favor of *discrete* multi-scale analysis, which remains up to now the only tool with proven constructive power. For a general presentation of these views and of multi-scale renormalization, see [27].

Returning to the more limited and specific context of perturbative renormalization of Feynman amplitudes, multi-scale analysis was first developed systematically in [16] and [19]. Initially, these authors were motivated by the desire to understand and simplify the proof of uniform bounds on renormalized amplitudes implying "local Borel summability" [13], which had been soon followed also by the construction and Borel summability of *planar* asymptotically free renormalized theories, such as "wrong sign" planar ϕ_4^4 [31, 26].

Multi-scale analysis evolved over the years into a versatile technique to understand and analyze renormalization and the renormalization group in new contexts. It was suitably generalized to the condensed matter case in which the Fermionic propagator is sliced into a sequence of scales pinching closer and closer the Fermi surface [1, 17]. This technique provided the backbone for the rigorous analysis of correlated quantum Fermions at low temperature, such as Fermi and Luttinger liquids in one, two, and three spatial dimensions (see [28] and the many references therein for a recent review of this large, active, and mature field of mathematical physics).

More recently, the multi-slice analysis has been used to prove perturbative renormalizability at all orders for radically new quantum field theories, in which the interaction is non-local, and the usual intuition of zero momentum subtraction around local parts fails. Such new models include the first examples of renormalizable noncommutative quantum field theories (see [30], [22], [21], [36], and [20]) and of tensor group field theories [2, 6]. The latter models might be relevant for the long term goal of quantization of gravity [29], but also for the more concrete analysis of statistical physics in random geometry [4] or with long range interactions, such as spin glasses [5].

Multi-scale analysis is characterized by the fact that the contraction and subtraction operations that implement renormalization are not effectuated blindly. They make physical sense only for so-called *high* subgraphs, i.e., connected subgraphs which have all their internal scales higher than any of their external scales. It is solely for such subgraphs that the comparison of their amplitudes to a local part makes sense[4]. It is this distinction which in its turn launches the renormalization group flow, hence the motion of effective constants with scale. Assigned graphs allow one to rigorously define such subgraphs which need to be renormalized, while general graphs do not; hence we feel they should become

[3]Let us remark at this point that (contrary to a belief sometimes heard in the mathematics community) the residues or the individual renormalized Feynman amplitudes do not correspond to any physical observables in QFT. Indeed, any measure always involves not a single Feynman amplitude but the infinite sum of such amplitudes compatible with a given set of external legs hence also with a certain *resolution power*. Only the value of this infinite sum, which is what constructive theory is after, has physical meaning.

[4]Physically this is nothing else but the trivial observation that objects with a certain size look local only when observed through probes that do not distinguish their internal structure.

part of the combinatorial Hopf algebra framework used by mathematicians to describe renormalization.

In this paper, we therefore define a new Hopf algebra, which is meant to describe the combinatorial soul of this discrete multi-scale renormalization technique. In order to do that, we define *assigned graphs* as Feynman graphs together with a scale assignment of their edges. The desired combinatorial Hopf algebra is then defined on the space freely generated by these assigned graphs. The coproduct has then to take into account the supplementary scale information of the assigned graphs: we only sum over the particular class of *high subgraphs*. For example, in the case of the scalar ϕ^4 model, we do not need to sum over *all* subgraphs with two- or four-external edges, as is done in the standard definition of the Connes–Kreimer coproduct.

Let us also mention that in this paper we deal with the ϕ^4 model, even though our results can be generalized in a straightforward manner to more general renormalizable QFTs.

2. Feynman graph expansion and multi-scale renormalization

In this section, we give a short overview of Feynman graphs and multi-scale expansions in quantum field theory.

2.1. **From path integrals to Feynman graphs.** In its most general acceptance, QFT can be defined as the study of quantum (or stochastic) dynamical systems involving continuous degrees of freedom. In the Euclidean path integral approach, one has to define the path integral representing the expectation value of an observable, heuristically written as

$$\langle \mathcal{O} \rangle = \int \frac{d\mu(\phi)\, \mathcal{O}[\phi]\, \exp -\mathcal{S}[\phi]}{\int d\mu(\phi)\, \exp -\mathcal{S}[\phi]}. \tag{2.1}$$

The integration is over a suitable space of fields $\phi : \mathbb{R}^D \to \mathbb{R}$, $\mathcal{S}[\phi]$ is the action and $\mathcal{O}$ an observable.

In the simplest case (the so called ϕ^N_D Euclidean field theory), the action can be written as

$$\mathcal{S}[\phi] = \int_{\mathbb{R}^D} d^D x \left\{ \frac{1}{2}\phi(x)\big(-\Delta + m^2\big)\phi(x) - \frac{\lambda}{N!}\phi(x)^N \right\}, \tag{2.2}$$

with Δ the Laplacian and m and λ two positive real numbers, identified with the mass and the coupling constant[5] of the theory. The observables are usually taken to be products of the fields at different space-time points, $\mathcal{O}[\phi] = \phi(x_1)\cdots\phi(x_n)$, whose expectation value defines the n-point correlation function.

In the free field case $\lambda = 0$, the path integral is Gaußian and is readily computed using Wick's theorem. With a suitably normalized measure, the expectation value of a product of fields reads

$$\int d\mu_C(\phi)\phi(x_1)\cdots\phi(x_n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \displaystyle\sum_{\substack{\text{pairings of} \\ \{1,2,\ldots,n\}}} C(x_{i_1}, x_{i_2})\cdots C(x_{i_{\frac{n}{2}-1}}, x_{I_{\frac{n}{2}}}), & \text{if } n \text{ is even,} \end{cases} \tag{2.3}$$

[5]In order to simplify notation, we have introduced a minus sign in the coupling constant. Therefore the physical coupling constant is negative.

where the covariance $C(x,y)$ is the kernel of the inverse of $-\Delta + m^2$.

In the non-Gaussian case, we expand the integrand as a formal power series in λ and perform all the integrals using Wick's theorem. Each term we obtain in this way is called a Wick contraction. Wick contractions naturally define graphs, called Feynman graphs. Collecting Wick contractions that correspond to the same graph, we obtain an expansion of the correlation functions as a sum over graphs,

$$\int d\mu_C(\phi)\phi(x_1)\cdots\phi(x_n) = \sum_{\substack{G \text{ graph with} \\ n \text{ external edges}}} \frac{\lambda^{v(G)}}{\sigma(G)} A(G)[x_1,\ldots,x_n]. \tag{2.4}$$

The Feynman graphs have n labelled univalent vertices (associated with the variables $x_1,\ldots,x_n$) and $v(G)$ N-valent vertices corresponding to the interaction monomial $\phi^N(x)$. The edges incident to two N-valent vertices are called internal edges and the other edges are called external. Because of the variables $x_1,\ldots,x_n$, there are labels on the external edges, while the internal ones are unlabelled so that we are summing over isomorphism classes of graphs with fixes external edges. This is accounted for by the symmetry factor $\sigma(G)$ defined as follows.

When expanding the path integral, we label the vertices, and for each vertex we also label the half-edges emanating from it, so that the half-edges are labelled by pairs (v,p). Then a Wick contraction is just a partition of the indices of half-edges into pairs, and the symmetry factor $\sigma(G)$ is the subgroup of the group of permutations of all these labels of the internal edges that preserve this partition, with G the corresponding isomorphism class. In the general case, there are $v(G)!\,(N!)^{v(G)}$ labellings of the internal half-edges, so that there are $\frac{v(G)!\,(N!)^{v(G)}}{\sigma(G)}$ Wick pairings associated with a given isomorphism class.

From an analytic point of view, we have to remember that the kernel $C(x,y)$ is a distribution, and the Feynman graph amplitudes are not well defined since they involve products of distributions. In order to overcome this problem, we first regulate the theory, replacing the distribution by some function $C_\rho(x,y)$ depending on a regulator ρ, in such a way that we recover $C(x,y)$ by letting ρ tend to infinity, $C(x,y) = \lim_{\rho\to\infty} C_\rho(x,y)$. Then, for the so-called renormalizable theories $\phi^4_4, \phi^3_6, \phi^6_3, \ldots$, a recursive operation is performed on the Feynman graph amplitudes in such a way that they are well defined in the limit $\rho \to \infty$. For graphs without subdivergent graphs, this operation is additive but otherwise we have to first renormalize the subdivergences. The corresponding operation is polynomial and involves a sum over all the forests of the graphs, as will be made more precise later. We refer to [7] for a detailed overview of perturbative renormalization and the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) forest formula.

Even if very successful, the BPHZ forest formula has an important drawback: it does not implement Wilson's idea that path integrals must be computed by first integrating over small distance degrees of freedom. To implement this idea, it is convenient to use multi-scale analysis.

2.2. **Multi-scale renormalization in a nutshell.** As mentioned in the introduction, the multi-scale analysis of renormalization, which is at the core of the Wilsonian approach, relies on a geometrically growing sequence of discrete scales. There are two main technical ways to create the sequence of scales:

- block spinning of the field variables in direct space, that is, defining $\phi = \phi_f + \phi_b$, where the background field ϕ_b is the local *average* of ϕ with respect to a lattice

of cubes of side length M, and ϕ_f, the fluctuation field, is simply the difference between the field and the background field;
- slicing the propagator C as $C_f + C_b$, where C_f has both infrared and ultraviolet cutoff with fixed ratio M, and C_b has only an ultraviolet cutoff, which is the infrared cutoff of C_f; in that case, the slicing induces an orthogonal decomposition of the field as $\phi = \phi_f + \phi_b$, where ϕ_f is distributed according to C_f and ϕ_b according to C_b.

The first technique is more general and can apply to any statistical mechanics system, but requires a discretization through lattices. The second technique is the most elegant and clearly best adapted to perturbative renormalization theory around a propagator with non-trivial spectrum. More precisely, an excellent compromise for a propagator with a positive spectrum is the parametric slicing.

Definition 2.1 (Parametric Slicing)**.** Let $C = 1/H$ be the propagator of the theory. The parametric slicing is

$$C = \int_0^\infty e^{-\alpha H} d\alpha = \sum_{i=0}^\infty C^i, \quad \text{with} \tag{2.5}$$

$$C^i = \int_{M^{-2i}}^{M^{-2(i-1)}} e^{-\alpha H} d\alpha \ , \quad C^0 = \int_1^\infty e^{-\alpha H} d\alpha. \tag{2.6}$$

The natural ultraviolet cutoff on the theory is then

$$C_\rho = \sum_{i=0}^{\rho} C^i \tag{2.7}$$

for finite and large integer ρ. In the case of the Laplacian plus mass on $\mathbb{R}^d$, we get the following slices:

$$C^i = \int_{M^{-2i}}^{M^{-2(i-1)}} e^{-m^2\alpha - \frac{|x-y|^2}{4\alpha}} \frac{d\alpha}{\alpha^{d/2}}, \tag{2.8}$$

$$C^0 = \int_1^\infty e^{-m^2\alpha - \frac{|x-y|^2}{4\alpha}} \frac{d\alpha}{\alpha^{d/2}}. \tag{2.9}$$

α being dual to p^2, one should consider each propagator C^i as corresponding to a theory with both an ultraviolet and an infrared cutoff. They differ by the fixed multiplicative constant M, the momentum slice "thickness".

The decomposition (2.8) is the multi-slice representation. From the general definition of Gaussian measures follows an associated decomposition of the Gaussian measure $d\mu_\rho$ of covariance C_ρ into a product of independent Gaussian measures $d\mu^i$ with covariance C^i. Similarly, the random field ϕ_ρ distributed according to $d\mu_\rho$ is the sum of *independent* random variables ϕ^i distributed according to $d\mu^i$:

$$\phi_\rho = \sum_{i=0}^{\rho} \phi^i; \quad d\mu_\rho(\phi_\rho) = \bigotimes_{i=0}^{\rho} d\mu^i(\phi^i) \tag{2.10}$$

This independence of the fields at each scale in the perturbative analysis of the corresponding functional integral leads in its turn to a sum over *assigned graphs*, that is, graphs which have an integer associated with each edge, namely its scale.

Definition 2.2. A *scale assignment* μ for a Feynman graph with labelled internal edges is a list of positive integers i_ℓ, $\ell = 1, \ldots, E$, associated with the internal edges of the respective Feynman graph (where E is the number of internal edges of the graph).

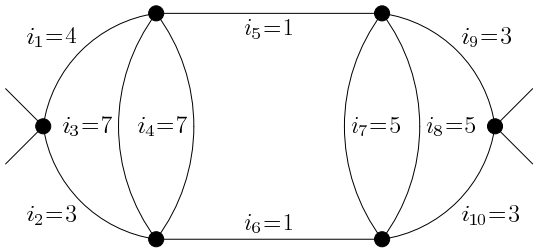


FIGURE 1. A Feynman graph with a scale assignment; it has 10 internal edges and 4 external edges.

Let us emphasize here that the integers in the definition above are bounded by the discrete cutoff ρ. Furthermore, we make the following definition.

Definition 2.3. An *assigned graph* (G, μ) is an isomorphism class of pairs formed by the one-particle irreducible (1PI) edge labelled Feynman graph G together with a scale assignment μ.

In physics, a 1PI graph is a graph that cannot be disconnected by cutting an arbitrary edge. In the mathematics literature, it is called 2-edge-connected.

Remark 2.4. Assigned graphs correspond to graphs whose edges are labelled by the scales. They can be seen as a particular class of decorated graphs.

Definition 2.5. An *assigned subgraph* (g, ν) of a given assigned graph (G, μ) is constructed in the following way (see the previous section). Consider a subgraph g of the Feynman graph G, in the usual QFT way. The scale assignment ν of G is given by the restriction of the scale assignment μ to the internal edges of g (which are also internal edges of G). Moreover, the external edges of g which are internal edges of G have the scale assignment which is attributed to them by μ. The same holds for the external edges of g which are external edges of G.

Furthermore, we can define the usual graph theoretical notions (number of edges, vertices, (independent) loops, etc.) of an assigned graph (G, μ) as the respective notions of the Feynman graph G. Moreover, we call (G, μ) an N-point assigned graph if G is an N-point 1PI Feynman graph.

We then define the internal and external index for a subgraph (g, ν) of an assigned graph (G, μ) as:

$$i_g(\mu) = \inf_{l \in g} \mu(l), \tag{2.11}$$

$$e_g(\mu) = \sup_{l \text{ external edge of } g} \mu(l), \tag{2.12}$$

(with the μ-dependence sometimes omitted for shortness).

Definition 2.6. Let (G, μ) an assigned graph. We say that a subgraph (g, ν) is a *high subgraph* if

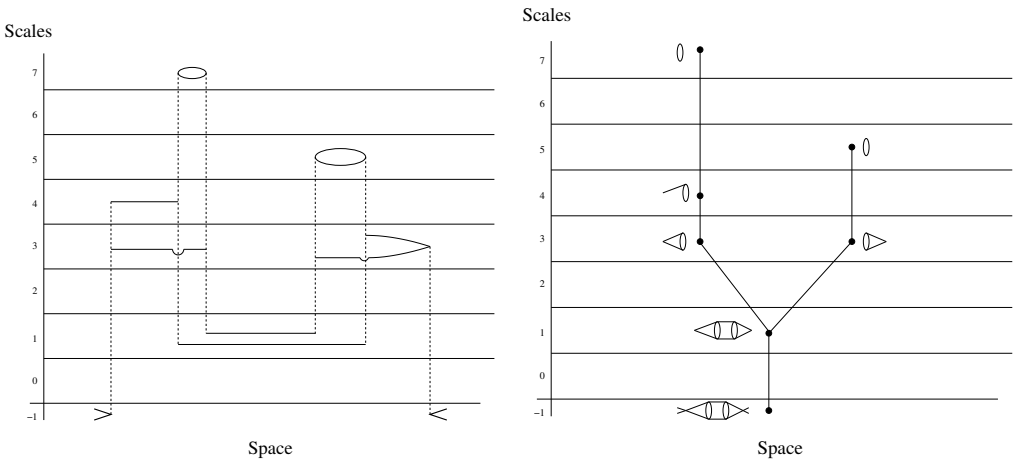
- g is connected;
- the internal index of g is higher than its external index:

$$e_g(\mu) < i_g(\mu) \qquad \text{(high condition)}. \tag{2.13}$$

With a connected assigned graph (G, μ), we associate the Gallavotti–Nicolò tree $T_{(G,\mu)}$, which is defined in the following way (see the book [27] for more details).

Definition 2.7. The Gallavotti–Nicolò tree $T_{(G,\mu)}$ is a rooted tree whose vertices at a distance i from the root are decorated with the connected high subgraphs G_c^i with scales $\geq i$ and whose arrows join the vertices decorated with G_c^i and $G_{c'}^{i-1}$ if and only if G_c^i is a high subgraph of $G_{c'}^{i-1}$.

In order to represent the Gallavotti–Nicolò tree, it is convenient to adopt a phase space representation with positions on the horizontal axis and scales on the vertical axis. The phase space representation of the graph in Figure 1 and its Gallavotti–Nicolò tree can be found in Figure 2.



a) Phase space representation. b) Gallavotti–Nicolò tree.

FIGURE 2. Phase space representation of the graph of Figure 1 and the associated Gallavotti–Nicolò tree

High subgraphs are partially ordered by inclusion. An essential result is that they form a (Zimmermann) forest in the following sense.

Lemma 2.8. *Let (G, μ) be a fixed graph and scale assignment. The set of high subgraphs is a forest, in the sense that, if g_1 and g_2 are both high, we have either $g_1 \subset g_2$, or $g_2 \subset g_1$, or $g_1 \cap g_2 = \emptyset$.*

Proof. Suppose we find S_1 and S_2 with a non-trivial intersection; in this case S_1 would have an external edge in S_2 and conversely; but the scale of any of these two edges must be both strictly larger and strictly smaller than the other, which is impossible. □

Usually, the final graph G is connected, and this inclusion forest of high subgraphs forms a tree which is nothing else but the celebrated "Gallavotti–Nicolò" tree [19].

3. Hopf algebras on assigned graphs and the combinatorics of multi-scale renormalization

3.1. **Some algebra.** In this section we briefly recall, following [34], the definitions of the algebraic notions that will be used in the sequel.

Definition 3.1 (Algebra)**.** A unital associative algebra $\mathcal{A}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-linear space endowed with two algebra homomorphisms:

- a product $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ satisfying the *associativity* condition

$$m \circ (m \otimes \mathrm{id})(G) = m \circ (\mathrm{id} \otimes m)(G), \text{ for all } G \in \mathcal{A}^{\otimes 3}, \tag{3.1}$$

- a unit $u : \mathbb{K} \to \mathcal{A}$ satisfying

$$m \circ (u \otimes \mathrm{id})(1 \otimes G) = G = m \circ (\mathrm{id} \otimes u)(G \otimes 1), \text{ for all } G \in \mathcal{A}. \tag{3.2}$$

Definition 3.2. A (coassociative, counital) *coalgebra* $\mathcal{C}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-linear space endowed with two linear homomorphisms:

- a coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ satisfying the *coassociativity* condition

$$(\Delta \otimes \mathrm{id}) \circ \Delta(G) = (\mathrm{id} \otimes \Delta) \circ \Delta(G), \text{ for all } G \in \mathcal{C}, \tag{3.3}$$

- a counit $\varepsilon : \mathcal{C} \to \mathbb{K}$ satisfying

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(G) = G = (\mathrm{id} \otimes \varepsilon) \circ \Delta(G), \text{ for all } G \in \mathcal{C}. \tag{3.4}$$

Definition 3.3. A *bialgebra* $\mathcal{B}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-linear space endowed with both an algebra and a coalgebra structure (see Definitions 3.1 and 3.2) such that the coproduct and the counit are unital algebra homomorphisms (or, equivalently, the product and unit are coalgebra homomorphisms):

$$\Delta \circ m_{\mathcal{B}} = m_{\mathcal{B}\otimes\mathcal{B}} \circ (\Delta \otimes \Delta), \ \Delta(1_{\mathcal{B}}) = 1_{\mathcal{B}} \otimes 1_{\mathcal{B}}, \tag{3.5a}$$

$$\varepsilon \circ m_{\mathcal{B}} = m_{\mathbb{K}} \circ (\varepsilon \otimes \varepsilon), \ \varepsilon(1_{\mathcal{B}}) = 1. \tag{3.5b}$$

Definition 3.4. A *graded bialgebra* is a bialgebra graded as a linear space,

$$\mathcal{B} = \bigoplus_{n=0}^{\infty} \mathcal{B}^{(n)}, \tag{3.6}$$

such that the grading is compatible with the algebra and coalgebra structures:

$$\mathcal{B}^{(n)}\mathcal{B}^{(m)} \subseteq \mathcal{B}^{(n+m)} \text{ and } \Delta\mathcal{B}^{(n)} \subseteq \bigoplus_{k=0}^{n} \mathcal{B}^{(k)} \otimes \mathcal{B}^{(n-k)}. \tag{3.7}$$

Definition 3.5. A *connected bialgebra* is a graded bialgebra $\mathcal{B}$ for which $\mathcal{B}^{(0)} = u(\mathbb{K})$.

Definition 3.6. A *Hopf algebra* $\mathcal{H}$ over a field $\mathbb{K}$ is a bialgebra over $\mathbb{K}$ equipped with an antipode map $S : \mathcal{H} \to \mathcal{H}$ obeying

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{3.8}$$

We end this section by recalling the following result.

Lemma 3.7 ([24])**.** *Any connected graded bialgebra is a Hopf algebra whose antipode is given by* $S(1_{\mathcal{H}}) = 1_{\mathcal{H}}$ *and, recursively, by any of the two following formulas for* $G \neq 1_{\mathcal{H}}$*:*

$$S(G) = -G - \sum_{(G)} S(G')G'', \tag{3.9a}$$

$$S(G) = -G - \sum_{(G)} G'S(G''), \tag{3.9b}$$

where we used Sweedler's notation.

It turns out that commutative Hopf algebras naturally lead to a group structure on the space of characters.

Definition 3.8. A character of a commutative Hopf algebra is a linear map α from $\mathcal{H}$ to the ground field $\mathbb{K}$ such that $\alpha(G_1G_2) = \alpha(G_1)\alpha(G_2)$ for any $G_1, G_2 \in \mathcal{H}$.

The group structure on the set of characters is given by the convolution product.

Proposition 3.9. *The set* G *of characters of* $\mathcal{H}$ *is a group for the multiplication law*

$$\alpha * \beta = (\alpha \otimes \beta)\Delta \tag{3.10}$$

with inverse $\alpha^{-1*} = \alpha \circ S$ *and unit* ϵ*.*

For graded connected Hopf algebras, characters form a Lie group, whose Lie algebra is made of infinitesimal characters, defined as follows.

Definition 3.10. An infinitesimal character δ is a linear map from $\mathcal{H}$ to $\mathbb{K}$ such that $\delta(G_1G_2) = \epsilon(G_1)\delta(G_2) + \delta(G_1)\epsilon(G_2)$ for any $G_1, G_2 \in \mathcal{H}$.

Infinitesimal characters define a Lie algebra $\mathcal{G}$ with bracket $[\delta_1, \delta_2] = \delta_1 * \delta_2 - \delta_2 * \delta_1$. This Lie algebra is the Lie algebra of the group of characters. The correspondence between $\mathcal{G}$ and G is given by the convolution exponential

$$\alpha = \exp_*(\delta) = \sum_n \frac{\overbrace{\delta * \cdots * \delta}^{n \text{ times}}}{n!}$$

$$\text{if and only if} \quad \delta = \log_* = \sum_{n \leq 1} (-1)^{n-1} \frac{\overbrace{(\alpha - \epsilon) * \cdots * (\alpha - \epsilon)}^{n \text{ times}}}{n}.$$

The Hopf algebra $\mathcal{H}$ can be understood as the algebra of functions from G to $\mathbb{K}$.

For further details on this topic, the interested reader is referred to [23] or [12], for example.

3.2. **Hopf algebra structures on the Gallavotti–Nicolò trees.**

Proposition 3.11. *Let* $T_{(G,\mu)}$ *be the Gallavotti–Nicolò tree associated with the assigned graph* (G, μ)*.*

1. *The root of* $T_{(G,\mu)}$ *is decorated with* G *itself.*
2. *The leaves of* $T_{(G,\mu)}$ *all are at distance* ρ *from the root, with* ρ *the ultraviolet cutoff.*
3. *If the scales* j *and* k *are such that the scales* i *with* $j \leq i \leq k$ *do not appear in* (G, μ)*, then any vertex whose distance from the root is* i*, with* $j \leq i \leq k$*, is of degree 2.*

Proof. The items above follow as a direct consequence of the definition of the Gallavotti–Nicolò trees given in Definition 2.7. □

In order to define the Hopf algebra underlying multi-scale renormalization on the Gallavotti–Nicolò trees, it is useful to introduce the following terminology. If T' is a subtree of $T_{(G,\mu)}$, we define its completion $\overline{T'} = T_{(G',\mu')}$ to be the Gallavotti–Nicolò tree associated with its root (G', μ'). Furthermore, we define an admissible cut C to be a non-empty subset of $|C|$ arrows of $T_{(G,\mu)}$ that join vertices decorated by two different graphs, the graph farther from the root having two of four external edges and such that any path from the leaves to the root contains at most one arrow in C. Removing the arrows in C, we get a subtree $T_<$ that contains the root and trees $T^n_>$ that do not contain the root.

Proposition 3.12. *The free commutative algebra $\mathcal{H}_{\mathrm{GN}}$ generated by all Gallavotti–Nicolò trees is a graded Hopf algebra whose counit and coproduct are defined on generators T by $\epsilon(T) = 0$ and*

$$\Delta(T) = T \otimes 1 + 1 \otimes T + \sum_{\substack{C \\ \text{admissible cut}}} \Big(\prod_{1 \le n \le |C|} \overline{T^n_>} \Big) \otimes \overline{T_<}. \tag{3.11}$$

Its grading is given by

$$n(T) = \#\{\text{arrows joining vertices decorated with different graphs}\} + 1.$$

Remark 3.13. Let us observe that the number $n(T)$ above is simply the number of vertices, counting those decorated by the same graph only once per connected component.

Proof. The only non-trivial assertions to check are the coassociativity of the coproduct and the existence of the grading and the antipode. The proof of coassociativity is analogous to the proof of the coassociativity of the coproduct in the algebra of rooted trees, see [8]. The assertion pertaining to the grading is easy to check as any cut reduces the number of arrows joining vertices with different graphs by the number of cut edges. Finally, for any graded commutative bigebra there is a recursive construction of the antipode, as given in [18]. □

3.3. **Operations on assigned graphs.** In this section, we define several operations which we need in the rest of the paper.

We define $\underline{(G,\mu)}$ to be the set of assigned graphs formed by high subgraphs of the assigned graph (G,μ) whose connected components are 1PI and have two or four external edges. An external edge of a subgraph is an edge of G attached to a vertex in g which is not an internal edge of g.

Definition 3.14. Let (g,ν) be a two- or a four-point assigned subgraph inside an assigned graph (G,μ). The *shrinking* of (g,ν) inside (G,μ) is obtained by shrinking the subgraph g inside the Feynman graph G in the usual QFT way, i.e., each connected component of the subgraph is replaced by a vertex (the internal structure of g vanishes); we obtain the cograph G/g. The scale assignment μ/ν of the cograph G/g is given by the initial scale assignment μ, where we have erased the scale assignment of the internal edges of g. If two external edges are added when shrinking a two-point function, they are always assigned a scale which is lower than any internal scale. We call the resulting assigned graph $(G/g, \mu/\nu)$ an assigned cograph.

Remark 3.15. The shrinking operation corresponds to the wave function or mass renormalization for a two-point subgraph, or to the coupling constant renormalization for a four-point subgraph. In the case of a wave-function renormalization, a decoration indicating the two derivative couplings of the Laplacian must be added to the shrinked two-point vertex to distinguish that renormalization from the mass renormalization.

Definition 3.16. The gluing data $\circ$ for the insertion of a two-point (respectively four-point) assigned graph (g,ν) on an edge e (respectively vertex v) of an assigned graph (G,μ) is given by a bijection between the external edges of g and the two half-edges of G defining e (respectively four have edges of G incident to v). It is defined only if the external assignment indices for (g,ν) coincide with the internal indices of the corresponding edges of (G,μ). In that case, the scale assignment of the resulting graph is obtained in the following way. The scale assignment for the internal edges of g is given by ν; the scale assignment for the external edges of g, identified through this operation with the internal edges of G, is given by their common value in (G,μ) and (g,ν).

3.4. **The assigned graph combinatorial Hopf algebra.** In this section, we define a Hopf algebra on assigned Feynman graphs. Subsequently, we exhibit the relation between this structure and the combinatorics of multi-scale renormalization.

Consider the unital associative algebra $\mathcal{H}$ freely generated by the assigned graphs, including the empty assigned graph, which we denote by $1_{\mathcal{H}}$.

The product $m((g_1,\mu_1),(g_2,\mu_2)) = (g,\mu)$ is given by the operation of disjoint union of assigned graphs. This means that the resulting 1PI Feynman graph g is given by the disjoint union of graphs, and each disjoint component g_i keeps its scale assignment μ_i, $i = 1,2$, the latter providing the resulting scale assignment μ. As in the case of the Connes–Kreimer product, this product is bilinear and commutative.

As we have already mentioned in Section 2.2, the integers of the scale assignment μ are bounded by some integer cutoff ρ. We have $\mathcal{H}_\rho \subset \mathcal{H}_{\rho+1} \subset \cdots \subset \mathcal{H}_\infty$. Since we do not deal here with distinct cutoffs, we abbreviate $\mathcal{H}_\rho$ as $\mathcal{H}$ in the rest of the paper.

We define the coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ as

$$\Delta(G,\mu) = (G,\mu) \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes (G,\mu) + \sum_{(g,\nu)\subset\underline{(G,\mu)}} (g,\nu) \otimes (G/g, \mu/\nu). \tag{3.12}$$

Note that, in this definition, the high subgraphs g are not necessarily connected. The coproduct can be written explicitly as

$$\Delta(G,\mu) = (G,\mu) \otimes 1 + \sum_{\substack{(g_i,\nu_i)\subset(G,\mu) \\ g_i \cap g_i = \emptyset}} \Big(\prod_i (g_i,\nu_i) \otimes (G,\mu) \Big) \Big/ \prod_i (g_i,\nu_i) + 1 \otimes (G,\mu), \tag{3.13}$$

where the sum runs over divergent and disjoint high 1PI subgraphs, excluding G itself.

In order to illustrate the definition of the coproduct, let us list all the connected 1PI high superficially divergent subgraphs (i.e., high 2- or 4-point subgraphs) of the graph of Figure 1:

$$\{1,2,3,4\}, \{7,8,9,10\}, \{3,4\}, \{7,8\}$$

Therefore, omitting the explicit expression of the scale assignment, the coproduct reads,

$$\begin{aligned}\Delta(G) = G \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes G + \{1,2,3,4,7,8,9,10\} \otimes G/\{1,2,3,4,7,8,9,10\} \\ + \{1,2,3,4\} \otimes G/\{1,2,3,4\} + \{7,8,9,10\} \otimes G/\{7,8,9,10\} \\ + \{1,2,3,4,7,8\} \otimes G/\{1,2,3,4,7,8\} + \{3,4,7,8,9,10\} \otimes G/\{3,4,7,8,9,10\} \\ + \{3,4\} \otimes G/\{3,4\} + \{7,8\} \otimes G/\{7,8\} + \{3,4,7,8\} \otimes G/\{3,4,7,8\}\end{aligned}$$

For example, the reduced graph $G/\{7,8,9,10\}$ is illustrated in Figure 3.

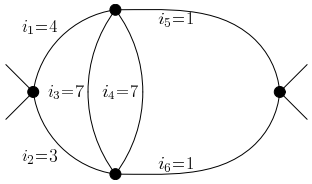


FIGURE 3. The reduced graph $G/\{7,8,9,10\}$

Observe that the vector space $\mathcal{H}$ is graded, as in the usual Connes–Kreimer case, by the number of independent loops, number of edges, or by the number of vertices minus one.

Let us recall the following result, which holds true for the combinatorial Connes–Kreimer Hopf algebra of Feynman graphs.

Lemma 3.17 (LEMMA 3.2 OF [34])**.** *Let G be a 1PI Feynman graph. Provided*

1. *for all $G \in \underline{G}$, $G' \in \underline{G}$, the graph G/G' is superficially divergent,*
2. *for all G_1, G_2 such that G_1 and G_2 are superficially divergent, there exist gluing data such that $(G_1 \circ G_2)$ is superficially divergent,*

the coproduct is coassociative,

$$\Delta G = G \otimes 1 + 1 \otimes G + \Delta' G, \tag{3.14a}$$

$$\Delta' G = \sum_{g \in \underline{G}} g \otimes G/g, \tag{3.14b}$$

where we have denoted by 1 the empty graph (the unit of the vector space freely generated by 1PI Φ^4 Feynman graphs). Moreover, the notation $\underline{G}$ stands for the set of superficially divergent subgraphs of G (i.e., the two- and four-point subgraphs of G, which are not necessarily connected but with connected components which are 1PI).

This result (naturally) extends to assigned graphs.

Lemma 3.18. *Let (G, μ) an assigned graph. Provided*

1. *for all $(g, \nu) \in \underline{(G, \mu)}$, $(g', \nu') \in \underline{(g, \nu)}$, the assigned cograph $(G/G', \nu/\nu')$ is a two- or four-point high assigned graph,*
2. *for all $(g_1, \nu_1), (g_2, \nu_2)$ such that (g_1, ν_1) and (g_2, ν_2) are two- or four-point high assigned graphs, there exists gluing data such that $(g_1 \circ g_2, \nu_1 \circ \nu_2)$ is a two- or four-point high assigned graph*

the coproduct given by formula (3.12) is coassociative.

Proof. The proof of Lemma 3.2 of [34] generalizes in a straightforward manner. □

Furthermore, we define the counit $\varepsilon : \mathcal{H} \to \mathbb{K}$ as

$$\varepsilon(1_{\mathcal{H}}) = 1,\ \varepsilon((G,\mu) = 0,\ \forall (G,\mu) \neq 1_{\mathcal{H}}. \tag{3.15}$$

Finally, the antipode is given recursively by

$$\begin{aligned} S : \mathcal{H} &\to \mathcal{H} \\ (G,\mu) &\mapsto -(G,\mu) - \sum_{(g,\nu)\in \underline{(G,\mu)}} S((g,\nu))(G/g, \mu/nu). \end{aligned} \tag{3.16}$$

This antipode can be computed as the inverse of the identity map for the convolution product. In the case of the Connes–Kreimer Hopf algebra of trees, this was done in [18]. For the graph algebra defined here, an analogous computation leads to

$$S(G,\mu) = \sum_{n=1}^{L} \sum_{i} c_{n-1}^{(i)} (-(G_1^{(i)}, \mu_1^{(i)})) \cdots (-(G_n^{(i)}, \mu_n^{(i)})), \tag{3.17}$$

where

$$\Delta'^{n}(G,\mu) = \sum_{i} c_n^{(i)} (G_1^{(1)}, \mu_1^{(i)}) \otimes \cdots \otimes (G_{n+1}^{(i)}, \mu_{n+1}^{(i)}), \tag{3.18}$$

and the $c_n^{(i)}$'s are the appropriate combinatorial coefficients obtained from the explicit coproduct computation. Observe that, from the explicit definition of the unit and of the counit map, one can prove that the sum over n in (3.17) has a finite number of terms, equal to the number of independent cycles of the respective graph, denoted here by L. The non-recursive formula (3.17) is then equivalent to the sum over Zimmermann forests of high superficially divergent graphs.

We can now state the main result of this section.

Theorem 3.19. *The quadruple $(\mathcal{H}, \Delta, \varepsilon, S)$ is a Hopf algebra.*

Proof. We first prove the coassociativity of the coproduct (3.12), using Lemma 3.18. Let us first check the first condition of this lemma. The fact that the resulting cograph has two or four external edges (the only thing to check in the usual Connes–Kreimer case) is trivial (since the shrinking does not affect the external structure of g, see Definition 3.14).

Let us now check in detail how the situation stands for the scales assignments. We denote by $i_{g'}$ the minimum of the scale assignments of the edges of g', and by $e_{g'}$ the maximum of the scale assignments of the edges of g'. Similarly, we denote by i_g the minimum of the scale assignments of the edges of g, and by e_g the maximum of the scale assignments of the edges of g. We also denote by $i_{g/g'}$ the minimum of the scale assignments of the edges of the cograph g/g', and by $e_{g/g'}$ the maximum of the scale assignments of the external edges of the cograph g/g'.

Since the external edges of g' are internal edges of g, using Definition 3.14, this means that

$$i_{g/g'}(\mu) > e_{g/g'}(\mu) = e_g(\mu), \tag{3.19}$$

because, as mentioned above, the shrinking does not affect the external structure of g. We have thus checked the first condition of Lemma 3.18.

The second condition of Lemma 3.18 is checked similarly, using Definition 3.16. This concludes the proof of the coassociativity of the coproduct. Since $\mathcal{H}$ is graded (see above),

connected, coassociativity of the coproduct (3.12), Definition (3.16) of the antipode, and Lemma 3.7 lead to the result. □

Let us notice that another way to prove the coassociativity (in a direct manner) is by the simple use of the isomorphism with the span of Gallavotti–Nicolò forests (see the following subsection).

Just as in the Connes–Kreimer case, there is a straightforward pre-Lie algebra structure, given by the operation of insertion of assigned graphs. Antisymmetrization of this operation leads to a Lie algebra of assigned graphs. Consider now the graded dual of the universal enveloping algebra of this Lie structure. This gives the renormalization Hopf algebra defined in this section.

3.5. **Combinatorial Hopf algebras homomorphisms.** We observe that the Gallavotti–Nicolò tree algebra is isomorphic to the algebra $\mathcal{H}$.

Proposition 3.20. *The algebra homomorphism $\pi : \mathcal{H} \to \mathcal{H}_{\mathrm{GN}}$ defined on the generators by $\pi_{\mathrm{GN}}(G,\mu) = T_{(G,\mu)}$ is a Hopf algebra isomorphism.*

Proof. The proof is done by a direct verification. □

On the other hand, the Hopf algebra of Gallavotti–Nicolò trees is a refinement of the Hopf algebra of rooted trees $\mathcal{H}_{\mathrm{RT}}$, as defined in [8]. Indeed, for any Gallavotti–Nicolò tree T, let us define $\widetilde{T}$ as the rooted tree obtained by contracting all the arrows joining vertices decorated with the same graphs and removing all the decorations.

Proposition 3.21. *The algebra homomorphism defined on the generators of $\mathcal{H}_{\mathrm{GN}}$ by $\pi_{\mathrm{RT}}(T) = \widetilde{T}$ extends to a surjective Hopf algebra homomorphism from $\mathcal{H}_{\mathrm{GN}}$ to $\mathcal{H}_{\mathrm{RT}}$.*

Proof. The proof is done by a direct verification. □

In [9], a graph renormalization Hopf algebra $\mathcal{H}_{\mathrm{CK}}$ was introduced. The relation between this Hopf algebra and the one presented here is the following.

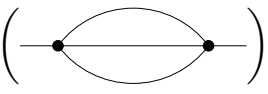
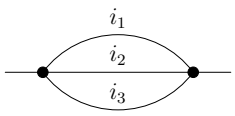
Proposition 3.22. *For every $\rho \in \mathbb{Z}_+$, the algebra homomorphism defined on the generators of $\mathcal{H}_{\mathrm{CK}}$ by*

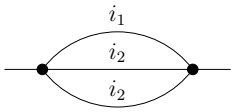
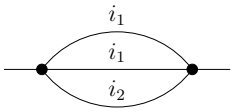
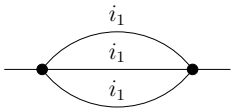
$$\pi^{\rho}_{\mathrm{CK}}(G) = \sum_{|\mu| \leq \rho} (G, \mu) \tag{3.20}$$

extends to a Hopf algebra homomorphism from $\mathcal{H}_{\mathrm{CK}}$ to $\widetilde{\mathcal{H}}$, where $\widetilde{\mathcal{H}}$ is identical to $\mathcal{H}$ as an algebra but equipped with a coproduct that extracts all assigned graphs with 2 or 4 external edges, not only high subgraphs.

Proof. The proof is done by a direct calculation. □

For example, for the "sunset" graph below, the homomorphism formula above leads to:

π_{CK}  $= 6 \sum_{0 \leq i_1 < i_2 < i_3 \leq \rho}$ 

$+ 3 \sum_{0 \leq i_1 < i_2 \leq \rho}$  $+ 3 \sum_{0 \leq i_1 < i_2 \leq \rho}$  $+ \sum_{0 \leq i_1 \leq \rho}$ 

Remark 3.23. This homomorphism allows us to evaluate Feynman amplitudes with cut-off ρ as

$$A^\rho(G) = A \circ \pi^\rho_{\mathrm{CK}}(G). \tag{3.21}$$

4. Multi-scale renormalization combinatorics

4.1. **Multi-scale forest formula as a Hopf coaction.** We now exhibit the relation between the Hopf algebra of the previous subsection and the combinatorics of multi-scale renormalization. This extends the relation between the Connes–Kreimer Hopf algebra and renormalization to multi-scale renormalization.

We first recall the Feynman rules. Given an assigned graph (G, μ), we associate a space-time variable in $\mathbb{R}^4$ to each vertex and a covariance $C^i(x_{s(l)}, y_{t(l)})$ to an edge with scale i joining the vertices $s(l)$ and $t(l)$. Then we integrate over all space-time variables but the ones attached to external edges to define the unrenormalized amplitude,

$$A(G,\mu) = \int \prod_{\text{internal vertices}} dx_v \prod_{\text{internal edges}} C^i(x_{s(l)}, y_{t(l)}). \tag{4.1}$$

These are the Feynman rules formulated in position space, as is usual in multi-scale analysis. Obviously, the evaluation of a disconnected graph is the product of the evaluation of its connected components, so that the evaluation map $A : \mathcal{H} \to \mathcal{A}$ which sends an assigned graph to its value $A(G, \mu)$, with $\mathcal{A}$ a suitable commutative algebra depending on the variables $x_1, \ldots, x_n$ attached to the external edges, is a character. The connected n-point correlation functions with cut-off ρ are computed as a sum over all connected Feynman graphs with n external edges and scales less than ρ,

$$W(x_1, \ldots, x_n) = \sum_{\substack{G \text{ connected assigned graph} \\ \text{with } n \text{ external edges}}} \sum_{|\mu| \leq \rho} A(G,\mu)[x_1, \ldots, x_n] \, \frac{(-\lambda)^{v(G)}}{\sigma(G,\mu)}, \tag{4.2}$$

with λ the coupling constant and $\sigma(G, \mu)$ the symmetry factor of the assigned graph (G, μ) (cardinality of the automorphism group of G preserving the scale assignment).

In order to simplify the analytic discussion and focus on combinatorics, from now on we restrict ourselves to the class of ϕ^4 graphs with at least four external edges that do not contain any non-trivial subgraph with two external edges. Following the terminology of [27], we call these graphs "biped-free graphs". Then, the multi-scale renormalization of biped-free graphs can be formulated in terms of Hopf algebras as follows.

We define the (useful) counterterms C_{U} recursively by

$$C_{\mathrm{U}}(G,\mu) = -\tau A(G,\mu) - \sum_{(G',\mu') \in \underline{(G,\mu)}} \tau A(G/G', \mu/\mu') \, C_{\mathrm{U}}(G',\mu'), \tag{4.3}$$

where the sum runs over all, not necessarily connected, high subgraphs whose connected components are biped-free quadrupeds (where a "quadruped" means a graph with exactly four external edges). A function of n variables $F(x_1, \ldots, x_n)$ that is invariant under translation, that is $F(x_1 + a, \ldots, x_n + a) = F(x_1, \ldots, x_n)$ for any $a \in \mathbb{R}^D$, thus becomes a function of only $n - 1$ variables $F(x_1 - x_n, x_2 - x_n, \ldots, x_n - x_n)$. We define τF as

$$\tau F = \int_{(\mathbb{R}^4)^{n-1}} dx_1 \cdots dx_{n-1} F. \tag{4.4}$$

We may integrate over any subset of $n-1$ variables to define τF, and the result is a constant.

We observe that $C_{\mathrm{U}}(G,\mu)$ does not depend on the space-time variable x attached to the external edges because of translation invariance, so that it is a constant function. It is also easy to see that it is multiplicative over disjoint unions and thus defines a character of $\mathcal{H}$. For graphs with bipeds, τ involves a Taylor expansion at order 2 so that the construction is more involved.

Let $\mathcal{K}_n$ be the vector space spanned by connected assigned Feynman graphs with n labelled external edges, and denote by $\mathcal{K}$ the direct sum $\mathcal{K}=\bigoplus_n \mathcal{K}_n$. We define a linear map $\underline{\Delta}:\mathcal{K}\to\mathcal{H}\otimes\mathcal{K}$ by

$$\underline{\Delta}(G,\mu)=(\overline{G},\mu)\otimes 1+\sum_{\substack{(g_i,\nu_i)\neq(G,\mu)\subset(G,\mu)\\ g_i\cap g_j=\emptyset}}\Big(\prod_i(g_i,\nu_i)\otimes(G,\mu)\Big)\Big/\prod_i(g_i,\nu_i), \tag{4.5}$$

where the sum runs over divergent and disjoint high subgraphs, and $\overline{G}$ is the Feynman graph obtained from G by erasing the labels on the external edges. $\underline{\Delta}$ is a Hopf coaction of $\mathcal{H}$ on $\mathcal{K}$, i.e., $m\circ(\epsilon\otimes\mathrm{id})\circ\delta=\mathrm{id}$ and $(\mathrm{id}\otimes\underline{\Delta})\circ\delta=(\Delta\otimes\mathrm{id})\circ\underline{\Delta}$.

Furthermore, let $\mathcal{D}_n$ be a vector space of suitable distributions on $(\mathbb{R}^D)^n$ in which the Feynman amplitudes take their values, and let $\mathcal{V}_n$ be the space of linear maps from $\mathcal{K}_n$ to $\mathcal{D}_n$. The previous coaction allows us to define an action of the group of characters of $\mathcal{H}$ on $\mathcal{V}=\bigoplus_n\mathcal{V}_n$ by $\alpha\cdot f=m\circ(\alpha\otimes f)\circ\underline{\Delta}$.

The multi-scale renormalization of biped-free graphs (see above) can then be formulated in terms of Hopf algebras by the following proposition. In the context of multiscale renormalization, only some counterterms are needed. These are called "useful counterterms" [27].

Proposition 4.1. *The usefully renormalized biped-free amplitudes are obtained as*

$$A_{UR}=C_U\cdot A, \tag{4.6}$$

where the useful counterterms are defined by

$$C_U=(\tau A)^{-1*}=(\tau A)\circ S. \tag{4.7}$$

This reformulation of multi-scale renormalization simply relies on the fact that

$$S(G,\mu)=\sum_{\mathcal{F}}\prod_{(G_i,\mu_i)\in\mathcal{F}}\big(-(G_i,\mu_i)\big), \tag{4.8}$$

where the sum runs over all dangerous forests, i.e., forests made of high subgraphs with four external edges.

Remark 4.2. It is crucial that τA is a character, which follows immediately from the fact that it is a constant function. For graphs with bipeds, τ involves a Taylor expansion at order 2 so that the construction is more involved and requires the introduction of new vertices of degree 2.

Remark 4.3. Let us emphasize that (4.8) is not the usual BPHZ forest formula. Indeed, the latter involves a sum over all forests, whereas the forests $\mathcal{F}$ in (4.6) are such that, if $G\subset G'$ in $\mathcal{F}$, then all edges of G have higher scales than those of G (the high scale condition, see above). The extra forests appearing in the BPHZ formula lead to new divergencies, called "renormalons," that no longer affect individual Feynman amplitudes

but the convergence or Borel summability of the power series in the coupling constant as a whole. This "renormalon" problem is cured by the multi-scale expansion at the expense of using multiple coupling constants known as the effective coupling constants, as discussed in the next subsection (see [27] for more details on this issue).

4.2. **Effective expansion.** The unrenormalized n-point connected correlation functions are expressed as a sum over connected Feynman graphs with n labelled external edges. Thus, the bare correlation functions read

$$A_{\mathrm{bf}}(x_1,\dots,x_n) = \sum_{\substack{(G,\mu),|\mu|\leq\rho \\ n \text{ labelled external edges}}} \frac{A(G,\mu)[x_1,\dots,x_n]}{\sigma(G,\mu)} (\lambda_\rho)^{v(G)}, \tag{4.9}$$

with $v(G)$ the number of vertices of G and $\sigma(G,\mu)$ its symmetry factor. Observe that we sum over assigned graph whose scales are bounded by ρ. The latter plays the role of an ultraviolet cut-off and we are ultimately interested in the limit $\rho\to\infty$. Because of the divergence of the sum over scales in (4.9) when $\rho\to\infty$, one has to renormalize the Feynman graph amplitudes and expand the correlation functions into powers of the renormalized coupling constant, conventionally denoted by λ_{-1}. Then, the correlation functions become

$$A_{\mathrm{bf}}(x_1,\dots,x_n) = \sum_{\substack{(G,\mu),|\mu|\leq\rho \\ n \text{ labelled external edges}}} \frac{A_{\mathrm{R}}(G,\mu)[x_1,\dots,x_n]}{\sigma(G,\mu)} (\lambda_{-1})^{v(G)}. \tag{4.10}$$

Here, $A_{\mathrm{R}}(G,\mu)[x_1,\dots,x_n]$ denotes the renormalized Feynman graph amplitude, involving a sum over all forests, not only those made of high subgraphs. The renormalized coupling constant λ_{-1} is computed as a sum over all graphs with four external legs,

$$\lambda_{-1}(\lambda_\rho) = \lambda_\rho + \sum_{\substack{(G,\mu),\,|\mu|\leq\rho,\,i_G(\mu)>i \\ \text{biped-free with four external edges}}} \frac{N(G,\mu)}{\sigma(G,\mu)}\,\tau A(G,\mu)\,(\lambda_\rho)^{v(G)}. \tag{4.11}$$

As usual, $\sigma(G,\mu)$ is the symmetry factor (cardinality of the automorphism group), while $N(G,\mu)$ is the number of inequivalent labellings of the external edges. These numbers do not depend on the scale assignment since they involve transformations that preserve the latter.

However as discussed above, the renormalization group formalism requires to expand the correlation functions not in a single coupling constant λ_{-1}, but in a series of $\rho+2$ effective coupling constants $\lambda_\rho,\lambda_{\rho-1},\dots,\lambda_{-1}$, one for each slice. This sequence interpolates between the bare coupling λ_ρ and the renormalized one λ_{-1}. This is formulated in the context of Hopf algebras as follows.

For every character α of $\mathcal{H}$, let define $\rho+2$ formal power series in $\rho+2$ variables by

$$\lambda_i'(\lambda_\rho,\dots,\lambda_{-1}) = \lambda_i + \sum_{\substack{(G,\mu),\,|\mu|\leq\rho,\,i_G(\mu)>i \\ \text{biped-free with four external edges}}} \frac{N(G,\mu)}{\sigma(G,\mu)}\,\alpha(G,\mu) \prod_{v\,\text{vertex}} \lambda_{e_v(\mu)}, \quad i\in\{-1,\dots,\rho\}, \tag{4.12}$$

where we recall that $i_G(\mu)$ is the lowest scale of the internal edges of (G,μ) and $e_v(\mu)$ the highest scale on the edges attached to v in (G,μ). The inclusion of this combinatorial factor is necessary because the graphs in (4.12) do not carry labels on their external edges.

In particular, we always have $\lambda'_\rho = \lambda_\rho$ since there are no assigned graphs with $i_G(\mu) > \rho$, while λ'^{-1} involves a sum over all assigned graphs.

Theorem 4.4. *The map $\Psi : G \to G_{formal}$ associating the formal power series $\lambda'_i(\lambda_\rho, \ldots, \lambda_{-1})$ with the character α is a group antihomomorphism from the group of characters G of $\mathcal{H}$ to the group of invertible formal power series in $\rho + 2$ variables*

$$\Psi(\beta) \circ \Psi(\alpha) = \Psi(\alpha * \beta). \tag{4.13}$$

Proof. Let us first notice that it is sufficient to prove the result at the Lie algebra level. Indeed, any character of $\mathcal{H}$ can be written in a unique way as the convolution exponential of an infinitesimal character in the Lie algebra of the group of characters (see Section 3.1). Therefore, there are infinitesimal characters δ and η such that $\alpha = \exp_* \delta$ and $\beta = \exp_* \eta$. Then, the group homomorphism follows from the integration of the Lie algebra homomorphism using the Campbell–Baker–Hausdorff formula. Since, for infinitesimal characters in the Lie algebra, the group antiautomorphism relation is linear, it is sufficient to check it for characters with $\delta(G, \mu) = 1$ if $(G, \mu) = (G_1, \mu_1)$, $\eta(G, \mu) = 1$ if $(G, \mu) = (G_2, \mu_2)$, and which vanish otherwise.

At the infinitesimal level, the relation (4.13) reads

$$\begin{aligned}\sum_G \frac{N(G,\mu)}{\sigma(G,\mu)} N((G_1,\mu_1),(G_2,(\mu)\mu_2),(G,\mu)) \prod_{v\in V(G)} \lambda_{e_v(\mu)} \\ = \frac{N(G_1,\mu_1)}{\sigma(G_1,\mu_1)} \Bigg(\prod_{v\in V(G_1)} \lambda_{e_v} \Bigg) \sum_{i_{G_1}(\mu_1)>i} \frac{\partial}{\partial \lambda_i} \Bigg(\frac{N(G_2,\mu_2)}{\sigma(G_2,\mu_2)} \prod_{v\in V(G_2)} \lambda_{e_v(\mu_2)} \Bigg),\end{aligned} \tag{4.14}$$

where all graphs are biped-free connected quadrupeds (G, μ) such that (G_1, μ_1) is a high subgraph of (G_2, μ_2), and $N((G_1, \mu), (G_2, \mu_2), (G, \mu))$ is the number of subgraphs of (G, μ) isomorphic to (G_1, μ_1) with $(G, \mu)/(G_1, \mu_1)$ isomorphic to (G_2, μ_2). The assertion of the theorem then follows from the combinatorial lemma given below. □

Lemma 4.5. *We have*

$$\sum_G \frac{N(G)}{\sigma(G)} N(G_1, G_2, G) = \frac{N(G_1)}{\sigma(G_1)} \frac{N(G_2)}{\sigma(G_2)} v(G_2) \tag{4.15}$$

Proof. To prove this lemma, first recall that $\frac{(4!)^{v(G)} v(G)!\, N(G)}{\sigma(G)}$ is the number of Wick contractions leading to the graph G in the expansion of the path integral (2.4), where $N(G)$ accounts for the number of labellings of the external edges. Then,

$$\sum_G \frac{(4!)^{v(G)} v(G)!\, N(G)}{\sigma(G)} N(G_1, G_2, G) = \frac{N(G_1)}{\sigma(G_1)} \frac{N(G_2)}{\sigma(G_2)} v(G_2)$$

is the number of Wick contractions leading to graphs G with a distinguished subgraph isomorphic to G_1 such that G/G_1 is isomorphic to G_2.

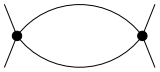
Equivalently, we can start with $v(G) = v(G_1) + v(G_2) - 1$ vertices and construct G_1. There are $\frac{(v(G_1)+v(G_2)-1)!}{v(G_1)!(v(G_2)-1)!}$ ways of choosing the vertices of G_1 and $\frac{(4!)^{v(G_1)} v(G_1)! N(G_1)}{\sigma(G_1)}$ Wick contractions leading to G_1. Next, we consider G_1 as a single vertex and construct G_2, which yields $\frac{(4!)^{v(G_2)-1} v(G_2)!\, N(G_2)}{\sigma(G_2)}$ Wick contractions leading to G_1. Observe that the counting involves $(4!)^{v(G_2-1)} v(G_2)!$ instead of $(4!)^{v(G_2)} v(G_2)!$ because of the labels of the external

edges of G_1. Accordingly,

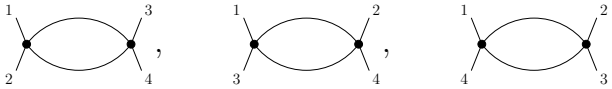
$$\sum_G \frac{(4!)^{v(G)} v(G)!\, N(G)}{\sigma(G)} N(G_1, G_2, G)$$
$$= \frac{(v(G_1)+v(G_2)-1)!}{v(G_1)!\,(v(G_2)-1)!} \frac{(4!)^{v(G_1)} v(G_1)!\, N(G_1)}{\sigma(G_1)} \frac{(4!)^{v(G_2)-1} v(G_2)!\, N(G_2)}{\sigma(G_2)},$$

which establishes the lemma. □

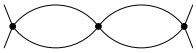
It is instructive to illustrate the combinatorics of the lemma in a simple example involving ordinary graphs. With two vertices, there is a single biped-free quadruped,


with $\sigma(\,) = \frac{1}{2}$ and $N(\,) = 3$.

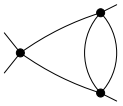
$N(\,) = 3$ corresponds to the following three inequivalent labellings of the external edges:



At order 3, we have two biped-free quadrupeds, namely


with $\sigma(\,) = \frac{1}{4}$, $N(\,) = 3$, and $N(\,,\,,\,) = 2$

and


with $\sigma(\,) = \frac{1}{2}$, $N(\,) = 6$ and $N(\,,\,,\,) = 1$.

In this case, the combinatorial lemma, Lemma (4.15), reads

$$\frac{3}{4} \times 2 + \frac{6}{2} = \frac{3}{2} \times \frac{3}{2} \times 2.$$

To alleviate the notations, we have proven this lemma for ordinary graphs, not for assigned ones. In the case of assigned graphs, all goes through except that we have to take the condition that (G_1, μ_1) is a high subgraph of (G_2, μ_2) into account, which restricts the possible insertions of (G_1, μ_1) into (G_2, μ_2).

Corollary 4.6. *As power series in λ_ρ, we have*

$$\sum_{\substack{(G,\mu), |\mu| \le \rho \\ n \text{ labelled external edges}}} \frac{A(G,\mu)[x_1, \dots, x_n]}{\sigma(G,\mu)} (\lambda_\rho)^{v(G)}$$
$$= \sum_{\substack{(G,\mu), |\mu| \le \rho \\ n \text{ labelled external edges}}} \frac{A_{UR}(G,\mu)[x_1, \dots, x_n]}{\sigma(G,\mu)} \prod_{v \text{ vertex}} \lambda_{e_v}(\mu), \quad (4.16)$$

where the effective couplings λ_i are computed using $\Psi(\tau A)$ evaluated at the bare coupling

$$\lambda_i(\lambda_\rho) = \lambda_\rho + \sum_{\substack{(G,\mu),\, |\mu| \le \rho,\, i_G(\mu) > i \\ \text{biped-free with four external edges}}} \frac{N(G,\mu)}{\sigma(G,\mu)} \tau A(G,\mu) (\lambda_\rho)^{v(G)}. \quad (4.17)$$

Proof. To derive this result, first compute the effective couplings λ_i in terms of λ_ρ using the homomorphism $\Psi(\tau A)$. Then, substitution of the effective couplings λ_i in terms of λ_ρ on the right-hand side amounts to an action of τA. However, the usefully renormalized amplitudes are precisely obtained by an action of the useful counterterms $C_{\mathrm{U}} = (\tau A)^{-1*}$. Thus, the action of $(\tau A)^{-1*}$ due to renormalization precisely cancels the action of τA due to the change of coupling constants. □

Remark 4.7. The counterterms defined by $C_{\mathrm{U}} = (\tau A)^{-1*}$ correspond to a given renormalization scheme which amounts to Taylor subtraction at zero momentum in Fourier space. This procedure renders the Feynman graph amplitude finite but this goal may be achieved by any other prescription. Indeed, at each step of the recursive definition of the counterterms, one can add a finite contribution $\alpha(G, \mu)$ to each Feynman graph amplitude. This amounts to applying the counterterm transformation $C_{\mathrm{U}} \to \alpha * C_{\mathrm{U}}$, which in turn may be compensated by the change of effective couplings induced by Ψ_α.

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