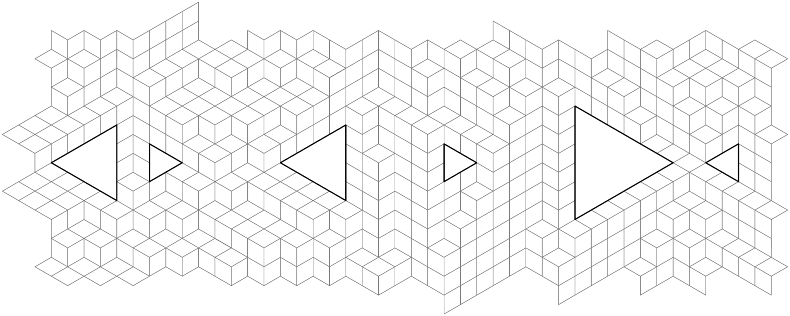
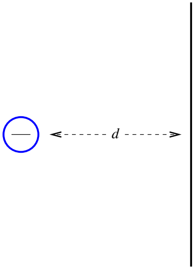
Symmetric rhombus tilings of holey hexagons and the method of images.



Tomack Gilmore
Universität Wien

The Method of Images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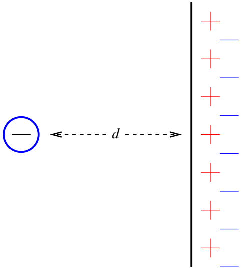
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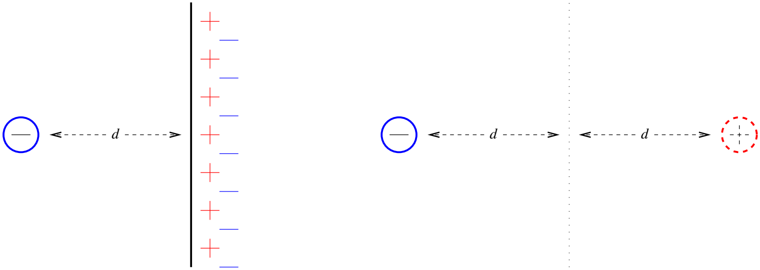
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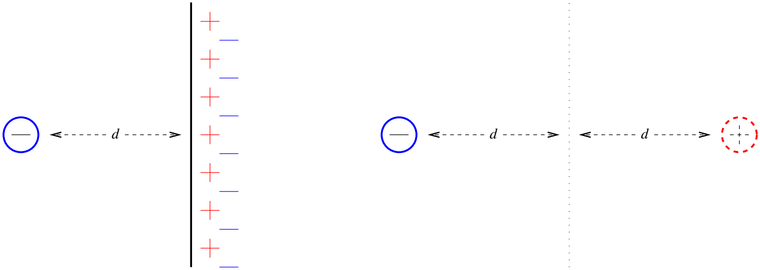
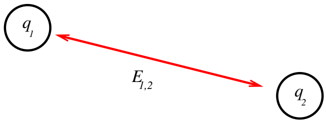
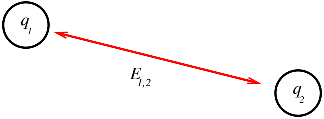


$$E_{\ominus,|} \quad = \quad \frac{1}{2} E_{\ominus,\oplus}$$


q_1
q_2


q_1
$E_{1,2}$
q_2

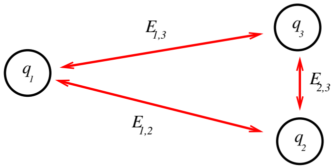


Coulomb's Law

The electrostatic energy of the system consisting of two point charges with signed magnitudes q_1 and q_2 is

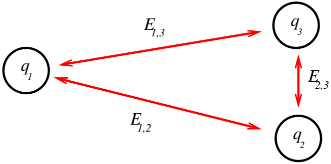
$$E_{1,2} = k_e \frac{q_1 q_2}{d(q_1, q_2)},$$

where k_e denotes Coulomb's constant and $d(q_1, q_2)$ is the Euclidean distance between the two charges.



The energy of a system of three point charges is given by

$$E_{1,2} + E_{1,3} + E_{2,3} = k_e \left(\frac{q_1 q_2}{d(q_1, q_2)} + \frac{q_1 q_3}{d(q_1, q_3)} + \frac{q_2 q_3}{d(q_2, q_3)} \right).$$

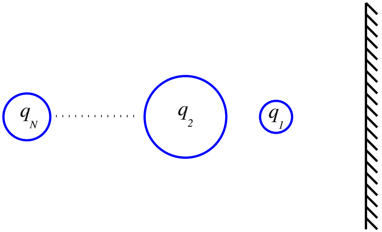


The energy of a system of three point charges is given by

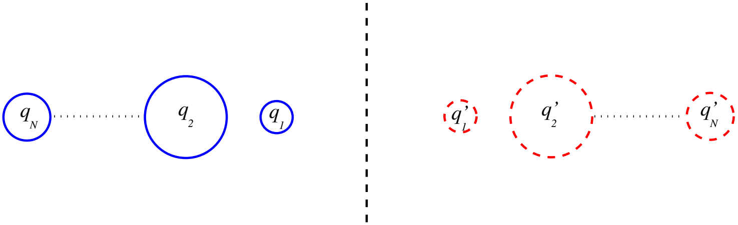
$$E_{1,2} + E_{1,3} + E_{2,3} = k_e \left(\frac{q_1 q_2}{d(q_1, q_2)} + \frac{q_1 q_3}{d(q_1, q_3)} + \frac{q_2 q_3}{d(q_2, q_3)} \right).$$

More generally, the electrostatic energy of a system of point charges $q_1, \ldots, q_n$ is given by

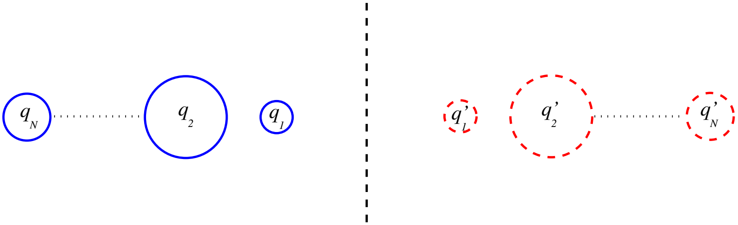
$$k_e \sum_{1 \le i < j \le n} \frac{q_i q_j}{d(q_i, q_j)}.$$



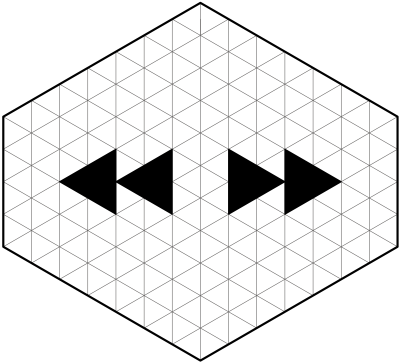
$E_{\{q_1,\ldots,q_N\},|}$

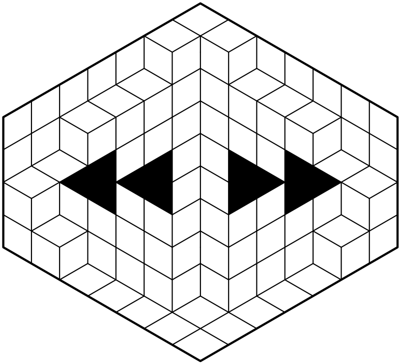


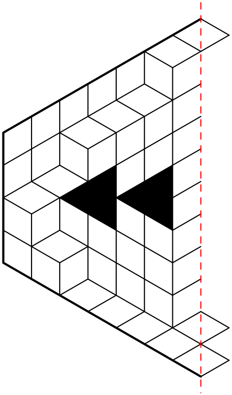
$$E_{\{q_1,\ldots,q_N\},|} = \frac{1}{2} E_{\{q_1,\ldots,q_N\},\{q'_1,\ldots,q'_N\}}$$

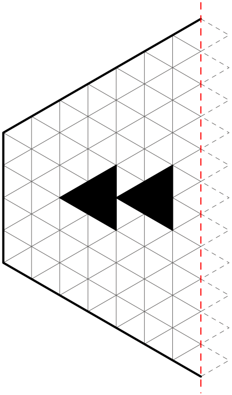


$$
\begin{aligned}
E_{\{q_1,\dots,q_N\},|} &= \frac{1}{2} E_{\{q_1,\dots,q_N\},\{q'_1,\dots,q'_N\}} \\
&= \frac{k_e}{2}\left(\sum_{1\leq i<j\leq N} \frac{q_i q_j}{d(q_i, q_j)} + \sum_{1\leq i<j\leq N} \frac{q'_i q'_j}{d(q'_i, q'_j)} \right. \\
&\qquad \left. - \sum_{1\leq i,j\leq N} \frac{|q_i q'_j|}{d(q_i, q'_j)} \right).
\end{aligned}
$$

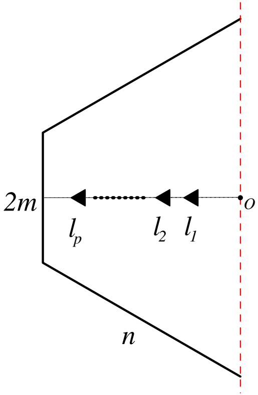






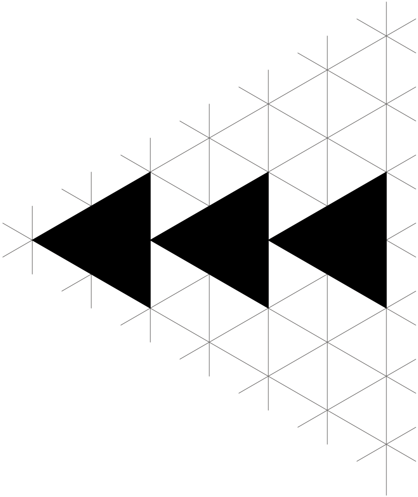


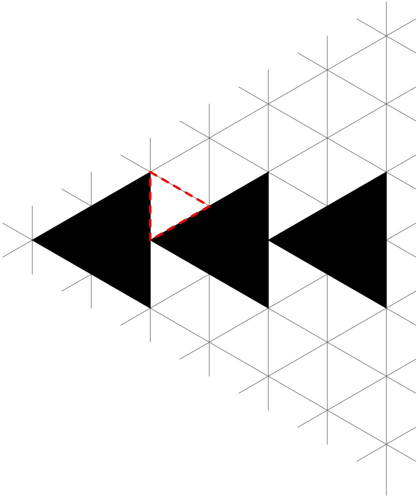
$$V_{7,4}^{\{-1,-3\}}$$

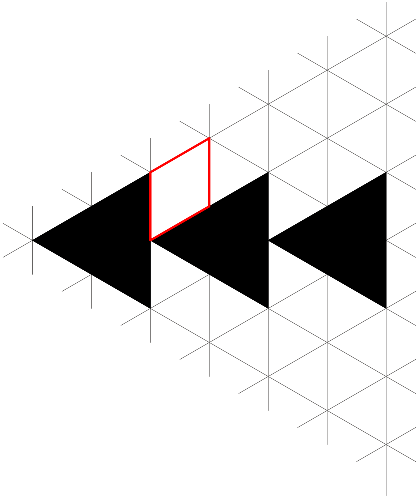


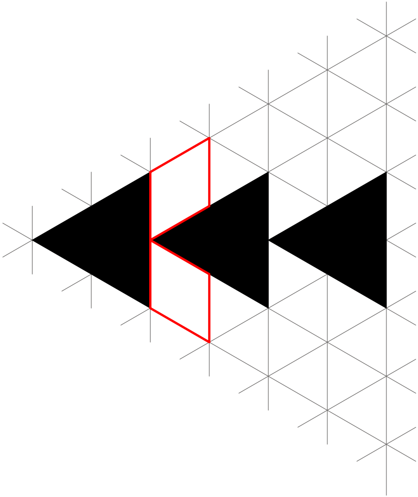
$$V^L_{n,2m}$$

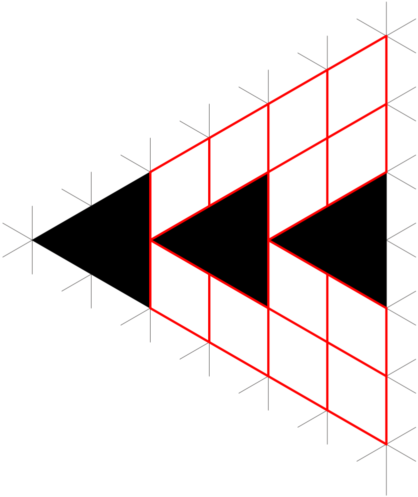
(here $L = \{l_1, \ldots, l_p\}$ indexes the left pointing triangular holes by their vertical lattice distance from the origin)

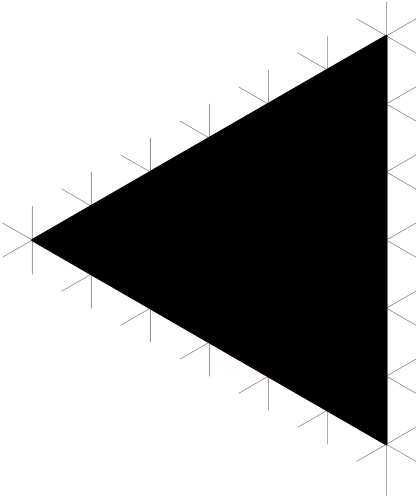












Theorem (TG 2016)

$$M(V_{n,2m}^{L}) = \left(\prod_{i=1}^{n} \frac{2i+2m-1}{2i-1} \prod_{1\leq i<j\leq n} \frac{i+j+2m-1}{i+j-1}\right) \cdot \det \widehat{E}_{R,L}$$

where $R = \{-l_1, \ldots, -l_{|L|}\}$, $\widehat{E}_{R,L} = (\hat{e}_{i,j})_{1\leq i,j\leq |L|}$ *with* (i,j)*-entries given by*

$$\hat{e}_{i,j} = \frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{n}{2}-\frac{l_i}{2}+\frac{3}{2})\Gamma(m+n+1)\Gamma(\frac{n}{2}+\frac{r_j}{2}+\frac{3}{2})}{2^{l_i-r_j-2}\pi\Gamma(m)\Gamma(\frac{n}{2}-\frac{l_i}{2}+2)\Gamma(m+n+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{r_j}{2}+2)} \times \sum_{s=0}^{\infty} \frac{(2+\frac{r_j}{2}-\frac{l_i}{2})_s(\frac{1}{2})_s(m+n+1)_s(1-m)_s}{(\frac{n}{2}-\frac{l_i}{2}+2)_s(\frac{n}{2}+\frac{r_j}{2}+2)_s(\frac{3}{2})_s(s!)}$$

(here $\Gamma(x) = (x-1)!$ *and* $(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}$ *for* $x \in \mathbb{Q}, y \in \mathbb{Z}$*).*

Theorem (TG 2016)

$$M(V^L_{n,2m}) = \left(\prod_{i=1}^{n} \frac{2i+2m-1}{2i-1} \prod_{1\leq i<j\leq n} \frac{i+j+2m-1}{i+j-1}\right) \cdot \det \widehat{E}_{R,L}$$

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Suppose $2m \sim \xi n$ for some real $\xi > 0$. The *interaction* of the holes indexed by L and the vertical free boundary is

$$\omega(\xi; R, L) = \lim_{n \to \infty} \frac{M(V_{n,2m}^{L})}{M(V_{n,2m}^{\emptyset})}$$

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As the distances between the sets of holes grows large,

$$\omega(\xi; R, L) \sim \det \left(\frac{(\xi(\xi+2))^{-1/2}}{\pi(r_j - l_i)} \left(\frac{2}{\xi+1}\right)^{r_j - l_i + 2} \right)_{1\le i,j\le |L|}$$

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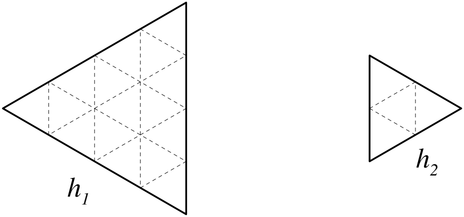
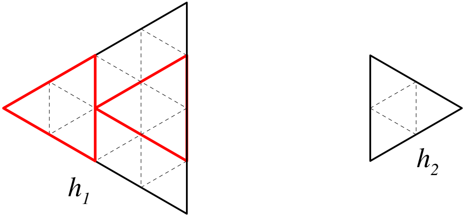
and so for $\xi = 1$,

$$\omega(1; R, L) \sim \left(\frac{1}{2\pi} \right)^{|L|} \frac{\prod_{i=2}^{|L|} \prod_{j=1}^{i-1} d(r_i, r_j) d(l_i, l_j)}{\prod_{1\le i,j\le |L|} d(r_i, l_j)}.$$



The *charge* of a hole h, denoted $q(h)$, is the number of left pointing unit triangles that comprise it minus the number of right pointing unit triangles.



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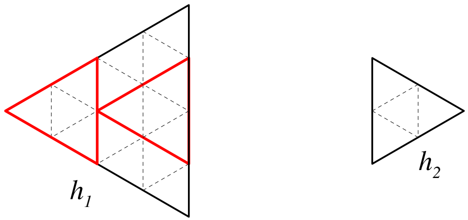
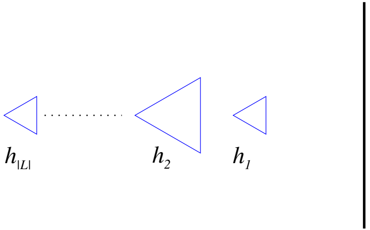


The *charge* of a hole h, denoted $q(h)$, is the number of left pointing unit triangles that comprise it minus the number of right pointing unit triangles.

$$\omega(1; R, L) \sim \prod_{h \in \mathcal{H}} C_h \prod_{1 \leq j < i \leq |\mathcal{H}|} d(h_i, h_j)^{\frac{1}{4} q(h_i) q(h_j)},$$

where $\mathcal{H}$ is the set of holes induced by the holes of side length two indexed by L and R.



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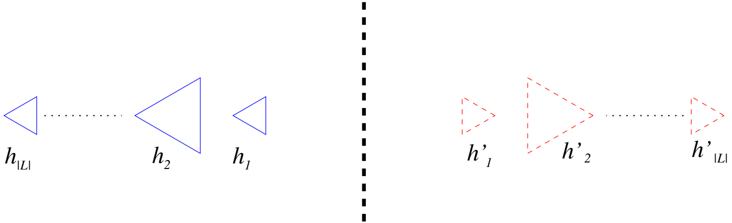
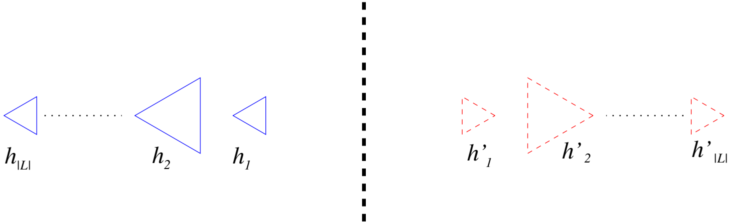


$$\omega(1;R,L) \sim C_{\mathcal{H}} \prod_{1\le i<j\le |L|} d(h_i,h_j)^{\frac{1}{4}q(h_i)q(h_j)} \cdot \prod_{1\le i<j\le |L|} d(h'_i,h'_j)^{\frac{1}{4}q(h'_i)q(h'_j)} \cdot \prod_{1\le i,j\le |L|} d(h_i,h'_j)^{\frac{1}{4}q(h_i)q(h'_j)}$$



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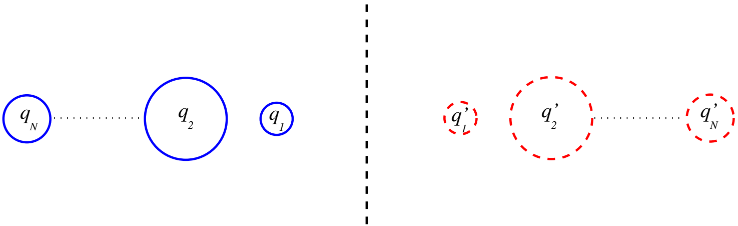

$$E_{\{q_1,\ldots,q_N\},|} = \frac{1}{2}E_{\{q_1,\ldots,q_N\},\{q'_1,\ldots,q'_N\}}$$
$$= \frac{k_e}{2}\left(\sum_{1\leq i<j\leq N}\frac{q_iq_j}{d(q_i,q_j)} + \sum_{1\leq i<j\leq N}\frac{q'_iq'_j}{d(q'_i,q'_j)} - \sum_{1\leq i,j\leq N}\frac{|q_iq'_j|}{d(q_i,q'_j)}\right).$$