Non-zero values in blocks of symmetric groups

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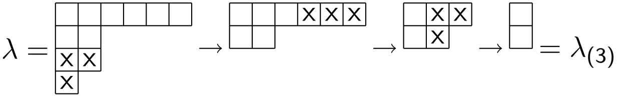
q-core partitions

Let λ be a partition and q be a positive integer. The q-core of λ is the partition $\lambda_{(q)}$ obtained by recursively removing from λ as many q-hooks as possible.

Example:



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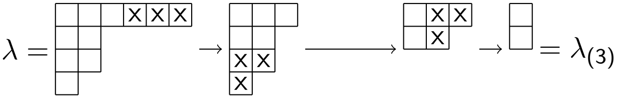

Remarks:

- q-cores are well defined (they do not depend on the order in which we remove q-hooks).

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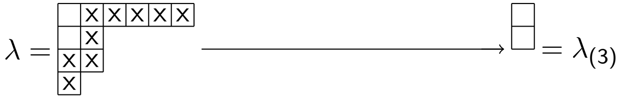

Remarks:

- q-cores are well defined (they do not depend on the order in which we remove q-hooks).
- If μ is obtained from λ by removing an hq-hook then $\mu_{(q)} = \lambda_{(q)}$, in particular $\lambda_{(q)}$ has no hook of length divisible by q.

Blocks of symmetric groups

Let $n \geq 0$, $q \geq 1$ and $\lambda = \lambda_{(q)} \vdash n - wq$ (for some $w \in \mathbb{N}$).
The q-block of S_n corresponding to λ is given by

$$B_\lambda := \{\chi^\alpha \in \mathrm{Irr}(S_n) : \alpha_{(q)} = \lambda\}.$$

Remarks:

- If q is prime then the q-blocks defined above coincide with q-blocks from modular representation (Nakayama conjecture).
- χ^α is contained in a unique q-block.
- χ^α and χ^β are contained in the same q-block if and only if $\alpha_{(q)} = \beta_{(q)}$.

q-regular partitions

A partition is q-regular if none of its parts is divisible by q.

Examples:

- $(8, 5, 1)$ is 3-regular.
- $(8, 5, 1)$ is not 4-regular.

Remark:

- If q is prime then λ is q-regular if and only if it is the cycle partition of a q-regular conjugacy class of S_n.

From now on $q \geq 2$ and $\lambda \vdash n - wq$ is a q-core.

Definition

For $\gamma \vdash n$ define $c_\lambda(\gamma) := |\{\chi^\alpha \in B_\lambda : \chi^\alpha_\gamma \neq 0\}|$.

Theorem

$\min\{c_\lambda(\gamma) : \gamma \vdash n$ *is q-regular and* $c_\lambda(\gamma) \neq 0\} = w + 1$.

$\min\{c_\lambda(\gamma) : \gamma \vdash n \text{ is } q\text{-regular and } c_\lambda(\gamma) \neq 0\} \leq w+1$

If $\lambda = ()$ let $\gamma = (n-1, 1) = (wq-1, 1)$.
If $\lambda \neq ()$ let $(h^\lambda_{1,1}, \ldots, h^\lambda_{r,r})$ be the diagonal hook lengths of λ and $\gamma = (h^\lambda_{1,1} + wq, h^\lambda_{2,2}, \ldots, h^\lambda_{r,r})$.

Then γ is q-regular and $c_\lambda(\gamma) = w+1$ as

if $\lambda = ()$ then

$$\{\alpha : \chi^\alpha \in B_\lambda \text{ and } \chi^\alpha_\gamma \neq 0\} = \{(n), (1^n)\} \cup \{(n-aq, 2, 1^{2-aq}) : 1 \leq a \leq w-1\}.$$

if $\lambda \neq ()$ then

$$\{\alpha : \chi^\alpha \in B_\lambda \text{ and } \chi^\alpha_\gamma \neq 0\} = \{(\lambda_1 + aq, \lambda_2, \ldots, \lambda_{\lambda'_1}, 1^{(w-a)q}) : 0 \leq a \leq w\}.$$

$\min\{c_\lambda(\gamma) : \gamma \vdash n \text{ is } q\text{-regular and } c_\lambda(\gamma) \neq 0\} \geq w+1$

Sketch (assuming $\chi^\alpha \in B_\lambda$ and $\chi^\alpha_\gamma \neq 0$):

- α has w hooks of length divisible by q.
- If $q|h^\alpha_{i,j}$ we can construct $f_{i,j} = \sum_{\chi^\beta \in B_\lambda} d^\beta_{i,j}\chi^\beta$ vanishing on the conjugacy class labeled by γ. Also $d^\alpha_{i,j} = \pm 1$.
- There exists $\beta_{i,j} \neq \alpha$ with $\chi^{\beta_{i,j}}_\gamma \neq 0$ and $d^{\beta_{i,j}}_{i,j} \neq 0$.
- $\beta_{i,j} \neq \beta_{k,l}$ for $(i,j) \neq (k,l)$ (with $q|h^\alpha_{i,j}, h^\alpha_{k,l}$).

Remarks

- It can happen that $c_\lambda(\gamma) = 0$, for example for $q = 2$, $\lambda = (4,3,2,1)$ and $\gamma = (11,1)$ (here $B_\lambda = \{\chi^{(6,3,2,1)}, \chi^{(4,3,2,1,1,1)}\}$).
- For $q \geq 2$ it looks that either $c_\lambda(\gamma) = 0$ or $c_\lambda(\gamma) \geq w + 1$ also when γ is not q-regular.
- For $q = 1$ we have that $()$ is the only 1-core and $B_{()} = \mathrm{Irr}(S_n)$. Here it looks that $c_{()}(\gamma) \geq n - 1$. Also $c_{()}(n-1,1) = n - 1$.