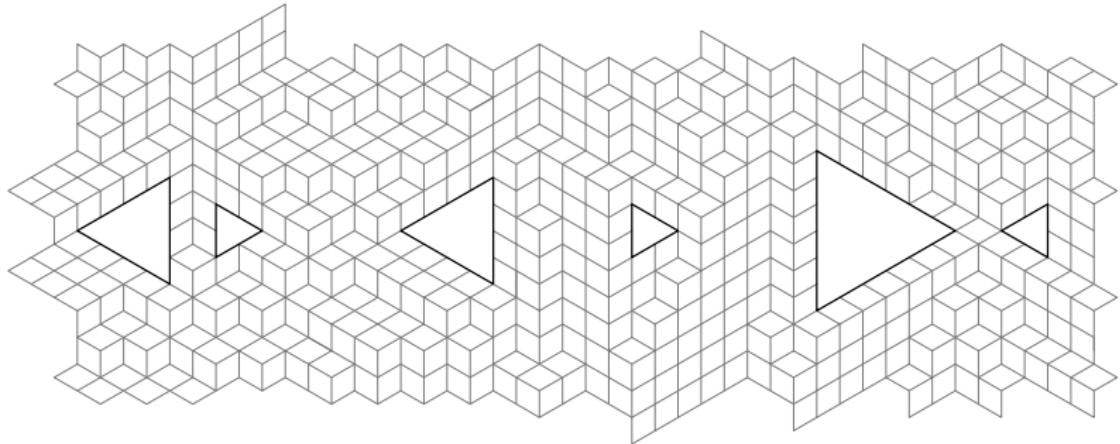


# *Symmetric rhombus tilings of holey hexagons and the method of images.*

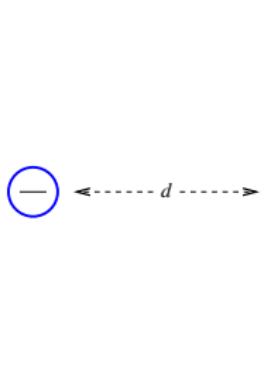


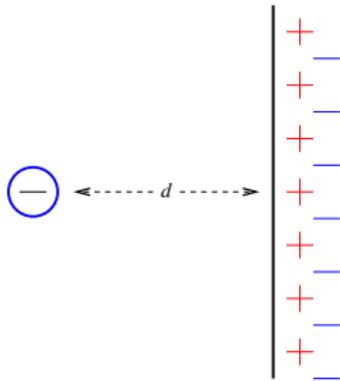
Tomack Gilmore  
Universität Wien

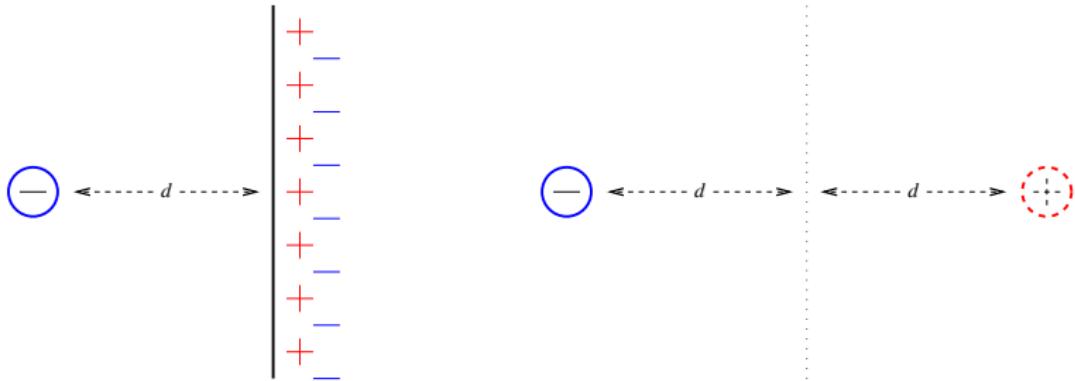
## The Method of Images

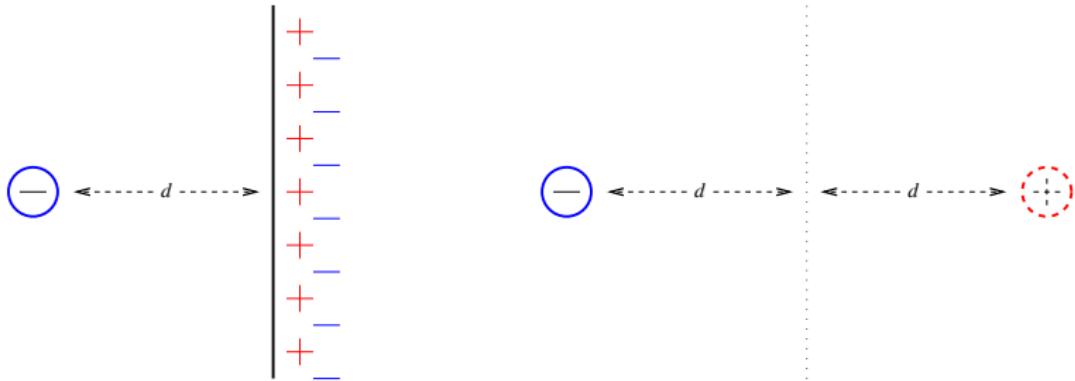






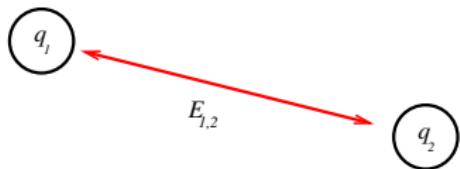


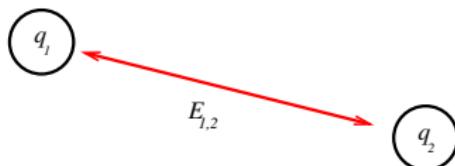




$$E_{\Theta,|} = \frac{1}{2} E_{\Theta,\oplus}$$





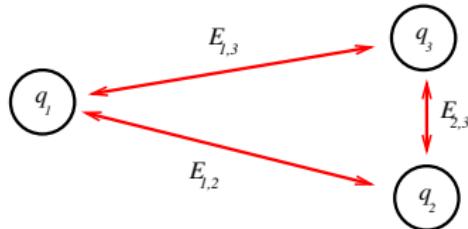


## Coulomb's Law

The electrostatic energy of the system consisting of two point charges with signed magnitudes  $q_1$  and  $q_2$  is

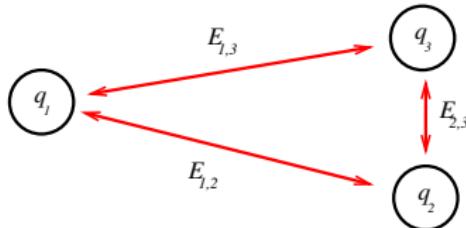
$$E_{1,2} = k_e \frac{q_1 q_2}{d(q_1, q_2)},$$

where  $k_e$  denotes Coulomb's constant and  $d(q_1, q_2)$  is the Euclidean distance between the two charges.



The energy of a system of three point charges is given by

$$E_{1,2} + E_{1,3} + E_{2,3} = k_e \left( \frac{q_1 q_2}{d(q_1, q_2)} + \frac{q_1 q_3}{d(q_1, q_3)} + \frac{q_2 q_3}{d(q_2, q_3)} \right).$$



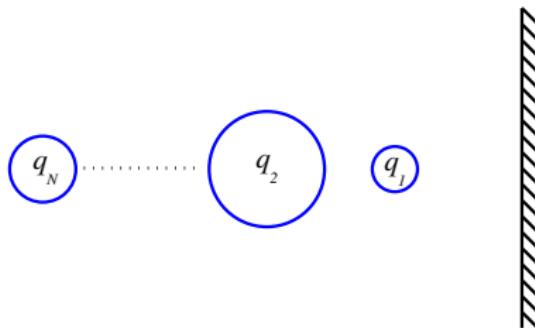
The energy of a system of three point charges is given by

$$E_{1,2} + E_{1,3} + E_{2,3} = k_e \left( \frac{q_1 q_2}{d(q_1, q_2)} + \frac{q_1 q_3}{d(q_1, q_3)} + \frac{q_2 q_3}{d(q_2, q_3)} \right).$$

More generally, the electrostatic energy of a system of point charges  $q_1, \dots, q_n$  is given by

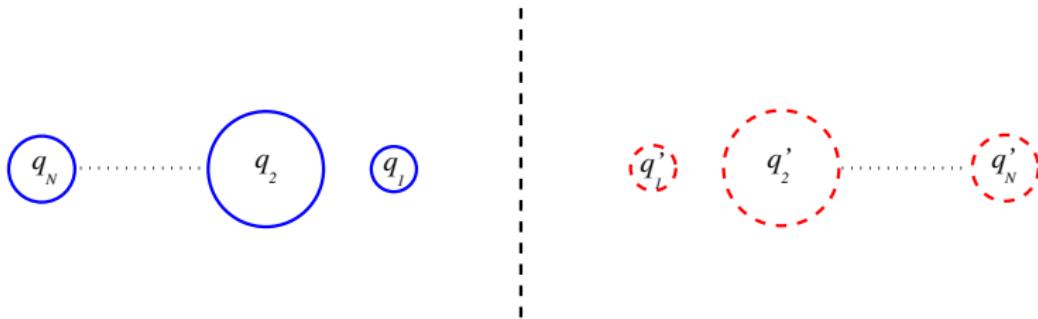
$$k_e \sum_{1 \leq i < j \leq n} \frac{q_i q_j}{d(q_i, q_j)}.$$

## Superposition Principle

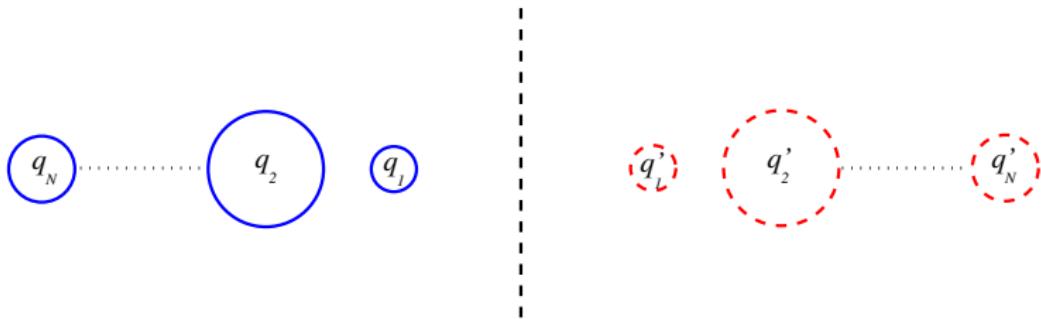


$$E_{\{q_1, \dots, q_N\}, |}$$

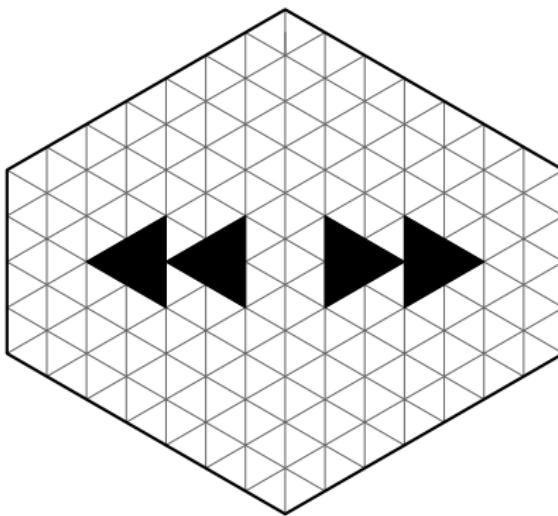
## Superposition Principle

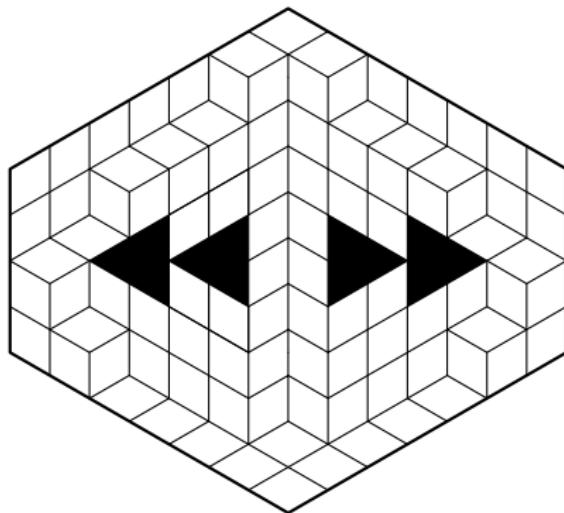


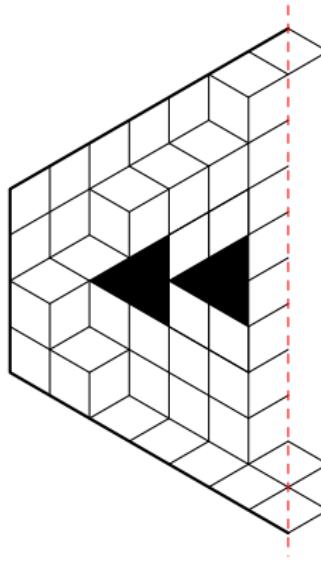
$$E_{\{q_1, \dots, q_N\}, |} = \frac{1}{2} E_{\{q_1, \dots, q_N\}, \{q'_1, \dots, q'_N\}}$$

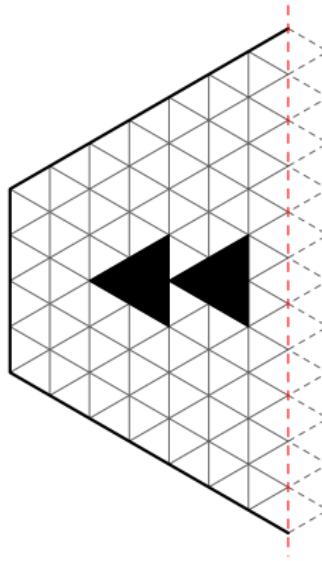


$$\begin{aligned}
 E_{\{q_1, \dots, q_N\}, |} &= \frac{1}{2} E_{\{q_1, \dots, q_N\}, \{q'_1, \dots, q'_N\}} \\
 &= \frac{k_e}{2} \left( \sum_{1 \leq i < j \leq N} \frac{q_i q_j}{d(q_i, q_j)} + \sum_{1 \leq i < j \leq N} \frac{q'_i q'_j}{d(q'_i, q'_j)} \right. \\
 &\quad \left. - \sum_{1 \leq i, j \leq N} \frac{|q_i q'_j|}{d(q_i, q'_j)} \right).
 \end{aligned}$$

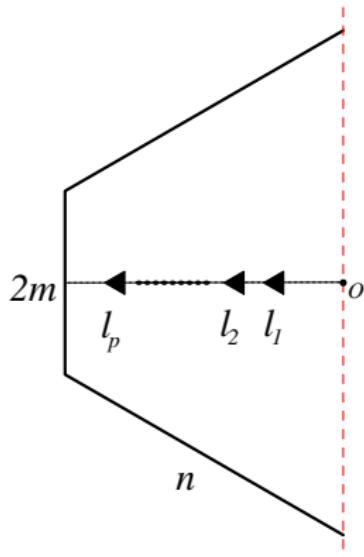






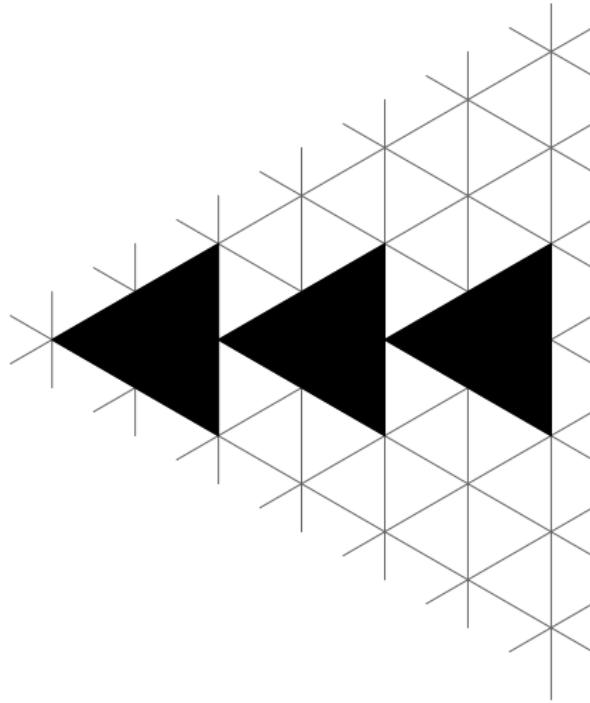


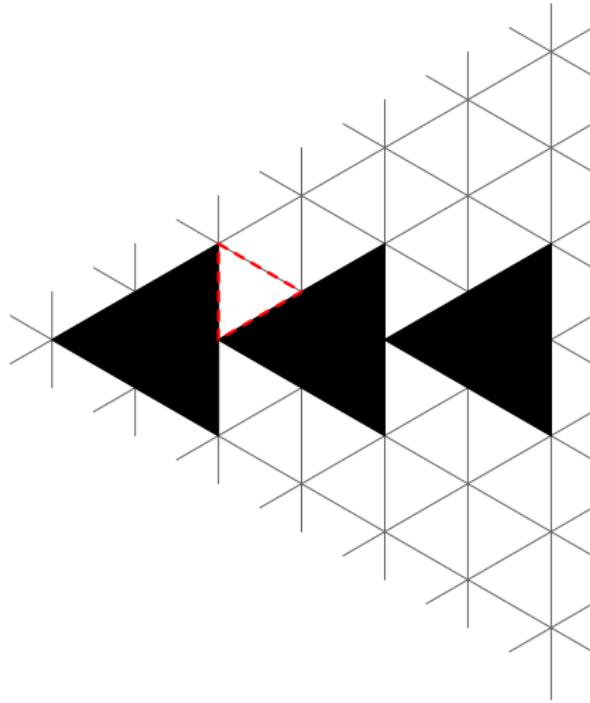
$$V_{7,4}^{\{-1,-3\}}$$

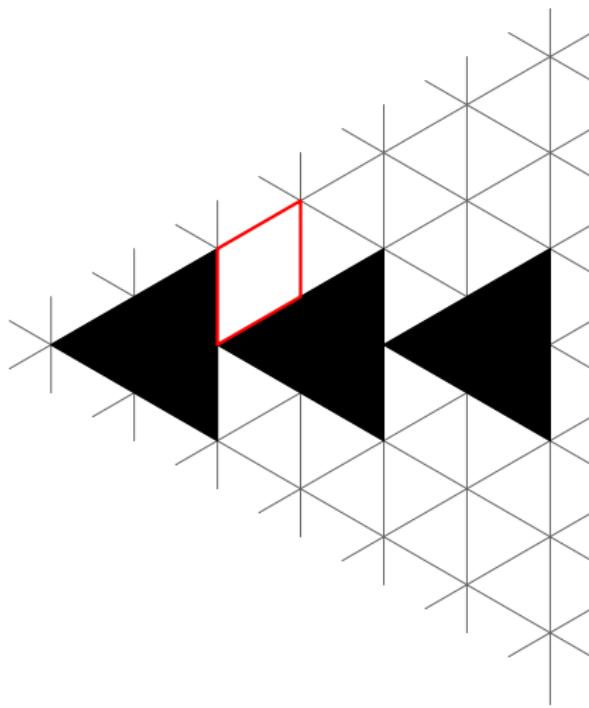


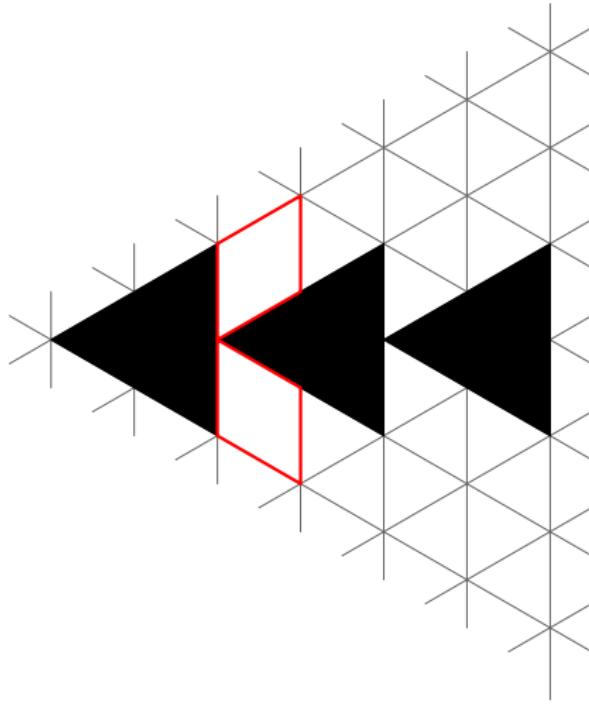
$$V_{n,2m}^L$$

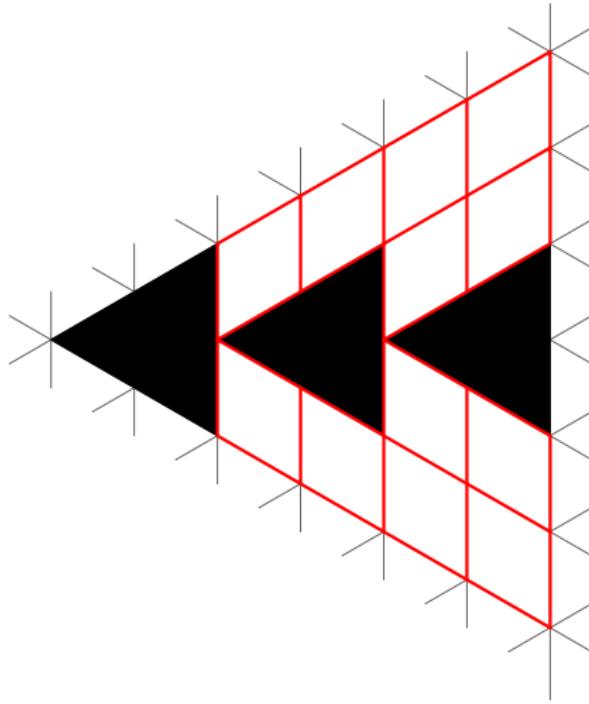
(here  $L = \{l_1, \dots, l_p\}$  indexes the left pointing triangular holes by their vertical lattice distance from the origin)

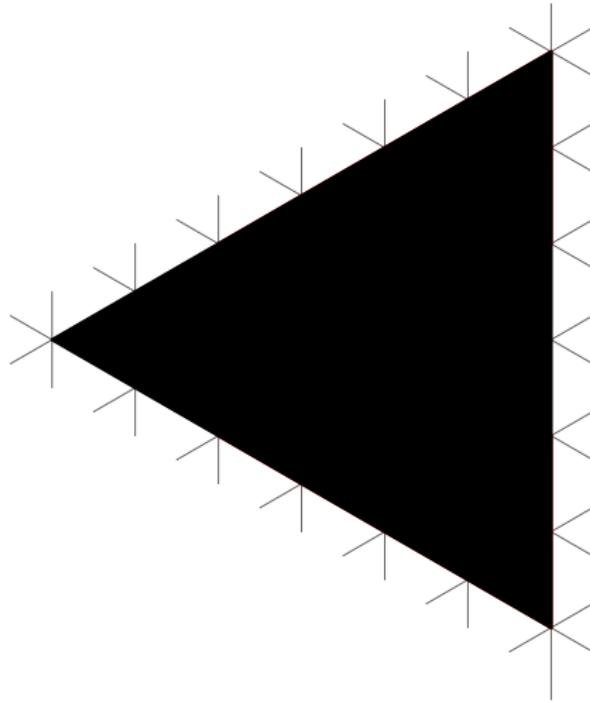












## Theorem (TG 2016)

$$M(V_{n,2m}^L) = \left( \prod_{i=1}^n \frac{2i+2m-1}{2i-1} \prod_{1 \leq i < j \leq n} \frac{i+j+2m-1}{i+j-1} \right) \cdot \det \widehat{E}_{R,L}$$

where  $R = \{-l_1, \dots, -l_{|L|}\}$ ,  $\widehat{E}_{R,L} = (\hat{e}_{i,j})_{1 \leq i,j \leq |L|}$  with  $(i,j)$ -entries given by

$$\begin{aligned} \hat{e}_{i,j} &= \frac{\Gamma(m + \frac{1}{2})\Gamma(\frac{n}{2} - \frac{l_i}{2} + \frac{3}{2})\Gamma(m+n+1)\Gamma(\frac{n}{2} + \frac{r_j}{2} + \frac{3}{2})}{2^{l_i-r_j-2}\pi\Gamma(m)\Gamma(\frac{n}{2} - \frac{l_i}{2} + 2)\Gamma(m+n+\frac{1}{2})\Gamma(\frac{n}{2} + \frac{r_j}{2} + 2)} \\ &\quad \times \sum_{s=0}^{\infty} \frac{(2 + \frac{r_j}{2} - \frac{l_i}{2})_s (\frac{1}{2})_s (m+n+1)_s (1-m)_s}{(\frac{n}{2} - \frac{l_i}{2} + 2)_s (\frac{n}{2} + \frac{r_j}{2} + 2)_s (\frac{3}{2})_s (s!)}. \end{aligned}$$

(here  $\Gamma(x) = (x-1)!$  and  $(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}$  for  $x \in \mathbb{Q}, y \in \mathbb{Z}$ ).

## Theorem (TG 2016)

$$M(V_{n,2m}^L) = \left( \prod_{i=1}^n \frac{2i+2m-1}{2i-1} \prod_{1 \leq i < j \leq n} \frac{i+j+2m-1}{i+j-1} \right) \cdot \det \widehat{E}_{R,\textcolor{blue}{L}}$$

where  $R = \{-l_1, \dots, -l_{|L|}\}$ ,  $\widehat{E}_{R,L} = (\hat{e}_{i,j})_{1 \leq i,j \leq |L|}$  with  $(i,j)$ -entries given by

$$\begin{aligned} \hat{e}_{i,j} &= \frac{\Gamma(m + \frac{1}{2})\Gamma(\frac{n}{2} - \frac{\textcolor{blue}{l}_i}{2} + \frac{3}{2})\Gamma(m+n+1)\Gamma(\frac{n}{2} + \frac{r_j}{2} + \frac{3}{2})}{2^{\textcolor{blue}{l}_i - r_j - 2}\pi\Gamma(m)\Gamma(\frac{n}{2} - \frac{\textcolor{blue}{l}_i}{2} + 2)\Gamma(m+n+\frac{1}{2})\Gamma(\frac{n}{2} + \frac{r_j}{2} + 2)} \\ &\quad \times \sum_{s=0}^{\infty} \frac{(2 + \frac{r_j}{2} - \frac{\textcolor{blue}{l}_i}{2})_s (\frac{1}{2})_s (m+n+1)_s (1-m)_s}{(\frac{n}{2} - \frac{\textcolor{blue}{l}_i}{2} + 2)_s (\frac{n}{2} + \frac{r_j}{2} + 2)_s (\frac{3}{2})_s (s!)} \end{aligned}$$

(here  $\Gamma(x) = (x-1)!$  and  $(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}$  for  $x \in \mathbb{Q}, y \in \mathbb{Z}$ ).

## Theorem (TG 2016)

$$M(V_{n,2m}^L) = \left( \prod_{i=1}^n \frac{2i+2m-1}{2i-1} \prod_{1 \leq i < j \leq n} \frac{i+j+2m-1}{i+j-1} \right) \cdot \det \widehat{E}_{\textcolor{red}{R}, \textcolor{blue}{L}}$$

where  $\textcolor{red}{R} = \{-l_1, \dots, -l_{|L|}\}$ ,  $\widehat{E}_{R,L} = (\hat{e}_{i,j})_{1 \leq i,j \leq |L|}$  with  
 $(i,j)$ -entries given by

$$\begin{aligned} \hat{e}_{i,j} = & \frac{\Gamma(m + \frac{1}{2})\Gamma(\frac{n}{2} - \frac{\textcolor{blue}{l}_i}{2} + \frac{3}{2})\Gamma(m+n+1)\Gamma(\frac{n}{2} + \frac{\textcolor{red}{r}_j}{2} + \frac{3}{2})}{2^{\textcolor{blue}{l}_i - \textcolor{red}{r}_j - 2}\pi\Gamma(m)\Gamma(\frac{n}{2} - \frac{\textcolor{blue}{l}_i}{2} + 2)\Gamma(m+n+\frac{1}{2})\Gamma(\frac{n}{2} + \frac{\textcolor{red}{r}_j}{2} + 2)} \\ & \times \sum_{s=0}^{\infty} \frac{(2 + \frac{\textcolor{red}{r}_j}{2} - \frac{\textcolor{blue}{l}_i}{2})_s (\frac{1}{2})_s (m+n+1)_s (1-m)_s}{(\frac{n}{2} - \frac{\textcolor{blue}{l}_i}{2} + 2)_s (\frac{n}{2} + \frac{\textcolor{red}{r}_j}{2} + 2)_s (\frac{3}{2})_s (s!)} \end{aligned}$$

(here  $\Gamma(x) = (x-1)!$  and  $(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}$  for  $x \in \mathbb{Q}, y \in \mathbb{Z}$ ).

Suppose  $2m \sim \xi n$  for some real  $\xi > 0$ . The *interaction* of the holes indexed by  $L$  and the vertical free boundary is

$$\omega(\xi; R, L) = \lim_{n \rightarrow \infty} \frac{M(V_{n,2m}^L)}{M(V_{n,2m}^\emptyset)}$$

Suppose  $2m \sim \xi n$  for some real  $\xi > 0$ . The *interaction* of the holes indexed by  $L$  and the vertical free boundary is

$$\begin{aligned}\omega(\xi; R, L) &= \lim_{n \rightarrow \infty} \frac{M(V_{n,2m}^L)}{M(V_{n,2m}^\emptyset)} \\ &= \lim_{n \rightarrow \infty} \det \widehat{E}_{R,L}\end{aligned}$$

Suppose  $2m \sim \xi n$  for some real  $\xi > 0$ . The *interaction* of the holes indexed by  $L$  and the vertical free boundary is

$$\begin{aligned}\omega(\xi; R, L) &= \lim_{n \rightarrow \infty} \frac{M(V_{n,2m}^L)}{M(V_{n,2m}^\emptyset)} \\ &= \lim_{n \rightarrow \infty} \det \widehat{E}_{R,L}\end{aligned}$$

As the distances between the sets of holes grows large,

$$\omega(\xi; R, L) \sim \det \left( \frac{(\xi(\xi + 2))^{-1/2}}{\pi(r_j - l_i)} \left( \frac{2}{\xi + 1} \right)^{r_j - l_i + 2} \right)_{1 \leq i, j \leq |L|}$$

Suppose  $2m \sim \xi n$  for some real  $\xi > 0$ . The *interaction* of the holes indexed by  $L$  and the vertical free boundary is

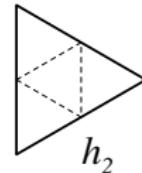
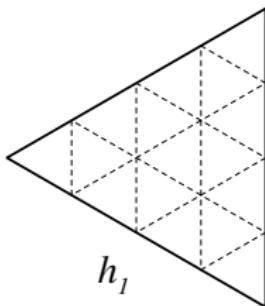
$$\begin{aligned}\omega(\xi; R, L) &= \lim_{n \rightarrow \infty} \frac{M(V_{n,2m}^L)}{M(V_{n,2m}^\emptyset)} \\ &= \lim_{n \rightarrow \infty} \det \widehat{E}_{R,L}\end{aligned}$$

As the distances between the sets of holes grows large,

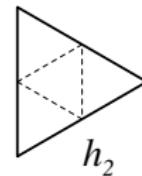
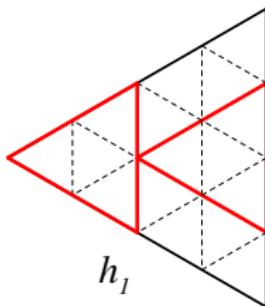
$$\omega(\xi; R, L) \sim \det \left( \frac{(\xi(\xi + 2))^{-1/2}}{\pi(r_j - l_i)} \left( \frac{2}{\xi + 1} \right)^{r_j - l_i + 2} \right)_{1 \leq i, j \leq |L|}$$

and so for  $\xi = 1$ ,

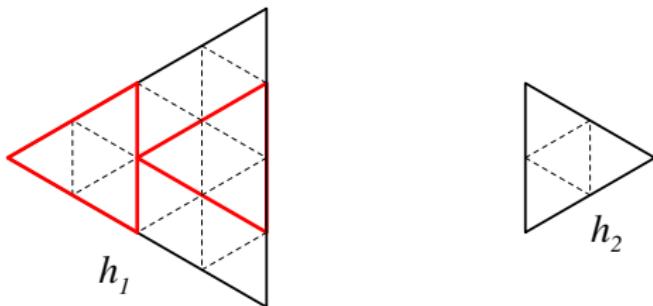
$$\omega(1; R, L) \sim \left( \frac{1}{2\pi} \right)^{|L|} \frac{\prod_{i=2}^{|L|} \prod_{j=1}^{i-1} d(r_i, r_j) d(l_i, l_j)}{\prod_{1 \leq i, j \leq |L|} d(r_i, l_j)}.$$



The *charge* of a hole  $h$ , denoted  $q(h)$ , is the number of left pointing unit triangles that comprise it minus the number of right pointing unit triangles.



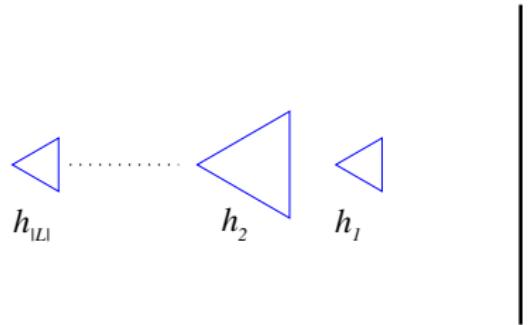
The *charge* of a hole  $h$ , denoted  $q(h)$ , is the number of left pointing unit triangles that comprise it minus the number of right pointing unit triangles.



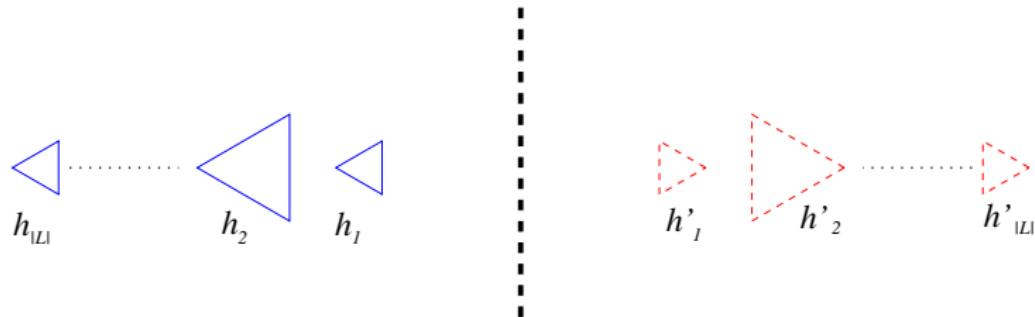
The *charge* of a hole  $h$ , denoted  $q(h)$ , is the number of left pointing unit triangles that comprise it minus the number of right pointing unit triangles.

$$\omega(1; R, L) \sim \prod_{h \in \mathcal{H}} C_h \prod_{1 \leq j < i \leq |\mathcal{H}|} d(h_i, h_j)^{\frac{1}{4}q(h_i)q(h_j)},$$

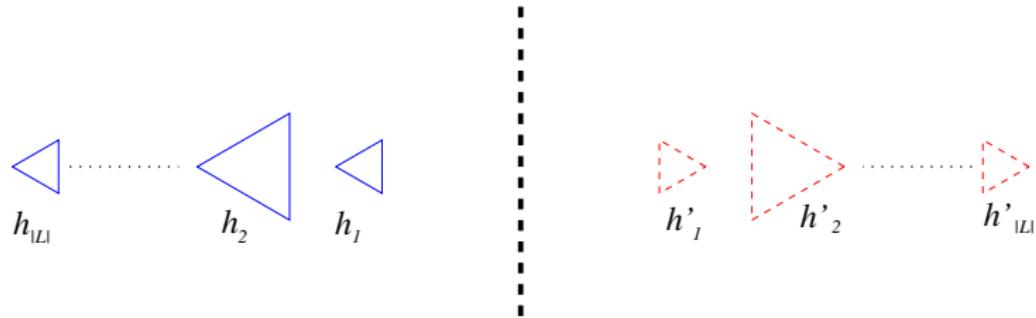
where  $\mathcal{H}$  is the set of holes induced by the holes of side length two indexed by  $L$  and  $R$ .



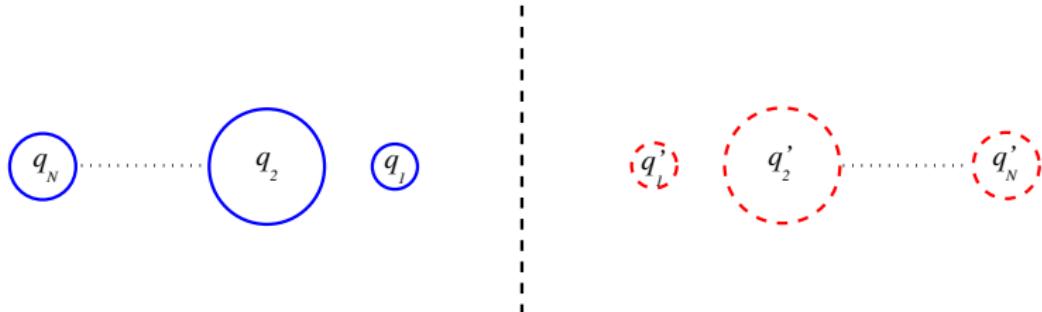
$$\omega(1; R, L) \sim \prod_{h \in \mathcal{H}} C_h \prod_{1 \leq j < i \leq |\mathcal{H}|} d(h_i, h_j)^{\frac{1}{4}q(h_i)q(h_j)}$$



$$\begin{aligned} \omega(1; R, L) \sim C_{\mathcal{H}} & \prod_{1 \leq i < j \leq |L|} d(h_i, h_j)^{\frac{1}{4}q(h_i)q(h_j)} \\ & \cdot \prod_{1 \leq i < j \leq |L|} d(h'_i, h'_j)^{\frac{1}{4}q(h'_i)q(h'_j)} \\ & \cdot \prod_{1 \leq i, j \leq |L|} d(h_i, h'_j)^{\frac{1}{4}q(h_i)q(h'_j)} \end{aligned}$$



$$\begin{aligned}\omega(1; R, L) \sim C_{\mathcal{H}} & \prod_{1 \leq i < j \leq |L|} d(\textcolor{blue}{h_i}, \textcolor{blue}{h_j})^{\frac{1}{4}q(\textcolor{blue}{h_i})q(\textcolor{blue}{h_j})} \\ & \cdot \prod_{1 \leq i < j \leq |L|} d(\textcolor{red}{h'_i}, \textcolor{red}{h'_j})^{\frac{1}{4}q(\textcolor{red}{h'_i})q(\textcolor{red}{h'_j})} \\ & \cdot \prod_{1 \leq i, j \leq |L|} d(\textcolor{blue}{h_i}, \textcolor{red}{h'_j})^{\frac{1}{4}q(\textcolor{blue}{h_i})q(\textcolor{red}{h'_j})}\end{aligned}$$



$$\begin{aligned}
 E_{\{q_1, \dots, q_N\}, |} &= \frac{1}{2} E_{\{q_1, \dots, q_N\}, \{q'_1, \dots, q'_N\}} \\
 &= \frac{k_e}{2} \left( \sum_{1 \leq i < j \leq N} \frac{q_i q_j}{d(\textcolor{blue}{q}_i, \textcolor{blue}{q}_j)} + \sum_{1 \leq i < j \leq N} \frac{q'_i q'_j}{d(\textcolor{red}{q}'_i, \textcolor{red}{q}'_j)} \right. \\
 &\quad \left. - \sum_{1 \leq i, j \leq N} \frac{|q_i q'_j|}{d(\textcolor{blue}{q}_i, \textcolor{red}{q}'_j)} \right).
 \end{aligned}$$