Algebraic properties of some statistics on permutations

V. Vong

Laboratoire d'Informatique Gaspard-Monge – Université Paris-Est Marne-la-Vallée

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Background

Some statistics on permutations

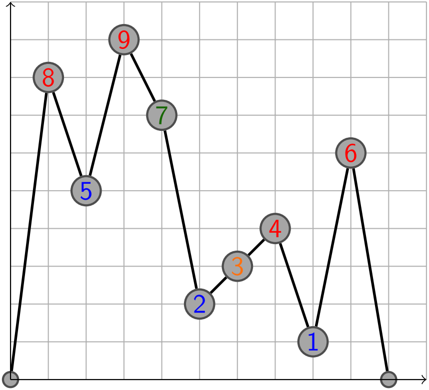
Let σ be in $\mathfrak{S}_n$. By convention, $\sigma(0) = 0$, and $\sigma(n+1) = 0$. Let i be a position between $\{1, \cdots, n\}$. The value σ_i is a:

$$\textit{peak} \longleftrightarrow \begin{array}{ccccc} & & \sigma_i & & \\ & / & & \backslash & \\ \sigma_{i-1} & & & & \sigma_{i+1} \end{array} \qquad \textit{valley} \longleftrightarrow \begin{array}{ccccc} \sigma_{i-1} & & & & \sigma_{i+1} \\ & \backslash & & / & \\ & & \sigma_i & & \end{array}$$

$$\textit{double rise} \longleftrightarrow \begin{array}{ccccc} & & & & \sigma_{i+1} \\ & & & / & \\ & & \sigma_i & & \\ & / & & & \\ \sigma_{i-1} & & & & \end{array} \qquad \textit{double descent} \longleftrightarrow \begin{array}{ccccc} \sigma_{i-1} & & & & \\ & \backslash & & & \\ & & \sigma_i & & \\ & & & \backslash & \\ & & & & \sigma_{i+1} \end{array}$$

Background

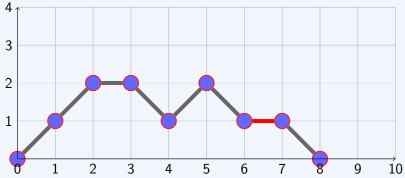
Example: $\sigma = 859723416$



$$
\begin{aligned}
P(\sigma) &= \{8, 9, 4, 6\} \\
V(\sigma) &= \{5, 2, 1\} \\
Dr(\sigma) &= \{3\} \\
Dd(\sigma) &= \{7\}
\end{aligned}
$$

Background

2-colored Motzkin path: example

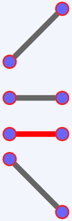


Background

Connection between statistics on permutations on 2-colored Motzkin paths

Consider the following map ϕ from permutations to paths. If σ is a permutation, the i-th step of $\phi(\sigma)$ is a:

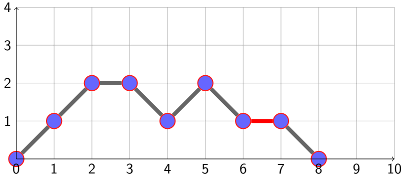
$$\text{if } i \text{ is a} \begin{cases} \text{a valley} & \longleftrightarrow \\ \text{a double rise} & \longleftrightarrow \\ \text{a double descent} & \longleftrightarrow \\ \text{a peak} & \longleftrightarrow \end{cases}$$



Background

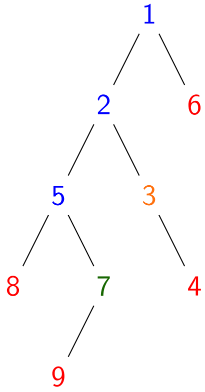
For $\sigma = 859723416$, here is $\phi(\sigma)$:

$$\begin{aligned} P(\sigma) &= \{8, 9, 4, 6\} \\ V(\sigma) &= \{5, 2, 1\} \\ Dr(\sigma) &= \{3\} \\ Dd(\sigma) &= \{7\} \end{aligned}$$



Background

Increasing binary tree of $\sigma = 859723416$:



peak	$\longleftrightarrow$	leaf
valley	$\longleftrightarrow$	node with two children
double rise	$\longleftrightarrow$	node with a right child
double descent	$\longleftrightarrow$	node with a left child

Background

The algebra **FQSym**

FQSym is a graded algebra whose components of weight n have dimensions $n!$. One can index the bases by permutations. The product on the basis F_σ is given by the shifted shuffle:

$$F_\sigma F_\tau = \sum_{s \in \sigma \overline{\sqcup\!\sqcup} \tau} F_s$$

Example

If $\sigma = 312$, and $\tau = 12$, we have:

$$F_{312} F_{12} = F_{31245} + F_{31425} + \cdots + F_{45312}$$

Background

Quotient by an equivalence relation

- Let $\sim$ be an equivalence relation on permutations.
- Consider the vector space $\mathcal{I}$ generated by $(F_\sigma - F_\tau)_{\sigma\sim\tau}$
- Is it a two-sided ideal ?
- If so, **FQSym**$/\mathcal{I}$ is a well-defined quotient algebra.

Proving that $\mathcal{I}$ is a two-sided ideal if and only if:

$$\text{if } \sigma\sim\tau \begin{cases} \exists\phi_s\colon\ \sigma \,\overline{\sqcup\!\sqcup}\, s \to \tau \,\overline{\sqcup\!\sqcup}\, s \text{ a bijection such that } \phi_s(p)\sim p, \\ \exists\psi_s\colon\ s \,\overline{\sqcup\!\sqcup}\, \sigma \to s \,\overline{\sqcup\!\sqcup}\, \tau \text{ a bijection such that } \psi_s(p)\sim p. \end{cases}$$

Background

Examples of equivalence relation

Notations	Definitions	Examples
$(P, V, Dr \cup Dd)$	same peaks, valleys, union of double rises and double descents sets	4132 and 2413
$(P \cup Dd, V \cup Dr)$	same union of peaks and double descents, same union of valleys and double rises sets	35142 and 13542

The different quotients and their properties

quotient by	dimensions	quotient algebras	free algebras
(P, V, Dr, Dd)	C_n	yes	yes
(P,V, Dr ∪ Dd)	M_{n-1}	yes	yes
(P,V ∪ Dr ∪ Dd)	$\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}$	no	no
(P ∪ V ∪ Dr, Dd)	2^{n-1}	no	no
(P ∪ V, Dr, Dd)	$\frac{3^{n-1}+1}{2}$	no	no
(P, Dr, V ∪ Dd)	A_{n-1}	no	no
(P ∪ V, Dr ∪ Dd)	2^{n-2}	no	no
(P ∪ Dd, V ∪ Dr)	2^{n-1}	yes	yes
(P ∪ V ∪ Dr ∪ Dd)	1	yes	yes

Sketch of the proof

What do we have to prove?

- Step 1: if $\sigma \sim \tau$, find a bijection ϕ from $\sigma \,\overline{\sqcup\!\sqcup}\, s$ to $\tau \,\overline{\sqcup\!\sqcup}\, s$ such that $\phi(p) \sim p$.
- Step 2: if $\sigma \sim \tau$, find a bijection ψ from $s \,\overline{\sqcup\!\sqcup}\, \sigma$ to $s \,\overline{\sqcup\!\sqcup}\, \tau$ such that $\psi(p) \sim p$.

Step 1

- Interpretation of the shifted shuffle in term of trees
- Example of construction of the bijection

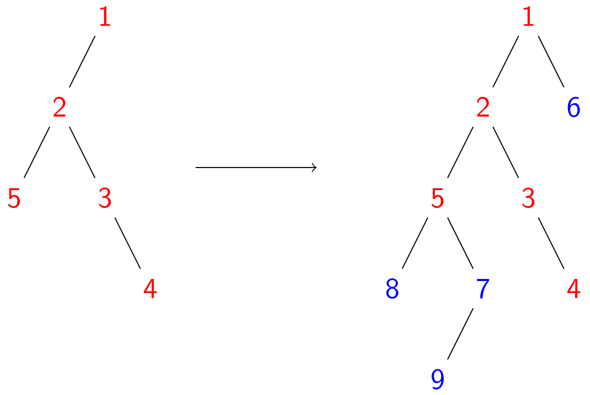
Step 2

- Factorization of permutations and statistics
- Example of construction of the bijection

Sketch of the proof

shifted shuffle and increasing binary trees: example

For $\sigma_1 = 52341$, $s = 3421$, $\sigma = 859723416 \in \sigma_1 \,\overline{\sqcup\!\sqcup}\, s$, we have:



Sketch of the proof

The grafting operation and the bijection ϕ_s

Thanks to the element σ, we have a decomposition of s, and graft locations in the increasing tree of σ_1. In the tree of σ_2, we have the same graft locations. So we graft at the places the blocks of s.

$\sigma_1 = 52341$, $s = 3421$

```
        1
       / \
      2   6
     / \
    5   3
   / \   \
  8   7   4
     /
    9
```

$\sigma = 859723416$

$\sigma_2 = 35241$, $s = 3421$, $\phi(\sigma)$:

```
        1
       / \
      2   6
     / \
    3   4
     \
      5
     / \
    8   7
       /
      9
```

Sketch of the proof

The different steps of the bijection ψ_s

$\sigma_1 = 52341$ and $\sigma_2 = 35241$ and $s = 4132$, $\sigma = 964173852 \in s \,\overline{\sqcup\!\sqcup}\, \sigma_1$, and the construction of the corresponding τ:

1. the factorization of σ_1 by deleting letter of s in σ: $52|3|41$,
2. the corresponding factorization for σ_2: $3|52|41$,
3. the factorization of s by deleting letters of shifted σ_1 in σ: $|41|3|2$,
4. the corresponding τ: 741963852.

Sketch of the proof

A useful factorization on permutations (seen as words)

Let σ and τ two permutations having the same four statistics. Let $\sigma = v_1 \cdots v_r$. Then there exists a unique factorization of $\tau = w_1 \cdots w_r$ such that each letter i in v_k, has the same status in a w_l.

An example of the factorization algorithm:

$\sigma = 859723416$ $\qquad\qquad$ $\tau = 956138724$

Sketch of the proof

A useful factorization on permutations (seen as words)

Let σ and τ two permutations having the same four statistics. Let $\sigma = v_1 \cdots v_r$. Then there exists a unique factorization of $\tau = w_1 \cdots w_r$ such that each letter i in v_k, has the same status in a w_l.

An example of the factorization algorithm:

$\sigma = 85|9723416$ $\qquad\qquad$ $\tau = 95|6138724$

Sketch of the proof

A useful factorization on permutations (seen as words)

Let σ and τ two permutations having the same four statistics. Let $\sigma = v_1 \cdots v_r$. Then there exists a unique factorization of $\tau = w_1 \cdots w_r$ such that each letter i in v_k, has the same status in a w_l.

An example of the factorization algorithm:

$\sigma = 85|972|3416$ $\qquad\qquad$ $\tau = 95|613872|4$

Sketch of the proof

A useful factorization on permutations (seen as words)

Let σ and τ two permutations having the same four statistics. Let $\sigma = v_1 \cdots v_r$. Then there exists a unique factorization of $\tau = w_1 \cdots w_r$ such that each letter i in v_k, has the same status in a w_l.

An example of the factorization algorithm:

$\sigma = 85|972|3|416$ $\qquad\qquad$ $\tau = 95|613|872|4$

Sketch of the proof

A useful factorization on permutations (seen as words)

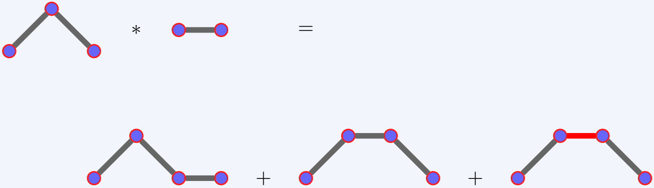
Let σ and τ two permutations having the same four statistics. Let $\sigma = v_1 \cdots v_r$. Then there exists a unique factorization of $\tau = w_1 \cdots w_r$ such that each letter i in v_k, has the same status in a w_l.

An example of the factorization algorithm:

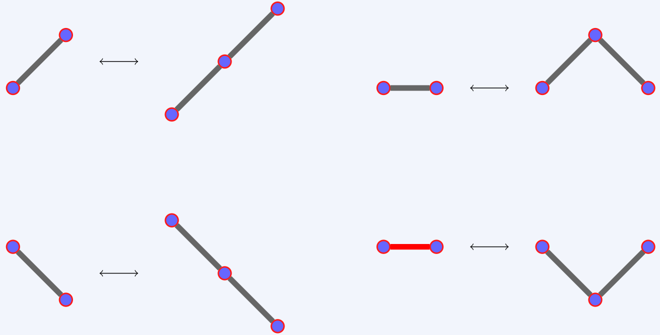
$\sigma = 85|972|3|41|6$ $\qquad$ $\tau = 95|61|3|872|4$

New products on 2-colored Motzkin paths and Dyck paths

Product on 2-colored Motzkin paths: example



Bijection between 2-colored Motzkin paths and Dyck paths



New products on 2-colored Motzkin paths and Dyck paths

Product on Dyck paths: example

For $C_1 = UUDUDD$ and $C_2 = UDUUDD$ we have the following product:

$$C_1 \cdot C_2 = \sum_{C = UU*U***D**DD} C$$