Box splines and lattice points in polytopes

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Séminaire Lotharingien de Combinatoire 70 – Ellwangen

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Variable polytopes

- $X = (x_1, \ldots, x_N) \subseteq \Lambda \cong \mathbb{Z}^d \subseteq U \cong \mathbb{R}^d$ list of vectors
- X totally unimodular and X spans U.

Definition (Variable polytopes)

$$\Pi_X(u) := \{\alpha \in \mathbb{R}^N_{\geq 0} : X\alpha = u\} \quad \text{and} \quad \Pi^1_X(u) := \Pi_X(u) \cap [0,1]^N$$

We assume $0 \notin \mathrm{conv}(X)$ for $\Pi_X(u)$.

Definition

$$\textit{box spline } B_X(u) := \frac{1}{\sqrt{\det(XX^T)}} \mathrm{vol}_{N-d}(\Pi^1_X(u))$$

$$\textit{multivariate spline } T_X(u) := \frac{1}{\sqrt{\det(XX^T)}} \mathrm{vol}_{N-d}(\Pi_X(u))$$

$$\textit{vector partition function } \mathcal{T}_X(u) := |\Pi_X(u) \cap \Lambda|$$

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Properties of the spline functions

Remark

- $\operatorname{supp}(B_X) = Z(X) := \left\{ \sum_{i=1}^N \lambda_i x_i : 0 \le \lambda_i \le 1 \right\}$ *zonotope*
- $\operatorname{supp}(T_X) = \operatorname{cone}(X) := \left\{ \sum_{i=1}^N \lambda_i x_i : 0 \le \lambda_i \right\}$ *cone*

Proposition (Dahmen-Micchelli, 1980s)

$$T_X = B_X *_d \mathcal{T}_X := \sum_{\lambda \in \Lambda} B_X(\cdot - \lambda) \mathcal{T}_X(\lambda)$$

Theorem (Khovanskii-Pukhlikov, 1992)

Let $u \in \Lambda$ and let p_Ω be the polynomial that agrees with T_X near u. Then

$$\left| \Pi_X(u) \cap \mathbb{Z}^d \right| = \mathcal{T}_X(u) = \operatorname{Todd}(X) p_\Omega(u).$$

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$\mathcal{P}$-spaces

- $u = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \rightsquigarrow p_u := \alpha_1 s_1 + \ldots + \alpha_d s_d \in \mathbb{R}[s_1, \ldots, s_d]$.
- $Y \subseteq X \rightsquigarrow p_Y := \prod_{y \in Y} p_y$.
- $Y = ((1,0),(1,2)) \rightsquigarrow p_Y = s_1^2 + 2 s_1 s_2$

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$$\textit{central } \mathcal{P}\textit{-space } \mathcal{P}(X) := \operatorname{span}\{p_Y : Y \subseteq X, \operatorname{rank}(X \setminus Y) = \operatorname{rank}(X)\}$$

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Example

- $X = ((1,0),(0,1),(1,1))$
- $\mathcal{P}(X) = \operatorname{span}\{1, s_1, s_2\}$
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The first theorem

$$p \in \mathbb{R}[s_1, \ldots, s_d] \rightsquigarrow p(D) := p(\tfrac{\partial}{\partial s_1}, \ldots, \tfrac{\partial}{\partial s_d})$$

Theorem (ML, conjectured by Holtz and Ron)

1. *$p(D)B_X$ is a continuous function for all $p \in \mathcal{P}_-(X)$.*
2. *Let z be an interior lattice point of the zonotope $Z(X)$. There exists a unique polynomial $p \in \mathcal{P}_-(X)$ s. t. $p(D)B_X$ equals 1 on z and 0 on the other interior lattice points.*

Example

- $X = (1, 1, 1)$
- $\mathcal{P}_-(X) = \mathrm{span}(1, s)$
- $q_1(s) = 1 + \frac{s}{2}$
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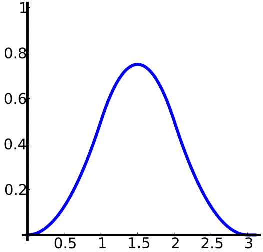

Todd operators

Remark

The Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}, \ldots$ *are defined by the equation:*

$$\frac{s}{e^s - 1} = \sum_{k \geq 0} \frac{B_k}{k!} s^k.$$

Definition

Let $z \in U$. We define the z-shifted *Todd operator*

$$\mathrm{Todd}(X, z) := e^{-z} \prod_{x \in X} \frac{x}{1 - e^{-x}} \in \mathbb{R}[[s_1, \ldots, s_d]].$$

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The Main Theorem

Definition (Cocircuit ideal)

$\mathcal{J}(X) := \mathrm{ideal}\{p_C : C \subseteq X \text{ and } \mathrm{rank}(X \setminus C) < \mathrm{rank}(X)\}$

It is known that $\mathbb{R}[s_1, \ldots, s_d] = \mathcal{P}(X) \oplus \mathcal{J}(X)$. Let

$$\psi_X : \mathbb{R}[s_1, \ldots, s_d] \to \mathcal{P}(X)$$

denote the projection.

Theorem (Main Theorem)

Let z be an interior lattice point of the zonotope $Z(X)$. Let $f_z := \psi_X(\mathrm{Todd}(X, z))$. Then

1. $f_z \in \mathcal{P}_-(X)$ *and*
2. $f_z(D)B_X = \delta_z$.

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A variant of the Khovanskii-Pukhlikov formula

Proposition (Dahmen-Micchelli, 1980s)

$$T_X = B_X *_d \mathcal{T}_X := \sum_{\lambda \in \Lambda} B_X(\cdot - \lambda)\mathcal{T}_X(\lambda)$$

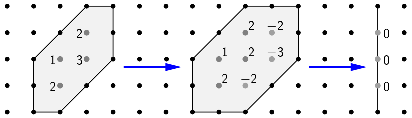
Corollary

Let $u \in \Lambda$ and let z be an interior lattice point of the zonotope $Z(X)$. Then

$$f_z(D)T_X(u) = \delta_z *_d \mathcal{T}_X = \mathcal{T}_X(u - z).$$

$$\Xi(X) := \{f : \mathcal{Z}_-(X) \to \mathbb{R}\}$$

$$\gamma_X : \mathcal{P}_-(X) \to \Xi(X)$$
$$p \mapsto \left[\, \mathbb{Z}^d \ni z \mapsto p(D)B_X(z) \,\right]$$



Proposition

Let $x \in X$ be neither a loop nor a coloop.
The following diagram commutative, the rows are exact and the vertical maps are isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}_-(X \setminus x) \arrow[r, "\cdot p_x"] \arrow[d, "\gamma_{X \setminus x}"] & \mathcal{P}_-(X) \arrow[r, "\pi_x"] \arrow[d, "\gamma_X"] & \mathcal{P}_-(X/x) \arrow[r] \arrow[d, "\gamma_{X/x}"] & 0 \\
0 \arrow[r] & \Xi(X \setminus x) \arrow[r, "\nabla_x"] & \Xi(X) \arrow[r, "\Sigma_x"] & \Xi(X/x) \arrow[r] & 0
\end{tikzcd}$$

Deletion-Contraction (II)

$$\Phi(X) := \operatorname{span}\{f_z : z \in \mathcal{Z}_-(X)\} \subseteq \mathcal{P}(X).$$

Let $q_z \in \mathcal{P}_-(X)$ s. t. $q_z B_X = \delta_u^\Lambda$

$$\phi_X : \mathcal{P}_-(X) \to \Phi(X) \qquad q_z \mapsto f_z.$$

Proposition

Let $x \in X$ be neither a loop nor a coloop. The following diagram is commutative, the rows are exact, and the vertical maps are isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}_-(X \setminus x) \arrow[r, "\cdot p_x"] \arrow[d, "\phi_{X \setminus x}"] & \mathcal{P}_-(X) \arrow[r, "\pi_x"] \arrow[d, "\phi_X"] & \mathcal{P}_-(X/x) \arrow[r] \arrow[d, "\phi_{X/x}"] & 0 \\
0 \arrow[r] & \Phi(X \setminus x) \arrow[r, "\cdot p_x"] & \Phi(X) \arrow[r, "\pi_x"] & \Phi(X/x) \arrow[r] & 0
\end{tikzcd}$$

End of talk

arXiv:1211.1187 and work in progress.