Proof of two conjectures by B. Klopsch, A. Stasinski and C. Voll

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(joint work with F. Brenti)

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Conjectures

The aim of this talk is to state and prove two conjectures by B. Klopsch, A. Stasinski and C. Voll, about a generating function, over arbitrary quotients of the symmetric and hyperoctahedral groups, involving a new statistic.

We call this statistic, defined on both S_n and B_n, Odd Length.

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Notation

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- $\mathbb{P} := \{1, 2, \ldots\}$
- $\mathbb{N} := \mathbb{P} \cup \{0\}$
- $[m, n] := \{m, m+1, \ldots, n\}$, for all $m, n \in \mathbb{Z}$, $m \leq n$
- $[n] := [1, n]$
- $[n]_q := \frac{1-q^n}{1-q}$
- $[n]_q! := \prod_{i=1}^{n} [i]_q$ $\qquad$ $[0]_q! := 1$.
- $\begin{bmatrix} n \\ n_1, \ldots, n_k \end{bmatrix}_q := \frac{[n]_q!}{[n_1]_q! \cdot \ldots \cdot [n_k]_q!}$, for $n_1, \ldots, n_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} n_i = n$.

Notation and preliminaries

The symmetric group S_n is the group of permutations of the set $[n]$. For $\sigma \in S_n$ we use the one-line notation $\sigma = [\sigma(1), \ldots, \sigma(n)]$.

We let $s_1, \ldots, s_{n-1}$ denote the standard generators of S_n, $s_i = (i, i+1)$.

We usually identify S with $[n-1]$, and for $I \subseteq S$, we write $I = [a_1, b_1] \cup \ldots \cup [a_s, b_s]$ and call $[a_i, b_i]$ *connected components* of I.

For (W, S) a Coxeter system we let ℓ be the Coxeter length and for $I \subseteq S$ we define the quotients:

$$W^I := \{w \in W \ : \ D(w) \subseteq S \setminus I\},$$

$$^I W := \{w \in W \ : \ D_L(w) \subseteq S \setminus I\},$$

where $D(w) = \{s \in S \ : \ \ell(ws) < \ell(w)\}$,

and $D_L(w) = \{s \in S \ : \ \ell(sw) < \ell(w)\}$, and the parabolic subgroup W_I to be the subgroup generated by I.

For subsets $X \subseteq W$ we let $X^I := X \cap W^I$.

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Proposition

Let (W, S) be a Coxeter system, $J \subseteq S$, and $w \in W$.
Then there exist unique elements $w^J \in W^J$ and $w_J \in W_J$ (resp., ${}^J w \in {}^J W$ and ${}_J w \in W_J$) such that

$$w = w^J w_J \quad (resp.,\ {}_J w\, {}^J w).$$

Furthermore
$\ell(w) = \ell(w^J) + \ell(w_J)$ *(resp.,* $\ell({}_J w) + \ell({}^J w)$*).*

Definition (Klopsch - Voll)

Let $n \in \mathbb{N}$. The statistic $L_A : S_n \to \mathbb{N}$ is defined as follows. For $\sigma \in S_n$

$$L_A(\sigma) = |\{(i,j) \in [n] \times [n] \mid i < j,\, \sigma(i) > \sigma(j),\, i \not\equiv j \pmod 2\}|$$

For example let $n = 5$, $\sigma = [4,2,1,5,3]$. Then

$\ell(\sigma) = |\{, (1,3), (1,5), , \}| = 5,$

while the L_A counts only inversions between positions with different parity, that is:

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Odd Length on S_n

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In the paper

B. Klopsch, C. Voll

Igusa-type functions associated to finite formed spaces and their functional equations.

Trans. Amer. Math. Soc., **361** (2009), no. 8, 4405-4436.

the authors defined the statistic L_A and formulated the following conjecture.

Conjecture (Klopsch - Voll)

Let $n \in \mathbb{N}$ and $I \subseteq [n-1]$. Then

$$\sum_{\sigma \in S_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \prod_{k=\widetilde{m}+1}^{m} (1-x^{2k}) \begin{bmatrix} \widetilde{m} \\ \left\lfloor \frac{|I_1|+1}{2} \right\rfloor, \ldots, \left\lfloor \frac{|I_s|+1}{2} \right\rfloor \end{bmatrix}_{x^2}$$

if $n = 2m+1$,

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$$L_A(\sigma) := \sum_{I\subseteq[n-1]} (-1)^{|I|} 2^{n-2-|I|} \ell({}^I\sigma).$$

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Tools: Reduction of the support

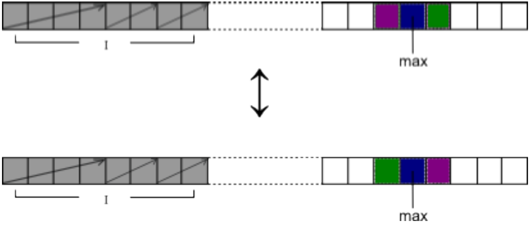
- Inversion around maximum

Inversion around maximum

Suppose that for $n \in \mathbb{P}$, $I \subseteq [n-1]$ we have a permutation $\sigma \in S_n^I$ such that $\sigma^{-1}(n)$ is *sufficiently far* from I.

Then we can define an involution $* : S_n^I \to S_n^I$ that switches the values around the maximum.

Clearly $\ell(\sigma) = \ell(\sigma^*) \pm 1$, while $L(\sigma) = L(\sigma^*)$.



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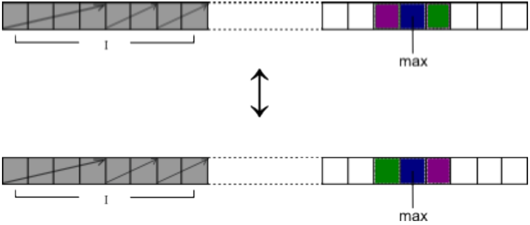

Suppose that for $n \in \mathbb{P}$, $I \subseteq [n-1]$ we have a permutation $\sigma \in S_n^I$ such that $\sigma^{-1}(n)$ is *sufficiently far* from I.

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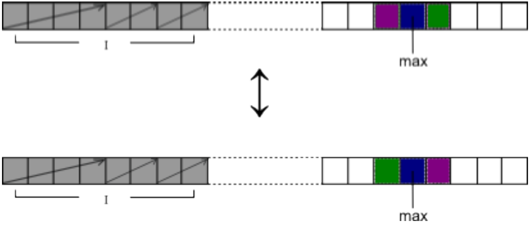

Lemma (Brenti - C.)

Let $I \subseteq [n-1]$ *and* $a \in [2, n-1]$ *be such that* $[a-2, a+1] \cap I = \emptyset$*. Then*

$$\sum_{\substack{\{\sigma \in S_n^I : \\ \sigma(a) = n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = 0$$

- Inversion around maximum
- Chessboard elements

Tools: Reduction of the support

- Inversion around maximum
- Chessboard elements

Chessboard elements

We define *chessboard elements* of the symmetric group as follows.

Let $n \in \mathbb{N}$. Set:

$$C_{n,+} := \{\sigma \in S_n \mid i + \sigma(i) \equiv 0 \pmod 2,\ i = 1, \dots, n\} \textit{ even}$$

$$C_{n,-} := \{\sigma \in S_n \mid i + \sigma(i) \equiv 1 \pmod 2,\ i = 1, \dots, n\} \textit{ odd}$$

$$C_n := C_{n,+} \cup C_{n,-}.$$

For $n = 2m + 1$ clearly $C_{n,-} = \emptyset$ so $C_n = C_{n,+}$.

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Proposition

Let $I \subseteq [n-1]$. *Then*

$$\sum_{\sigma \in S_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in C_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)}.$$

The other idea is to do some operations on the subset $I \subseteq S$ of generators in a way that changes the quotient but doesn't affect the the generating function that we are considering:

$$I \quad \rightsquigarrow \quad \widetilde{I} \quad \textit{such that} \quad \sum_{\sigma \in S_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in S_n^{\widetilde{I}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

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Shifting, fattening and compressing

Proposition (Brenti - C.)

Let $I \subseteq [n-1]$*, and* $i \in \mathbb{P}$*,* $k \in \mathbb{N}$ *be such that* $[i, i+2k]$ *is a connected component of* I *and* $i+2k+2 \notin I$*.*

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Type A Conjecture

Theorem (Brenti - C.)

Let $n \in \mathbb{N}$ and $I \subseteq [n-1]$. Then

$$\sum_{\sigma \in S_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \prod_{k=\widetilde{m}+1}^{m} (1-x^{2k}) \begin{bmatrix} \widetilde{m} \\ \left\lfloor \frac{|I_1|+1}{2} \right\rfloor, \ldots, \left\lfloor \frac{|I_s|+1}{2} \right\rfloor \end{bmatrix}_{x^2}$$

if $n = 2m+1$,

$$\sum_{\sigma \in S_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \prod_{k=\widetilde{m}+1}^{m-1} (1-x^{2k}) \begin{bmatrix} \widetilde{m} \\ \left\lfloor \frac{|I_1|+1}{2} \right\rfloor, \ldots, \left\lfloor \frac{|I_s|+1}{2} \right\rfloor \end{bmatrix}_{x^2} (1-x^m)$$

if $n = 2m > 2\widetilde{m}$,

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Sketch of the proof

Given n, I:

- Reduce to chessboard elements
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The hyperoctahedral group B_n is the group of signed permutations, or permutations σ of the set $[-n, n]$ such that $\sigma(j) = -\sigma(-j)$. We use the window notation $[\sigma(1), \ldots, \sigma(n)]$.

The Coxeter generating set of B_n is $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 = [-1, 2, 3, \ldots, n]$ and $s_1, \ldots, s_{n-1}$ are as for S_n.

Quotients and parabolic subgroups were already defined,

in particular when $J = [n-1]$ we have that $B_n = B_n^{[n-1]}(B_n)_{[n-1]}$ where the parabolic subgroup $(B_n)_{[n-1]}$ can be identified with S_n.

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Definition (Voll - Stasinski)

Let $n \in \mathbb{N}$. The statistic $L_B : B_n \to \mathbb{N}$ is defined as follows. For $\sigma \in B_n$

$$L_B(\sigma) = \frac{1}{2}|\{(i,j) \in [-n,\ n]^2 \mid i < j,\ \sigma(i) > \sigma(j),\ i \not\equiv j \pmod 2\}|$$

Let $n = 4$, $\tau = [-2, 4, 3, -1]$. Then

$$L_B(\tau) = \frac{1}{2}|\{(-4,-3),\ (-4,1),\ (-3,-2),\ (-1,0),\ (-1,4),\ (0,1),\ (2,3),\ (3,4)\}| = 4$$

Notice that if $\sigma \in S_n \subset B_n$ then $L_B(\sigma) = L_A(\sigma)$, so in the following we omit the type and write just L for both the statistics.

Odd Length on B_n

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Odd length in type B has the following characterization:

Proposition (Brenti - C.)

Let $\sigma \in B_n$. Then

$$L(\sigma) = oinv(\sigma) + oneg(\sigma) + onsp(\sigma).$$

where:

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Type B Conjecture

A. Stasinski, C. Voll

A new statistic on hyperoctahedral groups

Electronic J. Combin., **20** (2013), no. 3, Paper 50, 23 pp.

A. Stasinski, C. Voll

Representation zeta functions of nilpotent groups and generating functions for Weyl groups of type B,

Amer. J. Math., **136** (2) (2014), 501-550.

the authors defined L_B and formulated the following conjecture, arising in the field of representation zeta function of certain groups.

Type B Conjecture

Conjecture (Stasinski - Voll)

Let $n \in \mathbb{N}$ and $J \subseteq [0, n-1]$. Then

$$\sum_{\sigma \in B_n^J} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \frac{\prod_{j=a+1}^{n}(1-x^i)}{\prod_{i=1}^{\widetilde{m}}(1-x^{2i})} \begin{bmatrix} \widetilde{m} \\ \left\lfloor \frac{|J_1|+1}{2} \right\rfloor, \ldots, \left\lfloor \frac{|J_s|+1}{2} \right\rfloor \end{bmatrix}_{x^2}$$

where J_0 is the (possibly empty) connected component to 0, $J_1, \ldots, J_s$ are the remaining connected components of J, $\widetilde{m} := \sum_{i=1}^{s} \left\lfloor \frac{|J_i|+1}{2} \right\rfloor$ and $a := \min\{[0, n-1] \setminus J\}$.

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The results of reduction of the support that hold for S_n can be analogously stated and proved for B_n.

In particular, one can give the definition of chessboard elements also in B_n, and the following holds:

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Shifting, fattening and compressing

Given $J \subseteq [0, n-1]$ the results of shifting, fattening and compressing still hold, for quotients in type B, for the connected components of J which do not contain the 0.

Define $J_0 \subseteq J$ to be the (possibly empty) connected component of J which contains the 0. Then the following holds:

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Let $J \subseteq [0, n-1]$ and $a \in [0, n-1]$ be such that $[0, a-1] \subseteq J$, $a, a+1 \notin J$. Then

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By repeated applications of this result we can eliminate the connected component that contains 0 and use the following results of factorization.

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Proposition (Brenti - C.)

Let $J \subseteq [0, n-1]$ and $a \in [0, n-1]$ be such that $[0, a-1] \subseteq J$, $a, a+1 \notin J$. Then

$$\sum_{\sigma \in B_n^J} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (1 - x^{a+1}) \sum_{\sigma \in B_n^{J \cup \{a\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}.$$

By repeated applications of this result we can eliminate the connected component that contains 0 and use the following results of factorization.

Proposition (Stasinski - Voll)

Let $n \in \mathbb{P}$ and $J \subseteq [n-1]$. If $n \equiv 1 \pmod 2$ or $n \equiv 0 \pmod 2$ and $[n-1] \setminus J \subseteq 2\mathbb{N}$ then

$$\sum_{\sigma \in B_n^J} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \left(\sum_{\sigma \in B_n^{[n-\mathbf{1}]}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right) \left(\sum_{\sigma \in S_n^J} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right). \tag{1}$$

Proposition (Stasinski - Voll)

Let $n \in \mathbb{P}$ be even, and $J \subseteq [0, n-1]$ be such that $[0, n-1] \setminus J \subseteq 2\mathbb{N}$. Then

$$\sum_{\sigma \in B_n^{J \setminus \{\mathbf{0}\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \left(\sum_{\sigma \in B_n^J} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right) \left(\sum_{\sigma \in B_i^{[i-\mathbf{1}]}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right), \tag{2}$$

where $i := \min\{[0, n] \setminus J\}$.

Proposition (Stasinski - Voll)

Let $n \in \mathbb{P}$ *and* $J \subseteq [n-1]$. *If* $n \equiv 1 \pmod 2$ *or* $n \equiv 0 \pmod 2$ *and* $[n-1] \setminus J \subseteq 2\mathbb{N}$ *then*

$$\sum_{\sigma \in B_n^J} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \left(\sum_{\sigma \in B_n^{[n-\mathbf{1}]}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right) \left(\sum_{\sigma \in S_n^J} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right). \tag{1}$$

Proposition (Stasinski - Voll)

Let $n \in \mathbb{P}$ *be even, and* $J \subseteq [0, n-1]$ *be such that* $[0, n-1] \setminus J \subseteq 2\mathbb{N}$. *Then*

$$\sum_{\sigma \in B_n^{J \setminus \{\mathbf{0}\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \left(\sum_{\sigma \in B_n^J} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right) \left(\sum_{\sigma \in B_i^{[i-\mathbf{1}]}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right), \tag{2}$$

where $i := min\{[0, n] \setminus J\}$.

Sketch of the proof

- Use shifting, compressing and the result for the connected component which contains 0

  to get in the hypotheses of the results of factorization of L
- Combine with the result for type A

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  to get in the hypotheses of the results of factorization of L
- Combine with the result for type A

Type B Conjecture

Theorem (Brenti - C.)

Let $n \in \mathbb{N}$ and $J \subseteq [0, n-1]$. Then

$$\sum_{\sigma \in B_n^J} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \frac{\prod_{j=a+1}^{n} (1-x^i)}{\prod_{i=1}^{\widetilde{m}} (1-x^{2i})} \begin{bmatrix} \widetilde{m} \\ \left\lfloor \frac{|J_1|+1}{2} \right\rfloor, \ldots, \left\lfloor \frac{|J_s|+1}{2} \right\rfloor \end{bmatrix}_{x^2}$$

where J_0 is the (possibly empty) connected component to 0, $J_1, \ldots, J_s$ are the remaining connected components of J, $\widetilde{m} := \sum_{i=1}^{s} \left\lfloor \frac{|J_i|+1}{2} \right\rfloor$ and $a := \min\{[0, n-1] \setminus J\}$.

Thank you