New Foundations of Combinatorial Theory

Part 3. What is a nice bijection?

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What is a nice bijection?

You have: Two equinumerous sets of combinatorial objects.

You want: A *nice* bijection Φ between them.
Make Φ a *canonical* bijection if possible.

What is *nice*: A set of combinatorial or geometric constraints on Φ.

What is *canonical*: Constraints which uniquely determine Φ.

Examples of nice bijections: too many to list...

Examples of canonical bijections: generating trees [West, 1996], geometric partitions (P. 2005) + some I will show today (3 interrelated stories).

First Story

Tutte's external activities and MST

Theorem [Tutte, 1954]: $|\mathcal{C}(G)| = \sum_{\tau \in G} 2^{\mathrm{ea}(\tau)}$, where

- G is a connected graph with a fixed ordering $\prec$ of edges,
- τ are spanning trees in G,
- $\mathrm{ea}(\tau)$ is the number of *externally active* edges in τ,
- $\mathcal{C}(G)$ is the set of connected subgraphs in G.

Combinatorial proof: [Crapo, 1969]

- Let $\varphi : H \to \tau$ be the minimal spanning tree (MST) map.
- Observe that edges in $H - \tau$ are externally active edges (to τ).
- Conclude that $\left|\varphi^{-1}(\tau)\right| = 2^{\mathrm{ea}(\tau)}$.

Applications:

Canonical bijection proving

$$\sum_{H \in \mathcal{C}(G)} y^{|H|} = \sum_{\tau \in G} (1+y)^{\mathrm{ea}(\tau)} y^{n-1},$$

where G has n vertices, $|H|$ denotes the number of edges of H.

Formally, let $E(G,\tau) \subset G$ denote the set of eternally active edges in $G - \tau$.

$$\Phi : \mathcal{C}(G) \to \{(\tau, S), \text{s.t.}\, S \subseteq E(G,\tau)\}$$

Number of edges statistics: if $\Phi : H \to (\tau, S)$, then $|H| = |\tau| + |S|$.

Note: Crapo's original proof is stated for the whole Tutte polynomial $T_G(x,y)$.
The above bijection easily extends to this case. In fact, it further extends to all matroids.

Tree inversions and DFS

Theorem [Mallows & Riordan, 1968]: $C_n = \sum_{\tau \in K_n} 2^{\mathrm{inv}(\tau)}$, where

- K_n is a complete graph on $\{1, \ldots, n\}$,
- τ are spanning trees in K_n,
- $\mathrm{inv}(\tau)$ is the number of *inversions* in τ,
- $C_n = |\mathcal{C}(K_n)|$ is the number of connected subgraphs in K_n.

Bijection: [Gessel & Wang, 1979]

- Let $\varphi : H \to \tau$ be the *depth first search* (DFS) tree.
- Observe that edges in $K_n - \tau$ correspond to inversions in τ.
- Conclude that $\left|\varphi^{-1}(\tau)\right| = 2^{\mathrm{inv}(\tau)}$.

Note: We can convert this argument into a bijection for the Crapo's proof. This gives us a similar identity

$$\sum_{H \in \mathcal{C}(K_n)} y^{|H|} = \sum_{\tau \in K_n} (1+y)^{\mathrm{inv}(\tau)} y^{n-1},$$

Cane paths and the Neighbor-First Search (NFS)

NFS Algorithm [folklore]

Input: graph G on $\{1, \ldots, n\}$.

Start at n. Make node n *active*. Do:

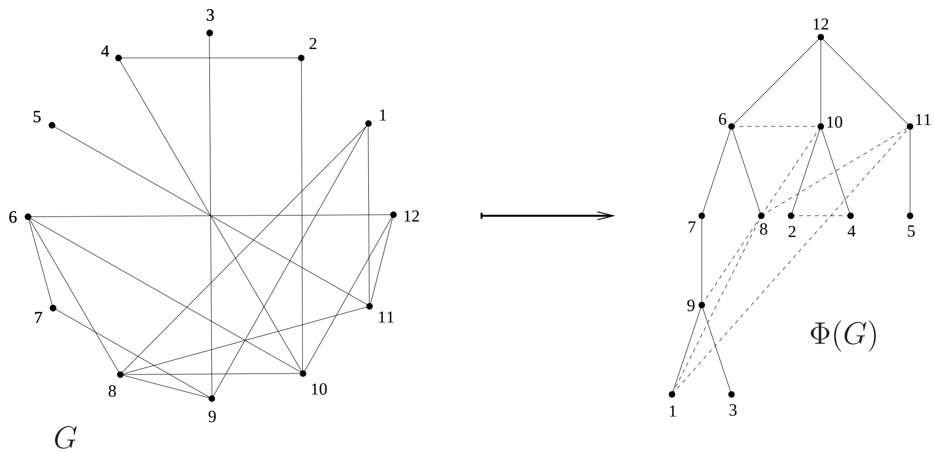
- Visit unvisited neighbors of the active node in decreasing order of their labels; make the one with the smallest label the new active vertex.
- If all the neighbors of the active vertex have been visited, backtrack to the last visited vertex that has not been an active vertex, and make it the new active vertex.

Repeat: until all vertices have been active.

Output: the resulting search tree $\tau = \Phi(G)$.

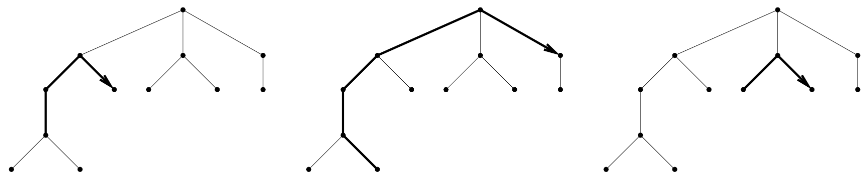
Remark: The NFS is a mixture of BFS and DFS. For more general class of search algorithms applied to G-parking functions, see [Chebykin-Pylyavskyy, 2005].

Example: Graph G and its NFS tree $\Phi(G)$.



Note: Dotted lines correspond to graph edges that are not in $\Phi(G)$.

Theorem [Gessel & Sagan, 1996]: $C_n = \sum_{\tau \in G} 2^{\alpha(\tau)}$, where $\alpha(\tau)$ is the number of *cane paths* in τ, defined as follows:



Key observation: *The number of graphs G with a given NFS search tree τ, is equal to $2^{\alpha(\tau)}$.*

Remark: See also [Gilbert, 1959] and [Kreweras, 1980].

The theorem similarly extends to:

$$\sum_{H \in \mathcal{C}(K_n)} y^{|H|} = \sum_{\tau \in G} (1+y)^{\alpha(\tau)} y^{n-1}.$$

Second Story

Cayley's Theorem (1857)

The number of integer sequences $(a_1, \ldots, a_n)$ *such that*

$$1 \le a_1 \le 2\,, \quad \textit{and} \quad 1 \le a_{i+1} \le 2a_i \quad \textit{for} \;\; 1 \le i < n\,,$$

is equal to the total number of partitions of integers $\le 2^n - 1$ *into parts* $1, 2, 4, \ldots, 2^{n-1}$.

These are called *Cayley compositions* $\mathcal{A}_n$ and *Cayley partitions* $\mathcal{B}_n$.

Example: $n = 2$, $|\mathcal{A}_2| = |\mathcal{B}_2| = 6$

$\mathcal{A}_2 = \big\{ (1,1), (1,2), (2,1), (2,2), (2,3), (2,4) \big\}$,

$\mathcal{B}_2 = \big\{ 21, 1^3, 2, 1^2, 1, \varnothing \big\}$.

Braun's Conjecture (2011)

Define *Cayley polytope* $\mathbf{C}_n \subset \mathbb{R}^n$ by inequalities:

$$1 \le x_1 \le 2, \quad \text{and } 1 \le x_i \le 2x_{i-1} \text{ for } i = 2, \ldots, n,$$

so that $\mathcal{A}_n$ are integer points in $\mathbf{C}_n$.

Theorem 1. [Konvalinka-P., formerly Braun's Conjecture]

$$\operatorname{vol} \mathbf{C}_n = C_{n+1}/n!,$$

where C_n *is the number of connected labeled graphs on* n *vertices.*

Remark: Polytope $\mathbf{C}_n$ is combinatorially equivalent to a n-cube.

$\{C_n\}$ is A001187 in Sloane's *Encyclopedia of Integer Sequences*:

$1, 1, 4, 38, 728, 26704, 1866256, 251548592, 66296291072, 34496488594816, \ldots$

Proof idea: an explicit triangulation into orthoschemes

Conjecture [Hadwiger, 1956]

Every convex polytope in $\mathbb{R}^d$ can be dissected into a finite number of orthoschemes.

Remark: Suffices to prove for simplices. Known for $d \leq 6$.

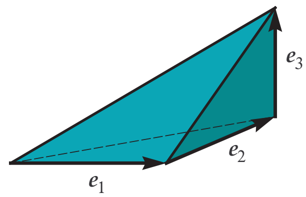
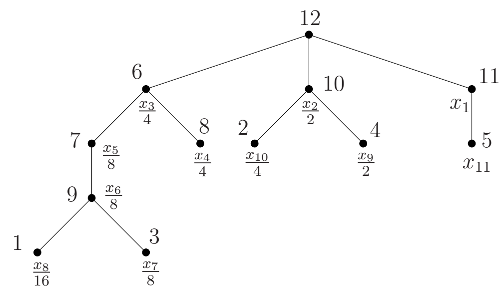


FIGURE 1. An example of an orthoscheme (path-simplex).

Triangulation Construction:



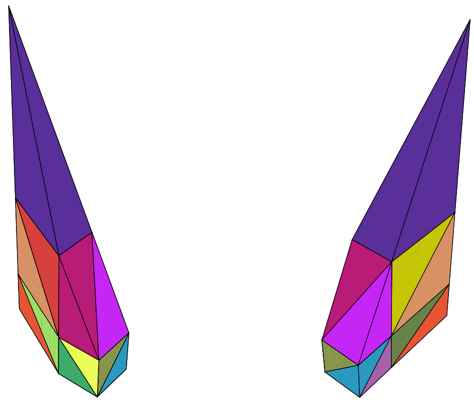
The simplex $\mathbf{S}_\tau \in \mathbb{R}^{11}$ corresponding to a labeled tree $\tau \in K_{11}$ is given by

$$1 \leq \frac{x_8}{16} \leq \frac{x_{10}}{4} \leq \frac{x_7}{8} \leq \frac{x_9}{2} \leq x_{11} \leq \frac{x_3}{4} \leq \frac{x_5}{8} \leq \frac{x_4}{4} \leq \frac{x_6}{8} \leq \frac{x_2}{2} \leq x_1 \leq 2.$$

We have $\alpha(\tau) = 21$, and $\text{vol}(\mathbf{S}_\tau) = 2^{21}/11!$.

Rules: Label nodes according to NFS. Take $x_i/2^{k_i}$ to be the coordinate corresponding to $v \in \tau$, where k_i is the number of cane paths in τ that start in v. For inequalities, use the original ordering of labels in τ.

Example: Our triangulation of Cayley polytope $\mathbf{C}_3$ from two different angles:



Note: There are 16 orthoschemes in the triangulation, each of volume $2^k/3!$, where k varies. In general, there are $(n+1)^{n-1}$ orthoschemes (*Cayley's formula*).

Sequel: extension to other values of the Tutte polynomial

$C_n = T_{K_n}(1,2)$, where $T_G(x,y)$ is the *Tutte polynomial* of graph G :

$$\mathrm{T}_G(x,y) = \sum_{H \subseteq G} (x-1)^{k(H)-k(G)} (y-1)^{e(H)-|V|+k(H)},$$

where $k(H)$ is the number of connected components in H. Also:

$$\mathrm{T}_G(x,y) = \sum_{\tau \in G} x^{ia(\tau)} y^{ea(\tau)},$$

where the summation is over all spanning trees τ in G,
ia(τ) is the number of *internally active* edges in τ,
ea(τ) is the number of *externally active* edges in τ.

Tutte polytope

For every $0 < q \leq 1$ and $t > 0$, define *Tutte polytope* $\mathbf{T}_n(q,t) \subset \mathbb{R}^n$ by inequalities:

$$x_n \geq 1 - q, \quad \text{and}$$

$$x_i \;\leq\; (1+t)x_{i-1} \;-\; \frac{t(1-q)}{q}(1-x_{j-1}), \text{ where } 1 \leq j \leq i \leq n \text{ and } x_0 = 1.$$

Theorem: *Tutte polytopes $\mathbf{T}_n(q,t)$ have 2^n vertices.*

Example: Compare the vertex coordinates of $\mathbf{C}_3$ and $\mathbf{T}_3(q,t)$:

2	4	8
2	4	1
2	1	2
2	1	1
1	2	4
1	2	1
1	1	2
1	1	1

$1+t$	$(1+t)^2$	$(1+t)^3$
$1+t$	$(1+t)^2$	$1-q$
$1+t$	1	$1+t$
$1+t$	$1-q$	$1-q$
1	$1+t$	$(1+t)^2$
1	$1+t$	$1-q$
1	1	$1+t$
$1-q$	$1-q$	$1-q$

Main Theorem [Konvalinka-P., 2013]

Let $\mathbf{T}_n(q,t) \subset \mathbb{R}^n$ *be the Tutte polytope defined above,* $0 < q \leq 1$*,* $t > 0$*. Then:*

$$\operatorname{vol} \mathbf{T}_n(q,t) \,=\, t^n \, \mathrm{T}_{K_{n+1}}(1 + q/t, 1 + t)/n!,$$

where $\mathrm{T}_H(x,y)$ *denotes the* Tutte polynomial *of graph* H*.*

Remark: Cayley polytopes are limits of Tutte polytopes:

$$\lim_{q \to 0+} \mathbf{T}_n(q,1) \,=\, \mathbf{C}_n .$$

This follows from the explicit form of vertex coordinates.

Since $\mathrm{T}_{K_n}(1,2) = C_n$, Main Theorem implies Braun's Conjecture.

Third Story

Back to Cayley's Theorem

The number of integer sequences $(a_1, \ldots, a_n)$ *such that*

$$1 \le a_1 \le 2\,, \quad \textit{and} \quad 1 \le a_{i+1} \le 2a_i \quad \textit{for} \;\; 1 \le i < n\,,$$

is equal to the total number of partitions of integers $\le 2^n - 1$ *into parts* $1, 2, 4, \ldots, 2^{n-1}$.

These are Cayley compositions $\mathcal{A}_n$ and Cayley partitions $\mathcal{B}_n$.

⋘ Now think of $\mathcal{A}_n, \mathcal{B}_n \subset \mathbb{R}^n$. ⋙

Example: $n = 2$, $|\mathcal{A}_2| = |\mathcal{B}_2| = 6$.

$\mathcal{A}_2 = \big\{\, (1,1),\ (1,2),\ (2,1),\ (2,2),\ (2,3),\ (2,4) \,\big\}$,

$\mathcal{B}_2 = \big\{\, (1,1),\ (0,3),\ (1,0),\ (0,2),\ (0,1),\ (0,0) \,\big\}$.

Polytope of Cayley partitions

Observe: $\mathcal{B}_n$ is the set of integer points in simplex $\mathbf{Q}_n$:

$$y_1, \ldots, y_n \geq 0, \quad 2^{n-1}y_1 + \ldots + 2y_{n-1} + y_n \leq 2^n - 1$$

Theorem: *Let* $\mathbf{P}_n \subset \mathbb{R}^n$ *be the convex hull of* $\mathcal{B}_n$.
Then $\mathrm{vol}\,\mathbf{P}_n = \mathrm{vol}\,\mathbf{C}_n$ (and thus $= C_{n+1}/n!$).

Sketch of proof: Define $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\varphi : (a_1, a_2, a_3, \ldots) \to (2 - a_1, 2a_1 - a_2, 2a_2 - a_3, \ldots).$$

Observe that φ is volume-preserving. Now check that $\varphi : \mathbf{C}_n \to \mathbf{P}_n$. $\square$

First application of map φ

Observe that $\varphi : \mathcal{A}_n \to \mathcal{B}_n$ is a *bijection*.
This proves Cayley's theorem.

Example: Bijection $\varphi : \mathcal{A}_2 \to \mathcal{B}_2$ is then as follows:

$$(1,1) \to (1,1) = \mathbf{21}, \ (1,2) \to (1,0) = \mathbf{2}, \ (2,1) \to (0,3) = \mathbf{1^3},$$
$$(2,2) \to (0,2) = \mathbf{1^2}, \ (2,3) \to (0,1) = \mathbf{1}, \ (2,4) \to (0,0) = \varnothing.$$

Corollary: *The number of Cayley partitions of m in $\mathcal{B}_n$ is equal to the number of Cayley compositions $(a_1, \ldots, a_n) \in \mathcal{A}_n$, such that $a_n = 2^n - m$.*

Parking functions polytope [Stanley & Pitman, 2002]

Let $\Pi_n(\theta_1, \ldots, \theta_n)$ be defined by the inequalities:

$$\Pi_n(\theta_1, \ldots, \theta_n) = \{(x_1, \ldots, x_n) : x_i \geq 0,\, x_1 + \ldots + x_i \leq \theta_1 + \ldots + \theta_i,\, \forall i\}.$$

Theorem [Stanley & Pitman, 2002]

$$\operatorname{vol}\Pi_n(\theta_1, \ldots, \theta_n) = \frac{1}{n!} \sum_{(a_1,\ldots,a_n)\in \mathrm{Park(n)}} \theta_{a_1} \cdots \theta_{a_n}$$

Corollary: $\operatorname{vol}\Pi_n(1, q, q^2 \ldots, q^{n-1}) = \dfrac{1}{n!2^{\binom{n}{2}}} \cdot T_{n+1}(1, 1/q)$

Second application of map φ

Observation: $\mathbf{P}_n$ is a scaled version of $\Pi_n\big(1, \frac{1}{2}, \frac{1}{4}, \ldots\big)$:

$$x_i \leftarrow 2^{i-1} x_i, \quad 1 \le i \le n$$

Corollary:

$$\operatorname{vol} \mathbf{P}_n \;=\; 2^{\binom{n}{2}} \operatorname{vol} \Pi_n\big(1, \frac{1}{2}, \frac{1}{4}, \ldots\big) \;=\; C_{n+1}/n!$$

The above volume theorem gives another proof of Braun's Conjecture.

Thank you!