Tropical Catalan Subdivisions

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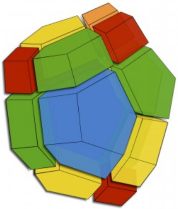
The 76th Séminaire Lotharingien de Combinatoire, Ottrott
April 6, 2016

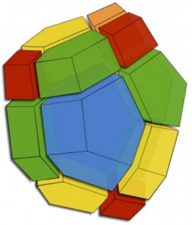
Vienna, December 18, 2015:

Frédéric Chapoton showed me a beautiful picture in François Bergeron's webpage



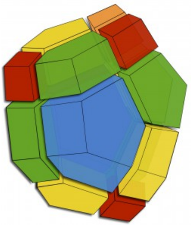
The 2-Tamari lattice for $n = 4$

Vienna, December 18, 2015:



Chapoton: Can you find a similar picture for all m-Tamari lattices?

Vienna, December 18, 2015:



Chapoton: Can you find a similar picture for all m-Tamari lattices?

Me: Wow . . . That is a really beautiful picture!!!

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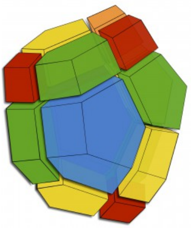


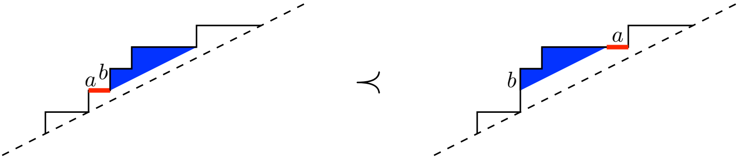
Chapoton: Can you find a similar picture for all m-Tamari lattices?

Me: Wow ... That is a really beautiful picture!!!

Me: Could you remind me what an m-Tamari lattice is?

Vienna, December 18, 2015:

Chapoton: The m-Tamari lattice is a poset (that turns out to be a lattice) on Fuss-Catalan paths determined by the following covering relation:



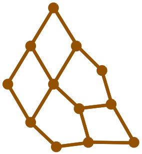
Fuss-Catalan path: lattice path from $(0,0)$ to (mn,n) that stays weakly above the main diagonal.

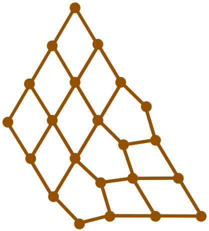
[Bergeron and Préville-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets, '12]

Vienna, December 18, 2015:

Me: Could you show me some examples?

Chapoton: 2-Tamari and 3-Tamari lattices for $n = 3$:





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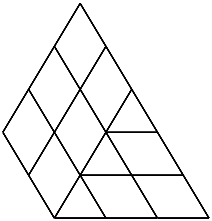

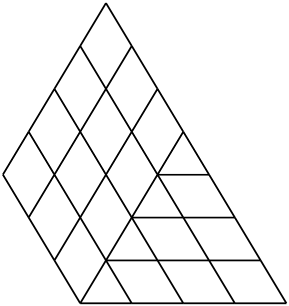

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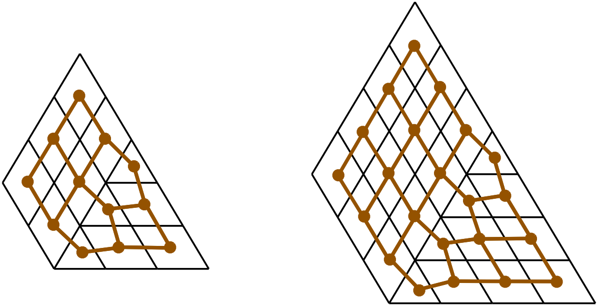
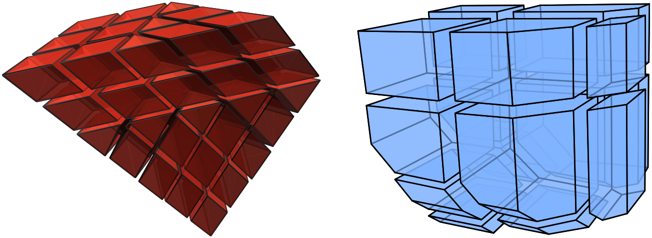


These are the kind of pictures that we can get today:



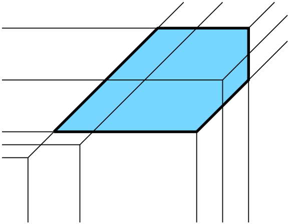
The 2-Tamari lattice for $n = 4$.

Goal of this talk

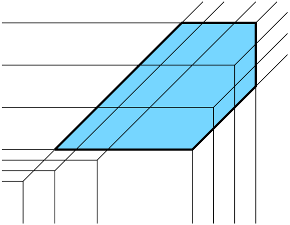
Explain these pictures.

Why Tropical Catalan Subdivisions?

These subdivisions come from regular triangulations of products of simplices. Their duals are obtained tropically (Develin–Sturmfels).



2-Tamari lattice for $n = 3$



3-Tamari lattice for $n = 3$

Theorem (CPS)

The m-Tamari lattice for n is the edge graph of a polytopal subdivision of an $(n-1)$-dimensional associahedron induced by a collection of tropical hyperplanes.

Why Tropical Catalan Subdivisions?

Colombia

Barcelona

Why Tropical Catalan Subdivisions?

Colombia

Barcelona

Tropical Catalan Subdivisions!

The associahedral triangulation

Consider the product of two simplices

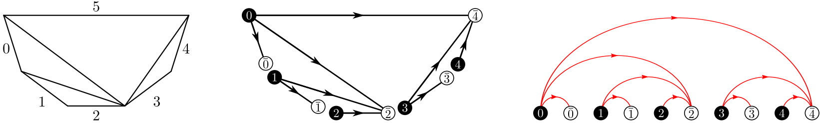
$$\Delta_n \times \Delta_{\overline{n}} = \mathrm{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}})\colon 0 \leq i, \bar{j} \leq n\right\}.$$

We want to triangulate the sub-polytope

$$\mathcal{C}_n = \mathrm{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}})\colon 0 \leq i \leq \bar{j} \leq n\right\}$$

The associahedral triangulation

The cells: indexed by triangulations of an $(n+2)$-gon

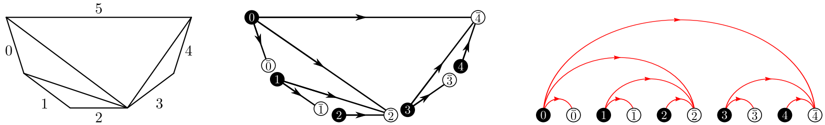


In this example, the cell is:

$$\operatorname{conv}\left\{(\mathbf{e}_0,\mathbf{e}_{\bar{0}}),(\mathbf{e}_0,\mathbf{e}_{\bar{2}}),(\mathbf{e}_0,\mathbf{e}_{\bar{4}}),(\mathbf{e}_1,\mathbf{e}_{\bar{1}}),\ldots,(\mathbf{e}_4,\mathbf{e}_{\bar{4}})\right\}$$

The associahedral triangulation

The cells: indexed by triangulations of an $(n+2)$-gon



In this example, the cell is:

$$\operatorname{conv}\{(\mathbf{e}_0, \mathbf{e}_{\bar{0}}), (\mathbf{e}_0, \mathbf{e}_{\bar{2}}), (\mathbf{e}_0, \mathbf{e}_{\bar{4}}), (\mathbf{e}_1, \mathbf{e}_{\bar{1}}), \ldots, (\mathbf{e}_4, \mathbf{e}_{\bar{4}})\}$$

Fact

- *These collection of cells triangulate the polytope $\mathcal{C}_n$.*
- *This triangulation is dual to an associahedron.*

The associahedral triangulation

This triangulation has appeared in many independent papers:

- Gelfand–Graev–Postnikov, Combinatorics of hypergeometric functions associated with positive roots, '97. (as a triangulation of a root polytope)
- Stanley–Pitman, A polytope related to empirical distributions, plane trees, parking functions, and the associahedron, '02.
- Petersen–Pylyavskyy–Speyer, A non-crossing standard monomial theory, '10.
- Santos–Stump–Welker, Noncrossing sets and a Grassmann associahedron. '14.
- ...

The associahedral triangulation

Example

The 1-dimensional associahedron is the dual of a triangulation of a 4-dimensional polytope $\mathcal{C}_2 \subset \Delta_2 \times \Delta_{\overline{2}}$.

The associahedral triangulation

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Why do you want to draw a 1-dim edge in 4 dimensions?

The associahedral triangulation

Example

The 1-dimensional associahedron is the dual of a triangulation of a 4-dimensional polytope $\mathcal{C}_2 \subset \Delta_2 \times \Delta_{\overline{2}}$.

Why do you want to draw a 1-dim edge in 4 dimensions?
This might look like a disadvantage.
But this approach is actually very powerful.

The $(I, \overline{J})$-triangulation

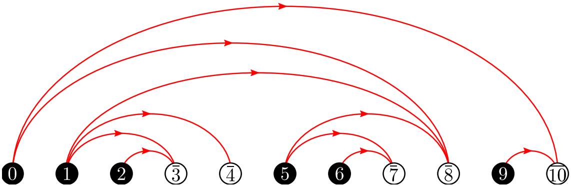
Let I, J be a partition of $[n]$ with $0 \in I$ and $n \in J$.
The restriction of the triangulation to the face

$$\Delta_I \times \Delta_{\overline{J}} = \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\overline{j}}) \colon i \in I \text{ and } j \in J\right\}$$

is called the $(I, \overline{J})$-triangulation.

The $(I, \overline{J})$-triangulation

The cells of this restricted triangulation are indexed by $(I, \overline{J})$-trees (maximal, non-crossing, increasing alternating graphs with support $I \cup \overline{J}$)

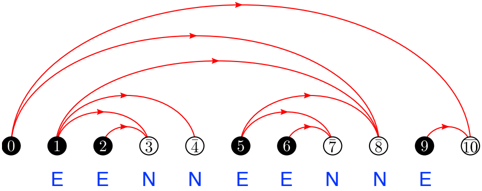


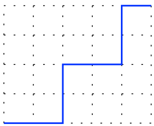
In this example,

$$I = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{6}, \mathbf{9}\} \qquad \overline{J} = \{\overline{3}, \overline{4}, \overline{7}, \overline{8}, \overline{10}\}$$

The $(I, \overline{J})$-triangulation

Given such a tree T we associate two paths $\nu(I, \overline{J})$ and $\rho(T)$:





ν

$\nu(I, \overline{J})$ replaces black and white balls by east and north steps respectively.

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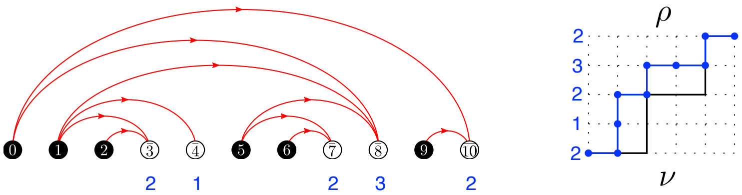

$\nu(I, \overline{J})$ replaces black and white balls by east and north steps respectively.
$\rho(T)$ counts the in-degrees of the white balls.

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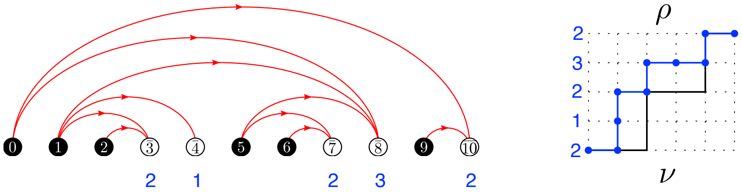
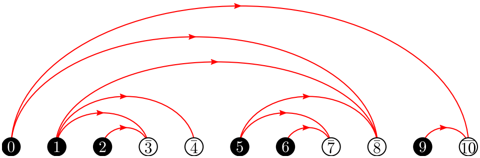
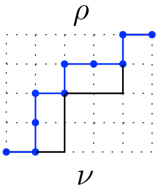


$\nu(I, \overline{J})$ replaces black and white balls by east and north steps respectively.
$\rho(T)$ counts the in-degrees of the white balls.

Note: the path $\rho(T)$ is weakly above ν.

The $(I, \overline{J})$-triangulation



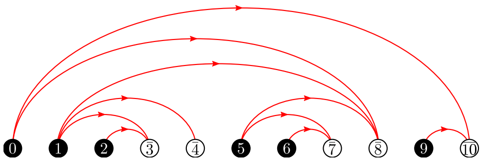


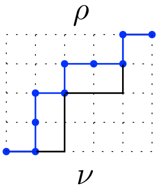
Proposition (CPS)

Let I, J be a partition of $[n]$ with $0 \in I$ and $n \in J$, and $\nu = \nu(I, \overline{J})$.

- ρ is a bijection from $(I, \overline{J})$-trees to ν-paths.
- two $(I, \overline{J})$-trees are related by a flip iff the corresponding ν-paths are related by a ν-Tamari relation.

The $(I, \overline{J})$-triangulation





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- two $(I, \overline{J})$-trees are related by a flip iff the corresponding ν-paths are related by a ν-Tamari relation.

this should be compared with a similar result in
[Préville-Ratelle and Viennot, An extension of Tamari lattices, '14.]

Tam(ν) as the dual of a triangulation

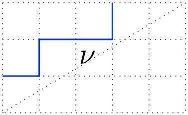
Theorem (CPS)

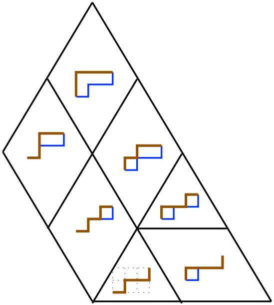
Let ν *be a lattice path from* $(0,0)$ *to* (a,b)*. The* ν*-Tamari lattice* Tam(ν) *can be realized geometrically as the dual of a regular triangulation of a subpolytope of* $\Delta_a \times \Delta_b$ *(in* $\mathbb{R}^{a+b}$*).*

Tam(ν) as the dual of a subdivision

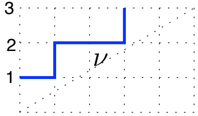
Corollary (CPS)

Let ν be a lattice path from $(0,0)$ to (a,b). Tam(ν) is the dual of a subdivision of a generalized permutahedron (in $\mathbb{R}^a$ and in $\mathbb{R}^b$).

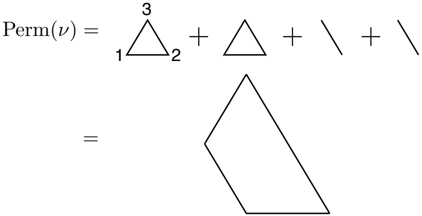




Tam(ν) as the dual of a subdivision



123+123+23+23



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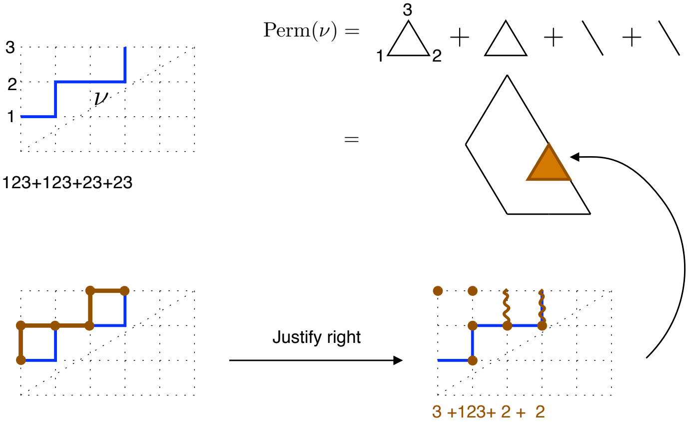
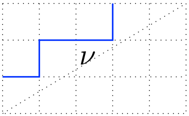
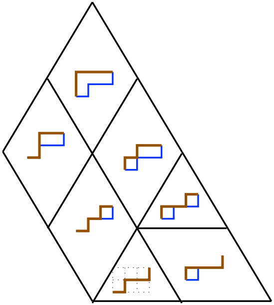


Tam(ν) as the dual of a subdivision

If you do it for all ν-paths you get

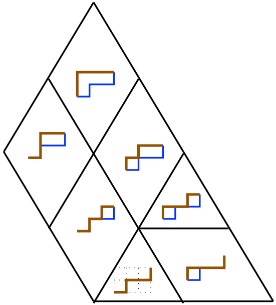


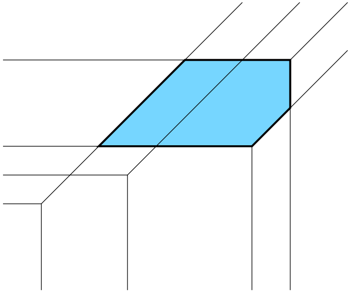


Two cells are adjacent iff the corresponding ν-paths are related by a ν-Tamari relation.

Tam(ν) as the dual of a subdivision

You can also obtain the dual tropically

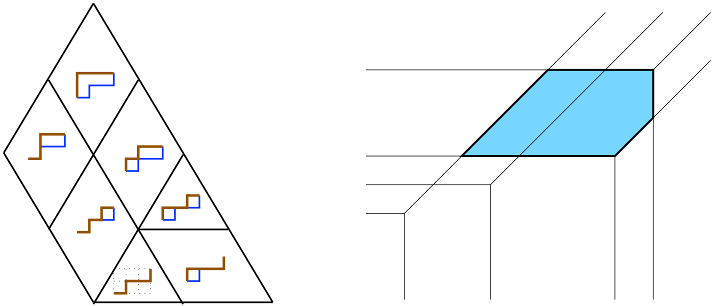




Tam(ν) as the graph of a tropical subdivision

Corollary (CPS)

Tam(ν) *is the edge graph of a polyhedral complex induced by a tropical hyperplane arrangement (in* $\mathbb{TP}^a \cong \mathbb{R}^a$ *and in* $\mathbb{TP}^b \cong \mathbb{R}^b$*).*



The rational Tamari lattice Tam$(3,5)$.

Tam(ν) as the graph of a tropical subdivision

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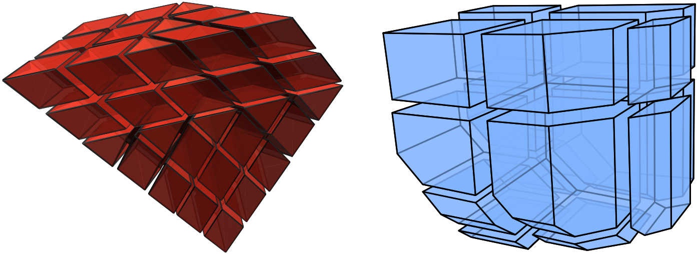
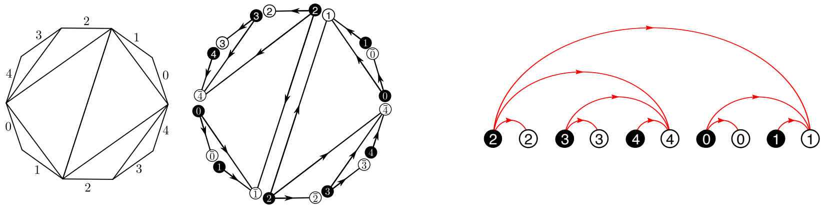


The 2-Tamari lattice for $n = 4$.

What about other types?

The cyclohedron triangulation

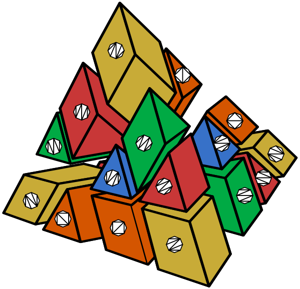
Consider the following trees indexed by cyclic symmetric triangulations of a $(2n+2)$-gon:

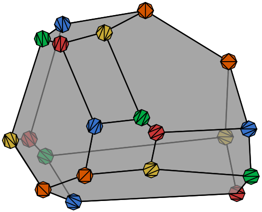


The cyclohedron triangulation

Theorem (CPS)

This collection of cells form a regular triangulation of $\Delta_n \times \Delta_{\overline{n}}$ *dual to an n-dimensional cyclohedron.*





Restricting to its faces, we obtain type B_n analogs of the realizations of Tam(ν).

Thank you!