Séminaire Lotharingien de Combinatoire **77** (2018), Article B77e

DIVISIONALLY FREE RESTRICTIONS OF REFLECTION ARRANGEMENTS

GERHARD RÖHRLE

Abstract. We study some aspects of divisionally free arrangements which were recently introduced by Abe. Crucially, Terao's conjecture on the combinatorial nature of freeness holds within this class. We show that while it is compatible with products, surprisingly, it is not closed under taking localizations. In addition, we determine all divisionally free restrictions of all reflection arrangements.

1. Introduction

The interplay between algebraic and combinatorial structures of hyperplane arrangements has been a driving force in the study of the field for many decades. At the very heart of these investigations lies Terao's fundamental Conjecture 1.1 which asserts that the algebraic property of freeness of an arrangement is determined by purely combinatorial data.

Conjecture 1.1 ([OT92, Conj. 4.138])**.** *For a fixed field, freeness of the arrangement $\mathscr{A}$ only depends on its lattice $L(\mathscr{A})$, i.e. is combinatorial.*

The conjecture is known to be sensitive to a change of the characteristic of the underlying field, cf. [Zie90, §4]. It is still open even for dimension 3.

Recently, T. Abe [Abe16] introduced a new class of free hyperplane arrangements, so called *divisionally free* arrangements $\mathcal{DF}$ (Definition 2.10). This properly encompasses the class of inductively free arrangements $\mathcal{IF}$ (Definition 2.5), cf. [Abe16, Thm. 1.6]. The relevance of this new notion is that Conjecture 1.1 is valid within $\mathcal{DF}$, cf. [Abe16, Thm. 4.4(3)].

Each of the classes of free, inductively free and recursively free arrangements is compatible with the product construction for arrangements ([OT92, Prop. 4.28], [HR15, Prop. 2.10], [HRS17, Thm. 2]). We show that this also is the case for $\mathcal{DF}$, cf. Proposition 2.12.

All the previously mentioned classes of free arrangements are known to be closed with respect to taking localizations ([OT92, Thm. 4.37], [HRS17, Thm. 1]). Unexpectedly, this fails for Abe's new class $\mathcal{DF}$, see Example 2.16.

Because of its relevance to Conjecture 1.1, it is important to know which arrangements from a given class belong to $\mathcal{DF}$, e.g. see [Abe16, §6]. Reflection arrangements have had a

2010 *Mathematics Subject Classification.* 20F55, 52C35, 14N20.

Key words and phrases. hyperplane arrangements, Terao's Conjecture, inductively free arrangements, reflection arrangements, restricted arrangements, divisionally free arrangements.

pivotal role in the theory of hyperplane arrangements ever since. In [Abe16, Cor. 4.7], Abe determined all irreducible divisionally free reflection arrangements, see Theorem 3.1. We extend this classification to all restrictions of reflection arrangements in Theorem 3.3.

For general information about arrangements and reflection groups we refer the reader to [OS82] and [OT92]. In this article we use the classification and labeling of the irreducible unitary reflection groups due to Shephard and Todd, [ST54].

2. Preliminaries

2.1. **Hyperplane Arrangements.** Let $\mathbb{K}$ be a field and let $V = \mathbb{K}^\ell$. By a hyperplane arrangement in V we mean a finite set $\mathscr{A}$ of hyperplanes in V. Such an arrangement is denoted $(\mathscr{A}, V)$ or simply $\mathscr{A}$. If $\dim V = \ell$ we call $\mathscr{A}$ an ℓ-arrangement. The number of elements in $\mathscr{A}$ is given by $|\mathscr{A}|$. The empty ℓ-arrangement is denoted by Φ_ℓ.

By $L(\mathscr{A})$ we denote the set of all nonempty intersections of elements of $\mathscr{A}$, [OT92, Def. 1.12]. For $X \in L(\mathscr{A})$, we have two associated arrangements, firstly the subarrangement $\mathscr{A}_X := \{H \in \mathscr{A} \mid X \subseteq H\} \subseteq \mathscr{A}$ of $\mathscr{A}$ and secondly, the *restriction of $\mathscr{A}$ to X*, $(\mathscr{A}^X, X)$, where $\mathscr{A}^X := \{X \cap H \mid H \in \mathscr{A} \setminus \mathscr{A}_X\}$, [OT92, Def. 1.13]. Note that V belongs to $L(\mathscr{A})$ as the intersection of the empty collection of hyperplanes and $\mathscr{A}^V = \mathscr{A}$.

If $0 \in H$ for each H in $\mathscr{A}$, then $\mathscr{A}$ is called *central*. We only consider central arrangements.

Let $H \in \mathscr{A}$ (for $\mathscr{A} \neq \Phi_\ell$) and define $\mathscr{A}' := \mathscr{A} \setminus \{H\}$, and $\mathscr{A}'' := \mathscr{A}^H$. Then $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ is a *triple* of arrangements, [OT92, Def. 1.14].

The *product* $\mathscr{A} = (\mathscr{A}_1 \times \mathscr{A}_2, V_1 \oplus V_2)$ of two arrangements $(\mathscr{A}_1, V_1), (\mathscr{A}_2, V_2)$ is defined by

$$\mathscr{A} := \mathscr{A}_1 \times \mathscr{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathscr{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathscr{A}_2\},$$

see [OT92, Def. 2.13]. In particular, $|\mathscr{A}| = |\mathscr{A}_1| + |\mathscr{A}_2|$.

An arrangement $\mathscr{A}$ is called *reducible*, if it is of the form $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$, where $\mathscr{A}_i \neq \Phi_0$ for $i = 1, 2$, else $\mathscr{A}$ is *irreducible*, [OT92, Def. 2.15].

If $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ is a product, then by [OT92, Prop. 2.14] there is a lattice isomorphism

$$L(\mathscr{A}_1) \times L(\mathscr{A}_2) \cong L(\mathscr{A}) \quad \text{by} \quad (X_1, X_2) \mapsto X_1 \oplus X_2.$$

It is easy to see that for $X = X_1 \oplus X_2 \in L(\mathscr{A})$, we have

$$\mathscr{A}^X = \mathscr{A}_1^{X_1} \times \mathscr{A}_2^{X_2}. \tag{2.1}$$

The *characteristic polynomial* $\chi(\mathscr{A}, t) \in \mathbb{Z}[t]$ of $\mathscr{A}$ is defined by

$$\chi(\mathscr{A}, t) := \sum_{X \in L(\mathscr{A})} \mu(X) t^{\dim X},$$

where μ is the Möbius function of $L(\mathscr{A})$, see [OT92, Def. 2.52].

If $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ is a product, then, thanks to [OT92, Lem. 2.50],

$$\chi(\mathscr{A}, t) = \chi(\mathscr{A}_1, t) \cdot \chi(\mathscr{A}_2, t). \tag{2.2}$$

2.2. **Free Arrangements.** Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* of V. If $\mathscr{A}$ is an arrangement in V, then for every $H \in \mathscr{A}$ we may fix $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. We call $Q(\mathscr{A}) := \prod_{H\in\mathscr{A}} \alpha_H \in S$ the *defining polynomial* of $\mathscr{A}$.

The *module of $\mathscr{A}$-derivations* is the S-submodule of $\mathrm{Der}(S)$, the S-module of $\mathbb{K}$-derivations of S, defined by

$$D(\mathscr{A}) := \{\theta \in \mathrm{Der}(S) \mid \theta(Q(\mathscr{A})) \in Q(\mathscr{A})S\}.$$

The arrangement $\mathscr{A}$ is said to be *free* if $D(\mathscr{A})$ is a free S-module.

If $\mathscr{A}$ is a free ℓ-arrangement, then $D(\mathscr{A})$ admits an S-basis of ℓ homogeneous derivations $\theta_1, \ldots, \theta_\ell$, by [OT92, Prop. 4.18]. The *exponents* of the free arrangement $\mathscr{A}$ are given by the multiset given by the polynomial degrees of the θ_i, $\exp \mathscr{A} := \{\mathrm{pdeg}\,\theta_1, \ldots, \mathrm{pdeg}\,\theta_\ell\}$.

Terao's basic *Addition-Deletion Theorem* plays a key role in the study of free arrangements.

Theorem 2.3 ([Ter80], [OT92, Thm. 4.51])**.** *Suppose $\mathscr{A} \neq \Phi_\ell$ and let $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ be a triple of arrangements. Then any two of the following statements imply the third:*

(i) *$\mathscr{A}$ is free with $\exp \mathscr{A} = \{b_1, \ldots, b_{\ell-1}, b_\ell\}$;*

(ii) *$\mathscr{A}'$ is free with $\exp \mathscr{A}' = \{b_1, \ldots, b_{\ell-1}, b_\ell - 1\}$;*

(iii) *$\mathscr{A}''$ is free with $\exp \mathscr{A}'' = \{b_1, \ldots, b_{\ell-1}\}$.*

The following is Terao's celebrated *Factorization Theorem* for free arrangements.

Theorem 2.4 ([OT92, Thm. 4.137])**.** *If $\mathscr{A}$ is free with $\exp \mathscr{A} = \{b_1, \ldots, b_{\ell-1}, b_\ell\}$, then*

$$\chi(\mathscr{A}, t) = \prod_{i=1}^{\ell} (t - b_i).$$

2.3. **Inductively Free Arrangements.** An iterative application of the addition part of Theorem 2.3 leads to the class of *inductively free* arrangements.

Definition 2.5 ([OT92, Def. 4.53])**.** The class $\mathcal{IF}$ of *inductively free* arrangements is the smallest class of arrangements subject to

(i) $\Phi_\ell \in \mathcal{IF}$ for each $\ell \geq 0$;

(ii) if there exists a hyperplane $H_0 \in \mathscr{A}$ such that both $\mathscr{A}'$ and $\mathscr{A}''$ belong to $\mathcal{IF}$, and $\exp \mathscr{A}'' \subseteq \exp \mathscr{A}'$, then $\mathscr{A}$ also belongs to $\mathcal{IF}$.

There is a hereditary version of $\mathcal{IF}$, cf. [OT92, §6.4].

Definition 2.6. The arrangement $\mathscr{A}$ is called *hereditarily inductively free* provided that $\mathscr{A}^X$ is inductively free for each $X \in L(\mathscr{A})$. Denote this class by $\mathcal{HIF}$.

Note that if $\mathscr{A}$ is hereditarily inductively free, it is inductively free as $V \in L(\mathscr{A})$ and $\mathscr{A}^V = \mathscr{A}$.

2.4. **Reflection Arrangements.** Let $W \subseteq \mathrm{GL}(V)$ be a finite, complex reflection group acting on the complex vector space $V = \mathbb{C}^\ell$. The *reflection arrangement* of W in V is the hyperplane arrangement $\mathscr{A}(W)$ consisting of the reflecting hyperplanes of the elements in W acting as reflections on V.

Terao [Ter80] has shown that every reflection arrangement $\mathscr{A}(W)$ is free and that the exponents of $\mathscr{A}(W)$ coincide with the coexponents of W, cf. [OT92, Prop. 6.59 and Thm. 6.60].

We recall the classification of the inductively free reflection arrangements from [HR15].

Theorem 2.7 ([HR15, Thms. 1.1 and 1.2]). *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A}(W)$. Then the following hold:*

(i) *$\mathscr{A}(W)$ is inductively free if and only if W does not admit an irreducible factor isomorphic to a monomial group $G(r,r,\ell)$ for $r, \ell \geq 3$, $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}$, or G_{34}.*

(ii) *$\mathscr{A}(W)$ is inductively free if and only if it is hereditarily inductively free.*

2.5. **Restrictions of Reflection Arrangements.** Orlik and Solomon defined intermediate arrangements $\mathscr{A}_\ell^k(r)$ in [OS82, §2] (cf. [OT92, §6.4]) which interpolate between the reflection arrangements of $G(r,r,\ell)$ and $G(r,1,\ell)$. They show up as restrictions of the reflection arrangement of $G(r,r,\ell)$, [OS82, Prop. 2.14] (cf. [OT92, Prop. 6.84]).

For $\ell \geq 2$ and $0 \leq k \leq \ell$ the defining polynomial of $\mathscr{A}_\ell^k(r)$ is given by

$$Q(\mathscr{A}_\ell^k(r)) = x_1 \cdots x_k \prod_{\substack{1 \leq i < j \leq \ell \\ 0 \leq n < r}} (x_i - \zeta^n x_j),$$

where ζ is a primitive r^{th} root of unity, so that $\mathscr{A}_\ell^\ell(r) = \mathscr{A}(G(r,1,\ell))$ and $\mathscr{A}_\ell^0(r) = \mathscr{A}(G(r,r,\ell))$. For $k \neq 0, \ell$, these are not reflection arrangements themselves.

We recall the classification of the inductively free restrictions of all reflection arrangements.

Theorem 2.8 ([AHR14, Thms. 1.2 and 1.3]). *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A}(W)$ and let $V \neq X \in L(\mathscr{A}(W))$.*

(a) *$\mathscr{A}(W)^X$ is inductively free if and only if one of the following holds:*

  (i) *$\mathscr{A}$ is inductively free;*

  (ii) *$W = G(r,r,\ell)$, $r \geq 3$ and $\mathscr{A}(W)^X \cong \mathscr{A}_p^k(r)$, for $p = \dim X$ and $p - 2 \leq k \leq p$;*

  (iii) *$W = G_{24}, G_{27}, G_{29}, G_{31}, G_{33}$, or G_{34} and $X \in L(\mathscr{A}(W))$ with $\dim X \leq 3$.*

(b) *$\mathscr{A}(W)^X$ is inductively free if and only if it is hereditarily inductively free.*

2.6. **Divisionally Free Arrangements.** First we recall the key result from [Abe16].

Theorem 2.9 ([Abe16, Thm. 1.1]). *Let $\mathscr{A} \neq \Phi_\ell$. Suppose there is a hyperplane H in $\mathscr{A}$ such that the restriction $\mathscr{A}^H$ is free and that $\chi(\mathscr{A}^H, t)$ divides $\chi(\mathscr{A}, t)$. Then $\mathscr{A}$ is free.*

Theorem 2.9 can be viewed as a strengthening of the addition part of Theorem 2.3. An iterative application leads to the class $\mathcal{DF}$.

Definition 2.10 ([Abe16, Def. 1.5])**.** An ℓ-arrangement $\mathscr{A}$ is called *divisionally free* if $\ell \leq 2$, $\mathscr{A} = \Phi_\ell$, or else there is a sequence of consecutive restrictions of arrangements starting with $\mathscr{A}$ and ending in a 2-arrangement such that the successive characteristic polynomials divide one another. That is, there is a sequence of arrangements $\mathscr{A} = \mathscr{A}_\ell, \mathscr{A}_{\ell-1}, \ldots, \mathscr{A}_2$ such that for each $i = 3, \ldots, \ell$ there is an H_i in $\mathscr{A}_i$ so that $\mathscr{A}_i^{H_i} = \mathscr{A}_{i-1}$ and $\chi(\mathscr{A}_i^{H_i}, t)$ divides $\chi(\mathscr{A}_i, t)$. Denote this class by $\mathcal{DF}$.

Thanks to Theorem 2.9 and the fact that any 2-arrangement is free, any $\mathscr{A}$ in $\mathcal{DF}$ is free.

In [Abe16, Thm. 1.6], Abe observed that $\mathcal{IF} \subsetneq \mathcal{DF}$. The reflection arrangement of the complex reflection group G_{31} is divisionally free but not inductively free.

Theorem 2.11 ([Abe16, Thm. 5.6])**.** *Let $r, \ell \geq 3$. Then $\mathscr{A}_\ell^k(r) \in \mathcal{DF}$ if and only if $k \neq 0$.*

It is immediate from Theorem 2.11 and [OS82, Prop. 2.11] (cf. [OT92, Prop. 6.82]) that the class $\mathcal{DF}$ is not closed under restrictions.

Each of the classes of free, inductively free and recursively free arrangements is compatible with the product construction for arrangements, cf. [OT92, Prop. 4.28], [HR15, Prop. 2.10], [HRS17, Thm. 2]. We observe that this also holds for the class $\mathcal{DF}$.

Proposition 2.12. *Let $(\mathscr{A}_1, V_1), (\mathscr{A}_2, V_2)$ be two arrangements. Then $\mathscr{A} = (\mathscr{A}_1 \times \mathscr{A}_2, V_1 \oplus V_2)$ is divisionally free if and only if both $\mathscr{A}_1$ and $\mathscr{A}_2$ are divisionally free.*

Proof. First suppose that both $\mathscr{A}_1$ and $\mathscr{A}_2$ are divisionally free. Then we claim that also $\mathscr{A}$ is so. We argue via induction on $|\mathscr{A}|$. If both $\mathscr{A}_1$ and $\mathscr{A}_2$ are empty, there is nothing to show. So suppose that $|\mathscr{A}| \geq 1$ and that the claim holds for any product of divisionally free arrangements with fewer than $|\mathscr{A}|$ hyperplanes. Without loss of generality there exists an H_1 in $\mathscr{A}_1$ such that $\mathscr{A}_1^{H_1}$ belongs to $\mathcal{DF}$ and $\chi(\mathscr{A}_1^{H_1}, t)$ divides $\chi(\mathscr{A}_1, t)$. Letting $H = H_1 \oplus V_2 \in \mathscr{A}$, by (2.1), $\mathscr{A}^H = \mathscr{A}_1^{H_1} \times \mathscr{A}_2$ is a product of divisionally free arrangements with $|\mathscr{A}^H| < |\mathscr{A}|$. So, by our induction hypothesis, $\mathscr{A}^H$ is divisionally free. In addition, since $\chi(\mathscr{A}^H, t) = \chi(\mathscr{A}_1^{H_1}, t) \cdot \chi(\mathscr{A}_2, t)$ divides $\chi(\mathscr{A}_1, t) \cdot \chi(\mathscr{A}_2, t) = \chi(\mathscr{A}, t)$, cf. (2.2), we infer that $\mathscr{A}$ belongs to $\mathcal{DF}$.

Conversely, suppose that $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ belongs to $\mathcal{DF}$. We claim that then both $\mathscr{A}_1$ and $\mathscr{A}_2$ also belong to $\mathcal{DF}$. Again we argue by induction on $|\mathscr{A}|$. If both $\mathscr{A}_1$ and $\mathscr{A}_2$ are empty, there is nothing to show. So suppose that $|\mathscr{A}| \geq 1$ and that the claim holds for any product in $\mathcal{DF}$ with fewer than $|\mathscr{A}|$ hyperplanes. Without loss of generality we may assume that there is an $H = H_1 \oplus V_2$ in $\mathscr{A}$ such that $\mathscr{A}^H$ belongs to $\mathcal{DF}$ and $\chi(\mathscr{A}^H, t)$ divides $\chi(\mathscr{A}, t)$. Since $|\mathscr{A}^H| < |\mathscr{A}|$ and $\mathscr{A}^H = \mathscr{A}_1^{H_1} \times \mathscr{A}_2$, both $\mathscr{A}_1^{H_1}$ and $\mathscr{A}_2$ belong to $\mathcal{DF}$, by our induction hypothesis. Moreover, since $\chi(\mathscr{A}^H, t) = \chi(\mathscr{A}_1^{H_1}, t) \cdot \chi(\mathscr{A}_2, t)$ and $\chi(\mathscr{A}, t) = \chi(\mathscr{A}_1, t) \cdot \chi(\mathscr{A}_2, t)$, cf. (2.2), it follows that $\chi(\mathscr{A}_1^{H_1}, t)$ divides $\chi(\mathscr{A}_1, t)$. Therefore, also $\mathscr{A}_1$ belongs to $\mathcal{DF}$. $\square$

There is a hereditary version of $\mathcal{DF}$.

Definition 2.13 ([Abe16, Def. 5.7(2)]). The arrangement $\mathscr{A}$ is called *hereditarily divisionally free* provided that $\mathscr{A}^X$ is divisionally free for each $X \in L(\mathscr{A})$. We denote this class by $\mathcal{HDF}$.

Remark 2.14. (i). Clearly, since $\mathcal{IF} \subseteq \mathcal{DF}$, we have $\mathcal{HIF} \subseteq \mathcal{HDF}$.

(ii). Note that $\mathcal{HDF} \subsetneq \mathcal{DF}$. For, in [HR15, Ex. 2.16], we constructed an inductively free arrangement $\mathscr{A}$ which admits a hyperplane H such that $\mathscr{A}^H$ is not free. In particular, $\mathscr{A} \in \mathcal{DF} \setminus \mathcal{HDF}$.

The compatibility with products from Proposition 2.12 descends to $\mathcal{HDF}$. The proof follows readily from Proposition 2.12 and (2.1).

Corollary 2.15. *Let $\mathscr{A}_1, \mathscr{A}_2$ be two arrangements. Then $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ is hereditarily divisionally free if and only if both $\mathscr{A}_1$ and $\mathscr{A}_2$ are hereditarily divisionally free.*

While the classes of free, inductively free and recursively free arrangements are closed under taking localizations, cf. [OT92, Thm. 4.37], [HRS17, Thm. 1], unexpectedly, this property fails for $\mathcal{DF}$, as the following example illustrates.

Example 2.16. Let $\mathscr{A} = \mathscr{A}_\ell^1(r)$ for $r \geq 3$ and $\ell \geq 4$. Let

$$X = \bigcap_{\substack{2 \leq i < j \leq \ell \\ 0 \leq n < r}} \ker(x_i - \zeta^n x_j),$$

where ζ is a primitive r^{th} root of unity. Then $\mathscr{A}_X \cong \mathscr{A}_{\ell-1}^0(r)$. By Theorem 2.11, $\mathscr{A}$ is divisionally free but $\mathscr{A}_X$ is not.

Remark 2.17. If $\mathscr{A} \neq \Phi_\ell$ is inductively free and $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ is a triple as in Definition 2.5(ii), then each member of $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ belongs to $\mathcal{IF}$. In contrast, if $\mathscr{A}$ and $\mathscr{A}''$ are consecutive members in a sequence as in Definition 2.10, then the deletion $\mathscr{A}'$ in the corresponding triple $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ need not belong to $\mathcal{DF}$. For instance, let $\mathscr{A} = \mathscr{A}_\ell^1(r)$ for $r, \ell \geq 3$ and $H = \ker(x_1)$. Then $\mathscr{A}^H \cong \mathscr{A}_{\ell-1}^{\ell-1}(r)$, by [OS82, Prop. 2.11]. It follows from [OS82, Prop. 2.13] (or else from [OT92, Prop. 6.85] and Theorem 2.4) that $\chi(\mathscr{A}^H, t)$ divides $\chi(\mathscr{A}, t)$. Consequently, thanks to Theorem 2.11, $\mathscr{A}$ and $\mathscr{A}''$ are successive terms in a sequence as in Definition 2.10. However, $\mathscr{A}' = \mathscr{A}_\ell^0(r)$ is not divisionally free, by Theorem 2.11.

3. Divisionally Free Reflection Arrangements

In view of Theorem 2.7(i) and Proposition 2.12, we can restate Abe's classification of the divisionally free reflection arrangements [Abe16, Cor. 4.7] as follows.

Theorem 3.1. *For W a finite, complex reflection group, its reflection arrangement $\mathscr{A}(W)$ is divisionally free if and only if W does not admit an irreducible factor isomorphic to a monomial group $G(r, r, \ell)$ for $r, \ell \geq 3$, $G_{24}, G_{27}, G_{29}, G_{33}$, or G_{34}.*

The counterpart to Theorem 2.7(ii) also holds in the setting of divisional freeness:

Theorem 3.2. *The reflection arrangement of a finite, complex reflection group is divisionally free if and only if it is hereditarily divisionally free.*

Proof. The forward implication follows from Proposition 2.12, Theorems 2.7(ii) and 3.1, and [Abe16, Prop. 5.8]. The reverse implication is obvious. ☐

We now consider restrictions of reflection arrangements. Thanks to Proposition 2.12 and (2.1), the question of divisional freeness of $\mathscr{A}^X$ reduces readily to the case when $\mathscr{A}$ is irreducible, so we may assume that W is irreducible. In view of Theorem 2.8(a), we can formulate our classification as follows:

Theorem 3.3. *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A}(W)$ and let $V \neq X \in L(\mathscr{A}(W))$. Then $\mathscr{A}(W)^X$ is divisionally free if and only if one of the following holds:*

(i) *$\mathscr{A}(W)^X$ is inductively free;*

(ii) *$W = G(r, r, \ell)$ and $\mathscr{A}(W)^X \cong \mathscr{A}_p^k(r)$, where $p = \dim X$ and $1 \leq k \leq p - 3$;*

(iii) *$W = G_{33}$ or G_{34} and $\dim X = 4$.*

Proof. If $\mathscr{A}(W)^X$ is inductively free, the result follows, since $\mathcal{IF} \subseteq \mathcal{DF}$.

Now suppose that $\mathscr{A}(W)^X$ is not inductively free. For $W = G(r, r, \ell)$, every restriction is of the form $\mathscr{A}(W)^X \cong \mathscr{A}_p^k(r)$, where $p = \dim X$ and $1 \leq k \leq p$, by [OS82, Prop. 2.14]. Thus (ii) follows from Theorems 2.8(a)(ii) and 2.11.

For $W = G_{33}$ or G_{34} the result follows from [OS82, Thm. 1.2], [OS82, Tables 10, 11], along with Theorem 2.8(a)(iii). ☐

In view of Proposition 2.12 and Theorem 2.8, we can restate Theorem 3.3 as follows.

Theorem 3.4. *Let W be a finite, irreducible, complex reflection group and let $V \neq X \in L(\mathscr{A}(W))$. Then $\mathscr{A}(W)^X$ is divisionally free unless $W = G_{34}$ and $\dim X = 5$.*

Finally, we show that Theorem 3.2 extends to restrictions.

Theorem 3.5. *Let W be a finite, complex reflection group with reflection arrangement $\mathscr{A}(W)$ and let $V \neq X \in L(\mathscr{A}(W))$. The restricted arrangement $\mathscr{A}(W)^X$ is divisionally free if and only if it is hereditarily divisionally free.*

Proof. The forward implication follows from Corollary 2.15, [OS82, Prop. 2.11, Prop. 2.14], and Theorems 3.2, 3.3, and 2.8(b). The reverse implication is obvious. ☐

Acknowledgments: We acknowledge support from the DFG-priority program SPP1489 "Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory".

References

[Abe16] T. Abe, *Divisionally free arrangements of hyperplanes*, Inventiones Math. **204(1)**, (2016), 317–346.

[AHR14] N. Amend, T. Hoge and G. Röhrle, *On inductively free restrictions of reflection arrangements*, J. Algebra **418** (2014), 197–212.

[HR15] T. Hoge and G. Röhrle, *On inductively free reflection arrangements*, J. reine u. angew. Math. **701** (2015), 205–220.

[HRS17] T. Hoge, G. Röhrle, and A. Schauenburg, *Inductive and Recursive Freeness of Localizations of Multiarrangements*, in: Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory, Springer-Verlag, 2017.

[OS82] P. Orlik and L. Solomon, *Arrangements Defined by Unitary Reflection Groups*, Math. Ann. **261**, (1982), 339–357.

[OT92] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, 1992.

[ST54] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*. Canadian J. Math. **6**, (1954), 274–304.

[Ter80] H. Terao, *Arrangements of hyperplanes and their freeness I, II*, J. Fac. Sci. Univ. Tokyo **27** (1980), 293–320.

[Zie90] G. M. Ziegler, *Matroid representations and free arrangements*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 525–541.

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany

E-mail address: `gerhard.roehrle@rub.de`