Factorisations of a group element, Hurwitz action and shellability

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joint work with **Henri Mühle** (École Polytechnique, France)

Outline

1. Framework and example:
   generated group, Hurwitz action on factorisations, shellability
2. Motivations:
   noncrossing partition lattices of reflection groups
3. Some results and a conjecture:
   compatible order on the generators, Hurwitz-transitivity, shellability

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Generated group and reduced decompositions

- (G, A) generated group
- $A \subseteq G$ generates G as a monoid
- Let $g \in G$. Write $g = a_1 a_2 \dots a_n$, with $a_i \in A$.
  Length of g: $\ell_A(g) := $ minimal such n.

Reduced decompositions of g

$\mathrm{Red}_A(g) := \{(a_1, \dots, a_n) \mid a_i \in A, g = a_1 \dots a_n\}, \quad$ where $n = \ell_A(g)$.

Example. $G = S_4 \qquad A = T := \{$all transpositions $(i\ j)\}$.

$g = (1\ 2\ 3\ 4) \qquad \ell_T(g) = 3 \qquad$ Reduced decompositions of g:

$$\begin{aligned} g &= (12)(23)(34) = (23)(13)(34) = (13)(12)(34) = (13)(34)(12) \\ &= (14)(13)(12) = (34)(14)(12) = (34)(12)(24) = (34)(24)(14) \\ &= (24)(23)(14) = (23)(34)(14) = (23)(14)(13) = (12)(34)(24) \\ &= (12)(24)(23) = (24)(14)(23) = (14)(12)(23) = (14)(23)(13) \end{aligned}$$

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Hurwitz action

Hurwitz moves

Fix $g \in G$. Take $(a_1, \dots, a_n) \in \mathrm{Red}_A(g)$. For $1 \leq i \leq n-1$ define:

$$\begin{array}{rl} \sigma_i \cdot & (a_1, \dots, a_{i-1}, \quad a_i \quad , \quad a_{i+1} \quad , a_{i+2}, \dots, a_n) \\ = & (a_1, \dots, a_{i-1}, \quad a_i a_{i+1} a_i^{-1} \quad , \quad a_i \quad , a_{i+2}, \dots, a_n) \end{array}$$

Assumption: For any $(a_1, \dots, a_n) \in \mathrm{Red}_A(g)$ and any $1 \leq i \leq n-1$, $a_i a_{i+1} a_i^{-1}$ and $a_{i+1}^{-1} a_i a_{i+1} \in A$. (e.g., A stable by conjugacy)

This defines an action on $\mathrm{Red}_A(g)$ by the braid group B_n [Hurwitz action].

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \rangle_{\mathrm{grp}}$$

$\rightsquigarrow$ **General Question 1:** Is the Hurwitz action transitive on $\mathrm{Red}_A(g)$?

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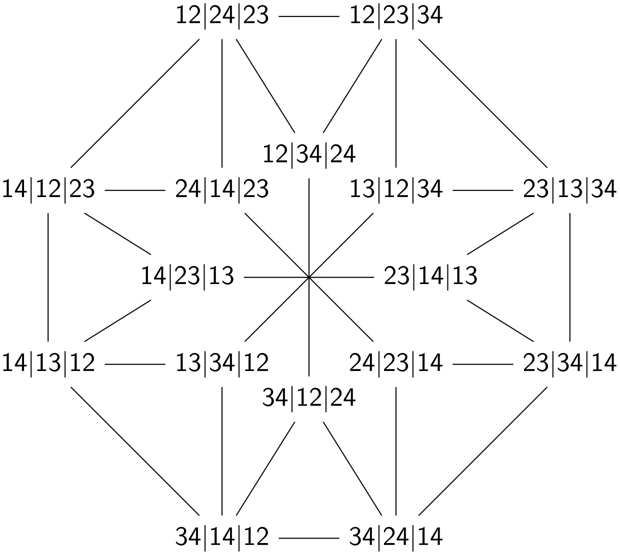
Example: Hurwitz graph of $\mathrm{Red}_T((1\ 2\ 3\ 4))$



The prefix poset

Prefix order

Equip G with a partial order $\leq_A$:

$$x \leq_A y \quad \Leftrightarrow \quad x \text{ is a } \textbf{prefix} \text{ of a reduced decomposition of } y$$
$$\Leftrightarrow \quad \ell_A(x) + \ell_A(x^{-1}y) = \ell_A(y)$$

Prefix poset of g

$$[e, g]_A := \{x \in G \mid x \leq_A g\}$$

- $[e, g]_A$ is a graded poset (by ℓ_A);
- Hasse diagram of the poset $[e, g]_A$ corresponds to **geodesics** from e to g in the Cayley graph of (G, A);
- for $x, y \in [e, g]_A$: $x \leq_A y$ if and only if a reduced decomposition of x is a **subword** of a reduced decomposition of y. [by assumption on conjugacy-stability]

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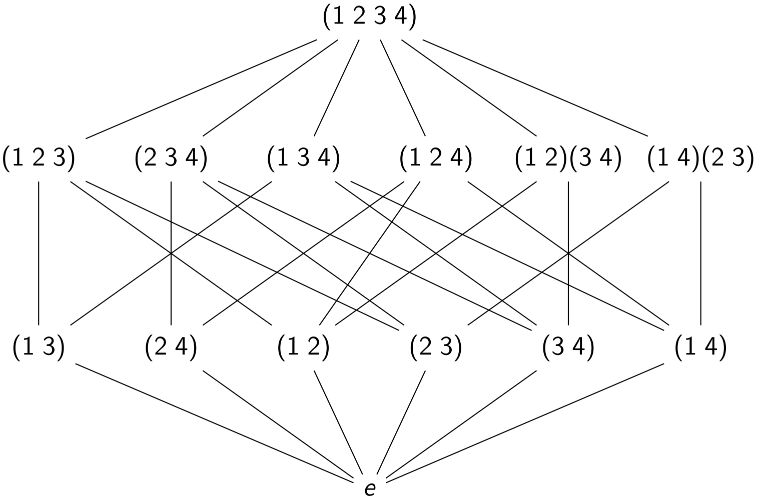
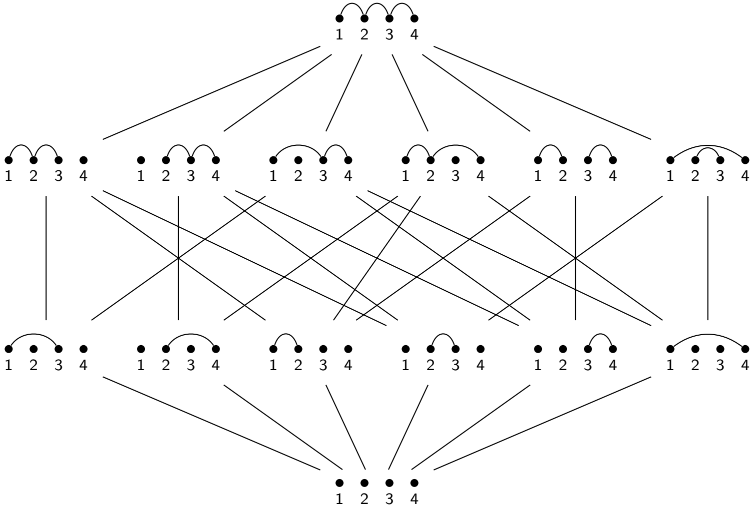
Example: $[e, (1\ 2\ 3\ 4)]_T$ in (S_4, T)



$[e, (1\ 2\ 3\ 4)]_T$ in $(S_4, T) \simeq$ Noncrossing partitions



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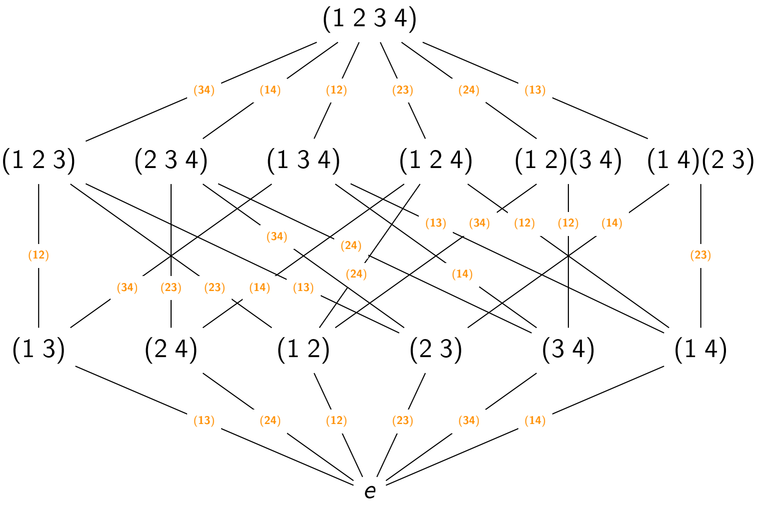

Notes: $\{\text{maximal chains of } [e,g]_A\} \longleftrightarrow \mathrm{Red}_A(g)$

$\forall x \leq_A y,\ [x,y]_A \simeq [e, x^{-1}y]_A$

Shellability

Definition

A graded poset P is EL-shellable if there exists a labelling of the edges (by a totally ordered set) such that for any interval $I \subseteq P$:

- there is a unique *increasingly labelled* maximal chain of I
- this is the lexicographically smallest among all maximal chains.

P EL-shellable $\Rightarrow$ P shellable [Björner-Wachs]
$\Rightarrow$ nice topology: the order complex is homotopy-equivalent to a wedge of spheres, ...

Definition

A graded poset P is shellable if its *order complex* is shellable, i.e.: there is a total order on the maximal chains $C_1 \prec \cdots \prec C_r$ such that $\forall i < j$, $\exists k < j$ with $C_i \cap C_j \subseteq C_k \cap C_j$, and the chains C_k and C_j differ by only one element.

$\rightsquigarrow$ **General question 2 :** Is $[e, g]_A$ EL-shellable?

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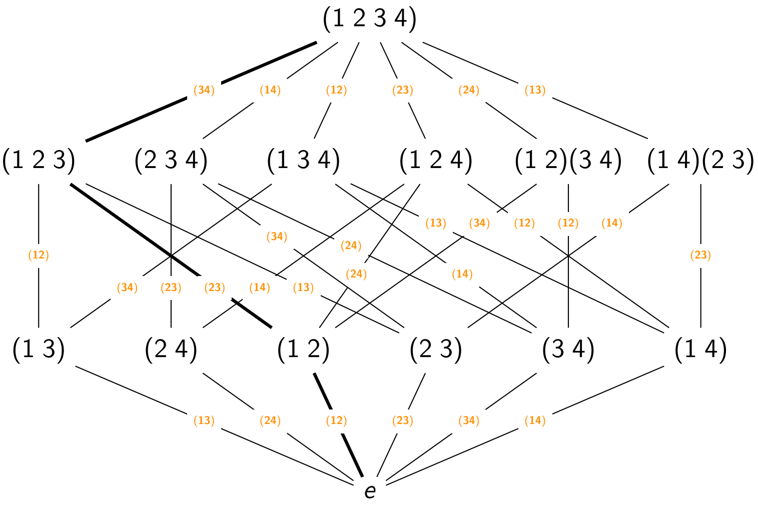

$(12) \prec (13) \prec (14) \prec (23) \prec (24) \prec (34)$

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Motivation

- W : finite Coxeter group, or well-generated complex reflection group
- T : set of all reflections of W
- c : Coxeter element of W
- W-noncrossing partitions: interval $[e, c]_T$ in $(W, \leq_T)$ $\rightsquigarrow NC_W(c)$

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Theorem (Deligne, 1974; Bessis & Corran, 2006; Bessis, 2006)

For any well-generated complex reflection group W, and any Coxeter element $c \in W$, the braid group $B_{\ell_T(c)}$ acts transitively on $Red_T(c)$.

- Uniform proof only for Coxeter groups!
- Crucial property used to construct a nice presentation of W, via its braid groups and its dual braid monoid [Bessis]

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- W : finite Coxeter group, or well-generated complex reflection group
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Theorem (Björner & Edelman, 1980; Reiner, 1997; Athanasiadis, Brady & Watt, 2007; Mühle, 2015)

For any well-generated complex reflection group W, and any Coxeter element $c \in W$, the poset $NC_W(c)$ is shellable.

- Uniform proof only for Coxeter groups! [ABW]

The Goal

- present a general framework to relate
  - transitivity of the Hurwitz action on $\mathrm{Red}_A(g)$ (General Question 1)
  - shellability of $[e, g]_A$ (General Question 2)
- help answering these questions by checking "simple" local criteria
- apply this to interesting examples

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Chain-connectedness

Definition

P graded poset. Define the chain graph of P to be the graph with vertices the maximal chains of P, and C connected to C' whenever they differ by only one element.
Say P is chain-connected if the chain graph is connected.

Observations:

- P shellable $\Rightarrow$ P chain-connected
- Hurwitz-transitivity on $\mathrm{Red}_A(g) \Rightarrow [e, g]_A$ chain-connected

Proposition

Assume

- $[e, g]_A$ *is chain-connected; and*
- *for all* $x \in [e, g]_A$, *with* $\ell_A(x) = 2$, *the Hurwitz action of* B_2 *on* $\mathrm{Red}_A(x)$ *is transitive (local Hurwitz transitivity)*

Then the Hurwitz action is transitive on $\mathrm{Red}_A(g)$.

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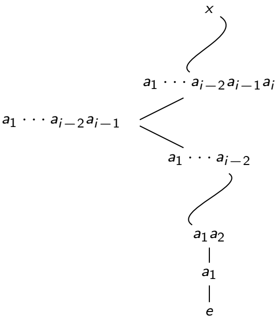
Hurwitz action corresponds to “taking detours”

$$x = a_1 \cdots a_{i-2} a_{i-1} a_i a_{i+1} \cdots a_n$$



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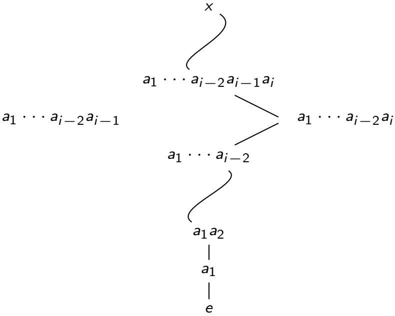

Compatible generator orders

- G, A, g as before
- assume from now on that $\mathrm{Red}_A(g)$ is finite
- $A_g := \{a \in A \mid a \leq_A g\}$ generators below g.

Definition (Mühle & R, 2015)

A total order $\prec$ on A_g is *g-compatible* if for any $x \leq_A g$ with $\ell_A(x) = 2$, there exists a unique $(s, t) \in \mathrm{Red}_A(x)$ with $s \preceq t$.

- inspired by definition of c-compatible reflection order for Coxeter groups [Athanasiadis, Brady & Watt, 2007], but forgetting the geometry;
- gives EL-shellability in rank 2.

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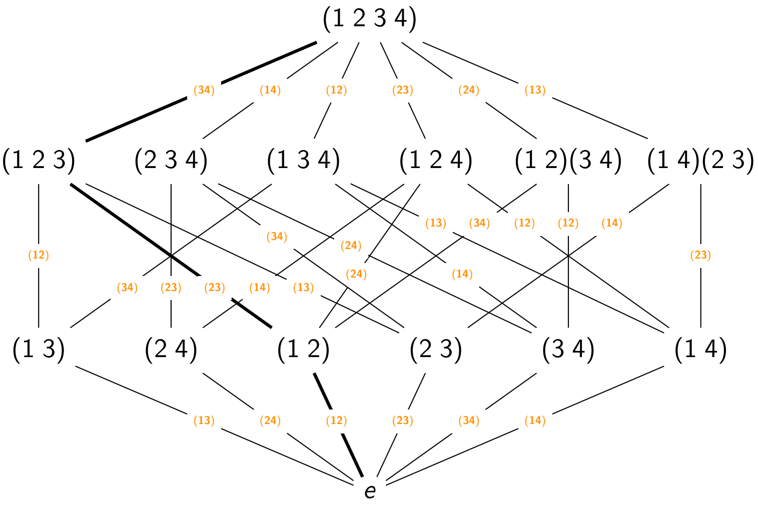
A total order $\prec$ on A_g is g-compatible if for any $x \leq_A g$ with $\ell_A(x) = 2$, there exists a unique $(s, t) \in \mathrm{Red}_A(x)$ with $s \preceq t$.

- inspired by definition of c-compatible reflection order for Coxeter groups [Athanasiadis, Brady & Watt, 2007], but forgetting the geometry;
- gives EL-shellability in rank 2.

Example: $[e,(1\ 2\ 3\ 4)]_T$ in (S_4, T)



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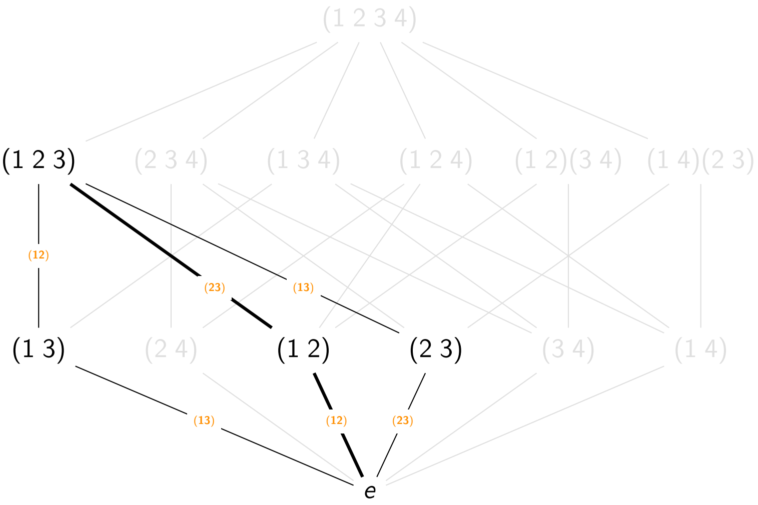

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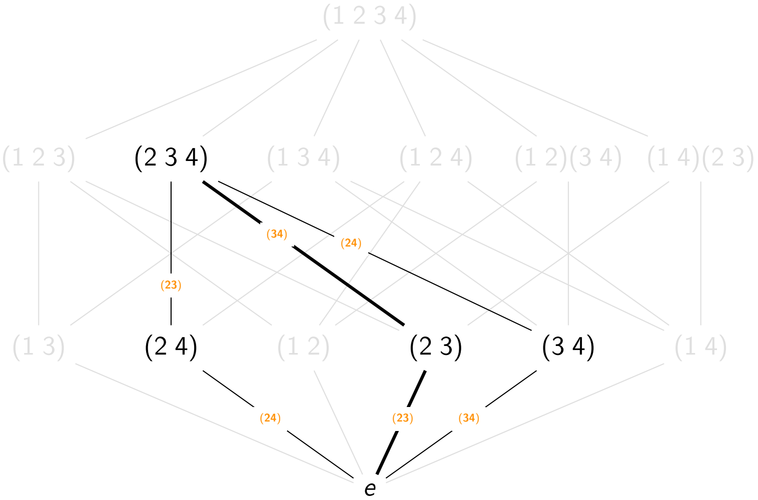

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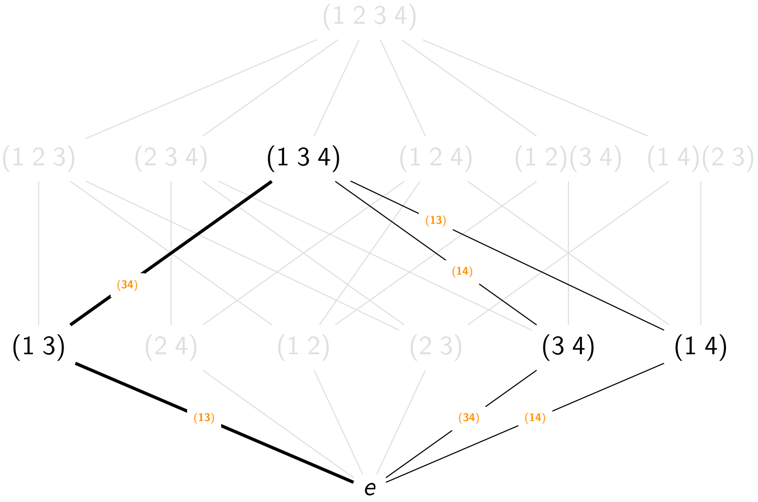

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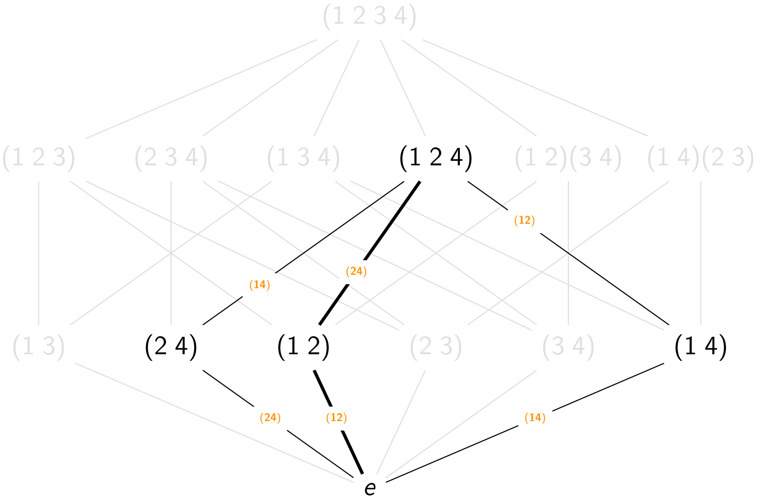

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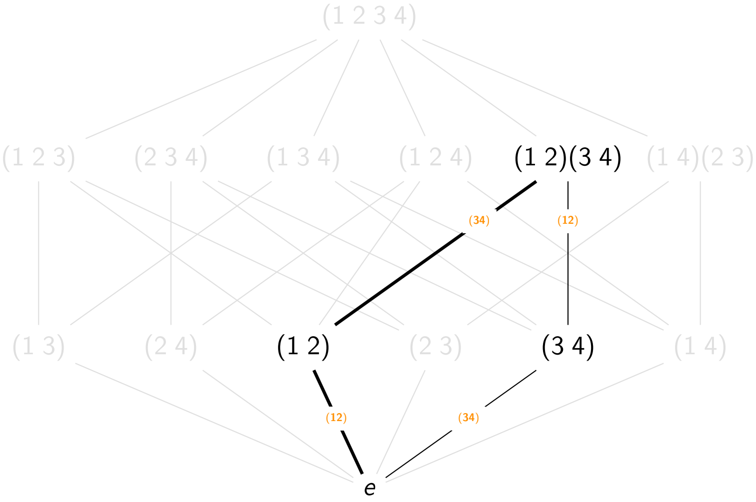

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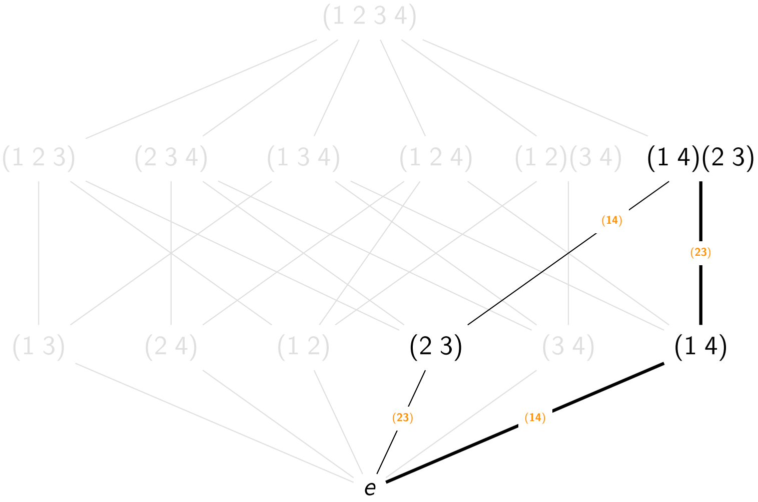

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Compatible orders and Hurwitz transitivity

Proposition (Rank 2 case)

Suppose $\ell_A(g) = 2$. Then:

$\exists$ *a g-compatible order on A_g* $\iff$ *the Hurwitz action of B_2 on $Red_A(g)$ is transitive.*

Corollary (arbitrary rank)

$\exists$ *a g-compatible order on $A_g \Longrightarrow$ local Hurwitz transitivity (i.e., for all $x \in [e, g]_A$ with $\ell_A(x) = 2$, the Hurwitz action of B_2 on $Red_A(x)$ is transitive).*

- the converse is false.
- Consequence of corollary: $\exists$ compatible order + chain-connectedness $\Rightarrow$ Hurwitz transitivity.
- Note: $\exists$ compatible order $\not\Rightarrow$ Hurwitz transitivity.

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Proof:

- In rank 2, any Hurwitz orbit has the form $g = a_1 a_2 = a_2 a_3 = \cdots = a_{s-1} a_s = a_s a_1$.
- Assume there is no rising decomposition, then $a_1 \prec a_s \prec a_{s-1} \prec \cdots \prec a_3 \prec a_2 \prec a_1$, impossible.
- so at least one rising decomposition for each orbit. $\square$

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$\exists$ a g-compatible order on $A_g \overset{?}{\Longrightarrow} [e,g]_A$ shellable ? No!

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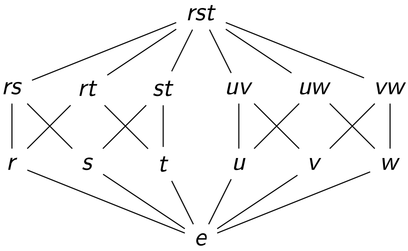

Compatible orders and shellability

Conjecture (Mühle & R, 2015)

Let G, A, g be as before. Suppose

- *there exists a g-compatible order on A_g;*
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Then $[e, g]_A$ is EL-shellable.
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We reduced the conjecture to:

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Same hypotheses.
Then for any generator a in A_g (excepted the $\prec$-smallest one), there exists another generator b in A_g such that

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Further questions

- Applications to specific groups:
  - complex reflection groups (need to construct uniformly a compatible order!);
  - (generalized) alternating groups;
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  - $GL_n(\mathbb{F}_q)$ [Huang-Lewis-Reiner]
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