A refinement of switching on ballot tableau pairs

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Plan

1. Ballot semistandard Young tableaux/Littlewood-Richardson tableaux
2. Switching of tableau pairs
3. Refinement of switching on ballot tableau pairs
    - Hidden features and LR commutators

Ballot semistandard Young tableaux or LR tableaux

- Ballot or Littlewood-Richardson tableaux (LR)

$$U = \begin{array}{|c|c|c|c|}\hline & & 1 & 2 \\ \hline & 1 & 3 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array} \qquad T = \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & 1 & 2 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} \qquad Y = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

21311   11213   11123

A semistandard Young tableau is ballot or LR if the content of each initial segment of the reading word (read right to left along rows, top to bottom) is a partition.

T and Y are ballot, U is not.

Littlewood-Richardson rule

- The Littlewood-Richardson (LR) rule (*D.E. Littlewood and A. Richardson 34; M.-P. Schützenberger 77; G.P. Thomas 74*) states that the coefficients appearing in the expansion of a product of Schur polynomials s_μ and s_ν

$$s_\mu(x)\ s_\nu(x) = \sum_\lambda\ c^\lambda_{\mu\nu}\ s_\lambda(x)$$

  are given by

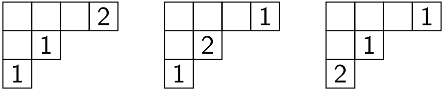
$$c^\lambda_{\mu\nu} = \#\{\text{ballot SSYT of shape } \lambda/\mu \text{ and content } \nu\}.$$

- Schubert structure coefficients of the product in $H^*(G(d,n))$, the cohomology of the Grassmannian $G(d,n)$ (as a $\mathbb{Z}$-module), are also given by the LR rule (*L. Lesier 47*),

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq d\times(n-d)} c^\lambda_{\mu\ \nu} \sigma_\lambda.$$

The structure coefficient $c^{\lambda}_{\mu,\nu}$ is

the cardinality of an explicit set of combinatorial objects.



$$c^{421}_{31,\,21} = 2$$

- Fixing λ, it is known that the number $c^{\lambda}_{\mu,\nu}$ is invariant under the switching of μ and ν.
- There are several bijections (involutions) exhibiting the commutativity

$$c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}.$$

The involutive nature is always quite hard and mysterious, very often, unfolded with the help of further theory.

Switching, B.S.S. (1996)

- Switching is an operation that takes two tableaux $S \cup T$ sharing a common border and moves them through each other giving another such pair $U \cup V$, in a way that preserves Knuth equivalence, $S \equiv V$ and $T \equiv U$, and the shape of their union.
- A second application of switching restores the original pair $U \cup V$. Switching is an involution.
- Benkart, Sottile and Stroomer (1996) have studied switching in a general context.

Switching moves

- A perforated tableau pair $S \cup T$ is a labeling of the boxes satisfying some restrictions: whenever x and x' are letters from S (T) and x is north-west of x', $x' \geq x$; within each column of T (S) the letters are distinct.
- The moves are such that if **s** and **t** are adjacent letters from S and T then a switch of **s** with **t**, $\mathbf{s} \underset{s}{\leftrightarrow} \mathbf{t}$, is a move such that the outcome pair is still perforated.

$$
\begin{array}{|c|c|c|c|c|c|}\hline \color{red}1 & \color{red}1 & 1 & \color{red}1 & \color{red}1 & 1 \\ \hline \color{red}2 & \color{red}2 & \color{red}2 & 2 & \color{red}2 \\ \cline{1-5} 1 & 2 & 3 & \color{red}3 \\ \cline{1-4} \end{array}
\;\leftrightarrow\;
\begin{array}{|c|c|c|c|c|c|}\hline 1 & \color{red}1 & 1 & \color{red}1 & \color{red}1 & 1 \\ \hline \color{red}1 & \color{red}2 & \color{red}2 & 2 & \color{red}2 \\ \cline{1-5} \color{red}2 & 2 & 3 & \color{red}3 \\ \cline{1-4} \end{array}
\;\leftrightarrow\;
\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & \color{red}1 & \color{red}1 & \color{red}1 \\ \hline \color{red}1 & \color{red}2 & \color{red}2 & 2 & \color{red}2 \\ \cline{1-5} \color{red}2 & 2 & 3 & \color{red}3 \\ \cline{1-4} \end{array}
$$

The Switching Procedure

- The Switching Procedure, B.S.S. (1996).
  - Start with the tableau pair $S \cup T$.
  - Switch integers from S with integers from T until it is no longer possible to do so. This produces a new pair $U \cup V$ where $U \equiv T$ and $S \equiv V$.

$$S \cup T = \begin{array}{|c|c|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & 1 & 1 \\ \hline {\color{red}2} & {\color{red}2} & {\color{red}2} & {\color{red}2} & 2 & \\ \cline{1-5} {\color{red}3} & 1 & 2 & 3 & & \\ \cline{1-4} \end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & 1 & {\color{red}1} & {\color{red}1} & 1 \\ \hline {\color{red}2} & {\color{red}2} & {\color{red}2} & 2 & {\color{red}2} & \\ \cline{1-5} 1 & 2 & 3 & {\color{red}3} & & \\ \cline{1-4} \end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|c|}\hline 1 & {\color{red}1} & 1 & {\color{red}1} & {\color{red}1} & 1 \\ \hline {\color{red}1} & {\color{red}2} & {\color{red}2} & 2 & {\color{red}2} & \\ \cline{1-5} {\color{red}2} & 2 & 3 & {\color{red}3} & & \\ \cline{1-4} \end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} & {\color{red}2} & \\ \cline{1-5} 3 & {\color{red}2} & {\color{red}2} & {\color{red}3} & & \\ \cline{1-4} \end{array}$$

- Let ρ_1 denote the map that the switching procedure calculates on ballot tableau pairs of partition shape.
- Imposing a certain order on switches on such pairs ($Y \cup T$ with Y Yamanouchi) reveals interesting features of the map ρ_1.

Basic ideas

- Switching on a two-row tableau pair:

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&3&3\\\hline 2&2&3&4\\\cline{1-4}\end{array} \to \begin{array}{|c|c|c|c|c|}\hline 1&1&3&3&1\\\hline 2&3&4&2\\\cline{1-4}\end{array} \to \begin{array}{|c|c|c|c|c|}\hline 1&3&3&3&1\\\hline 2&1&4&2\\\cline{1-4}\end{array} \to \begin{array}{|c|c|c|c|c|}\hline 1&3&3&3&1\\\hline 2&4&1&2\\\cline{1-4}\end{array}$$

- Comparision of the switching on one-row tableau pair with the switching on the augmented two-row tableau pair.

$$S \cup T = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline\end{array} \to U \cup V = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline\end{array}$$

Add the second row 212 to $S \cup T$.

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 2&1&2\\\cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 1&2&2\\\cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 1&2&2\\\cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 2&1&2\\\cline{1-3}\end{array}$$

Put 2 at the beginning of the second row of $U \cup V$; insert 1 in first row of $U \cup V$ by bumping the first 1 and then put it at the end of the second row; add at the end of the second row 2.

Switching on ballot tableau pairs

$$
Y_\mu \cup T = \begin{array}{cccccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1 \\
\mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & 2 & \\
\mathbf{3} & 1 & 2 & 3 & & \\
\mathbf{4} & 2 & 3 & 4 & &
\end{array}
\quad \to \quad
\begin{array}{cccccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1 \\
\mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & 2 & \\
1 & 2 & 3 & \mathbf{3} & & \\
2 & 3 & 4 & \mathbf{4} & &
\end{array}
$$

$$
\begin{array}{cccccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1 \\
\mathbf{2} & \mathbf{2} & 2 & \mathbf{2} & \mathbf{2} & \\
1 & 2 & 3 & \mathbf{3} & & \\
2 & 3 & 4 & \mathbf{4} & &
\end{array}
\quad \to \quad
\begin{array}{cccccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1 \\
1 & 2 & 2 & \mathbf{2} & \mathbf{2} & \\
\mathbf{2} & \mathbf{2} & 3 & \mathbf{3} & & \\
2 & 3 & 4 & \mathbf{4} & &
\end{array}
$$

Switching on ballot tableau pairs

$$\begin{array}{cccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1 \\ 1 & 2 & 2 & \mathbf{2} & \mathbf{2} & \\ 2 & 3 & 3 & \mathbf{3} & & \\ \mathbf{2} & \mathbf{2} & 4 & \mathbf{4} & & \end{array} \quad \to \quad \begin{array}{cccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1 \\ 1 & 2 & 2 & \mathbf{2} & \mathbf{2} & \\ 2 & 3 & 3 & \mathbf{3} & & \\ 4 & \mathbf{2} & \mathbf{2} & \mathbf{4} & & \end{array}$$

$$\begin{array}{cccccc} \mathbf{1} & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 2 & 2 & \mathbf{2} & \mathbf{2} & \\ 2 & 3 & 3 & \mathbf{3} & & \\ 4 & \mathbf{2} & \mathbf{2} & \mathbf{4} & & \end{array} \quad \to \quad \begin{array}{cccccc} 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 2 & 2 & 2 & \mathbf{2} & \mathbf{2} & \\ \mathbf{1} & 3 & 3 & \mathbf{3} & & \\ 4 & \mathbf{2} & \mathbf{2} & \mathbf{4} & & \end{array}$$

$$\begin{array}{cccccc} 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 2 & 2 & 2 & \mathbf{2} & \mathbf{2} & \\ 3 & 3 & \mathbf{1} & \mathbf{3} & & \\ 4 & \mathbf{2} & \mathbf{2} & \mathbf{4} & & \end{array} \quad = Y_\nu \cup U = \rho_1(Y_\mu \cup T) \quad U \equiv Y_\mu, \; Y_\nu \equiv T.$$

A recursive definition for ρ_1

- $$(Y_\mu \cup T)^- = \begin{array}{|c|c|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & 1 & 1 \\ \hline {\color{red}2} & {\color{red}2} & {\color{red}2} & {\color{red}2} & 2 \\ \cline{1-5} {\color{red}3} & 1 & 2 & 3 \\ \cline{1-4} \end{array} \quad \xrightarrow[\rho_1]{} \rho_1[(Y \cup T)^-] = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} & {\color{red}2} \\ \cline{1-5} 3 & {\color{red}2} & {\color{red}2} & {\color{red}3} \\ \cline{1-4} \end{array}$$

  $$Y_\mu \cup T = \begin{array}{|c|c|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & 1 & 1 \\ \hline {\color{red}2} & {\color{red}2} & {\color{red}2} & {\color{red}2} & 2 \\ \cline{1-5} {\color{red}3} & 1 & 2 & 3 \\ \cline{1-4} {\color{red}4} & 2 & 3 & 4 \\ \cline{1-4} \end{array} \quad \xrightarrow[\rho_1]{} \rho_1(Y \cup T) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ \hline 2 & 2 & 2 & {\color{red}2} & {\color{red}2} \\ \cline{1-5} 3 & 3 & {\color{red}1} & {\color{red}3} \\ \cline{1-4} 4 & {\color{red}2} & {\color{red}2} & {\color{red}4} \\ \cline{1-4} \end{array}$$

- Are $\rho_1(Y \cup T)$ and $\rho_1[(Y \cup T)^-]$ related?

$$\rho_1[(Y \cup T)^-] = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} & {\color{red}2} \\ \cline{1-5} 3 & {\color{red}2} & {\color{red}2} & {\color{red}3} \\ \cline{1-4} \end{array} \quad \xrightarrow[\bar\theta_{44}]{4} \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} & {\color{red}2} \\ \cline{1-5} 3 & {\color{red}2} & {\color{red}2} & {\color{red}3} \\ \cline{1-4} 4 \\ \cline{1-1} \end{array} \quad \xrightarrow[\bar\theta_{3,4}]{3} \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ \hline 2 & 2 & {\color{red}1} & {\color{red}2} & {\color{red}2} \\ \cline{1-5} 3 & 3 & {\color{red}2} & {\color{red}3} \\ \cline{1-4} 4 & {\color{red}2} \\ \cline{1-2} \end{array}$$

$$\xrightarrow[\bar\theta_{24}]{2} \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ \hline 2 & 2 & 2 & {\color{red}2} & {\color{red}2} \\ \cline{1-5} 3 & 3 & {\color{red}1} & {\color{red}3} \\ \cline{1-4} 4 & {\color{red}2} & {\color{red}2} \\ \cline{1-3} \end{array} \quad {\color{red}\xrightarrow[\chi_4]{4}} \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ \hline 2 & 2 & 2 & {\color{red}2} & {\color{red}2} \\ \cline{1-5} 3 & 3 & {\color{red}1} & {\color{red}3} \\ \cline{1-4} 4 & {\color{red}2} & {\color{red}2} & {\color{red}4} \\ \cline{1-4} \end{array} = \rho_1(Y \cup T)$$

- $$Y_\mu \cup T = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\ \hline 2&2&2&2&2& \\ \cline{1-5} 3&1&2&3&& \\ \cline{1-4} 4&2&3&4&& \\ \cline{1-4}\end{array} \quad \underset{\rho_1}{\rightarrow} \quad \rho_1[(Y \cup T)] = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\ \hline 2&2&2&2&2& \\ \cline{1-5} 3&3&1&3&& \\ \cline{1-4} 4&2&2&4&& \\ \cline{1-4}\end{array}$$

$$\downarrow \qquad\qquad\qquad\qquad \underset{\bar\theta_4}{\uparrow}\ \underset{\delta_4}{\downarrow}$$

$$(Y_\mu \cup T)^- = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\ \hline 2&2&2&2&2& \\ \cline{1-5} 3&1&2&3&& \\ \cline{1-4}\end{array} \quad \underset{\rho_1}{\rightarrow} \quad \rho_1[(Y \cup T)^-] = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\ \hline 2&2&1&2&2& \\ \cline{1-5} 3&2&2&3&& \\ \cline{1-4}\end{array}$$

- $$\rho_1(Y \cup T) = \underbrace{\chi_4 \bar\theta_{2,4} \bar\theta_{3,4} \bar\theta_{4,4}}_{\bar\theta_4}\ \rho_1[(Y \cup T)^-].$$

$$\delta_4\ \rho_1(Y \cup T) = \rho_1[(Y \cup T)^-], \qquad \delta_4 = \bar\theta_4^{-1}$$

An avatar of switching map ρ_1: $\overline{\rho}^{(n)}$

- $Y_\mu \cup T \to Y_\nu \cup U$, $T \equiv Y_\nu$, $U \equiv T_\mu$: use the GT pattern T_ν for internal insertion, and add μ_i boxes marked with i at the end of each row i.

$$Y_\mu \cup T = \begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&2&2&2 \\ 3&1&2&3 \\ 4&2&3&4 \end{array} \qquad T_\nu = \begin{array}{ccccccc} &&&2&&& \\ &&2&&1&& \\ &3&&2&&1& \\ 3&&3&&2&&1 \end{array}$$

-

$$\emptyset \to \begin{array}{llllll} 1&1&1&1&1&1 \end{array} \to \begin{array}{llllll} 1&1&1&1&1&1 \\ 2 \end{array} \to \begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&2&2&2 \end{array}$$

$$\begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&2&2&2 \\ 3 \end{array} \to \begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&2&2&2 \\ 3&2 \end{array} \to \begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&1&2&2 \\ 3&2&2 \end{array} \to \begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&1&2&2 \\ 3&2&2&3 \end{array}$$

$$\begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&1&2&2 \\ 3&2&2&3 \\ 4 \end{array} \to \begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&1&2&2 \\ 3&3&2&3 \\ 4&2 \end{array} \to \begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&2&2&2 \\ 3&3&1&3 \\ 4&2&2 \end{array} \to \begin{array}{llllll} 1&1&1&1&1&1 \\ 2&2&2&2&2 \\ 3&3&1&3 \\ 4&2&2&4 \end{array} = Y_\nu \cup U$$

The bijection $\bar{\rho}^{(n)}$ and its inverse $\rho^{(n)}$

- Let $Y_\mu \cup T$ be a ballot tableau pair of shape λ. Let ν be the content of T with GT pattern $T_\nu = (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(n-1)}, \nu^{(n)})$.
- Let $\nu^{(i)} - \nu^{(i-1)} = (V_1^{(i)}, \dots, V_{i-1}^{(i)}, \nu_i)$, $1 \le i \le n$. Then

$$\bar{\rho}^{(n)}(Y_\mu \cup T) = \bar{\theta}_n \cdots \bar{\theta}_2 \bar{\theta}_1 \emptyset,$$

$$\text{where } \theta_i = \chi^{\mu_i} \bar{\theta}_{1,i}^{V_1^{(i)}} \bar{\theta}_{2,i}^{V_2^{(i)}} \cdots \bar{\theta}_{i-1,i}^{V_{i-1}^{(i)}} \bar{\theta}_{i,i}^{\nu_i}, \quad , 1 \le i \le n.$$

- Let $\rho^{(n)}$ denote the inverse of $\bar{\rho}^{(n)}$. If $Y_\nu \cup U = \bar{\rho}^{(n)}(Y_\mu \cup T)$, then

$$\rho^{(n)}(Y_\nu \cup U) = \delta_1 \delta_2 \cdots \delta_n (Y_\nu \cup U)$$

produces the GT pattern of type ν consisting of the sequence of inner shapes in $Y_\nu \cup U$, and $\delta_i \cdots \delta_n (Y_\nu \cup U)$, $i = 2, \dots, n$.

Avatars of switching map ρ_1: $\bar{\rho}^{(n)}$ and its inverse $\rho^{(n)}$

$$\bar{\rho}^{(n)}(Y_\mu \cup T) = \bar{\theta}_n \bar{\rho}^{(n-1)}(Y_\mu \cup T)^-.$$

$$[\rho^{(n)}(Y_\mu \cup T)]^- = \rho^{(n-1)}\delta_n(Y_\mu \cup T).$$

- **Lemma**. Let $\mathcal{LR}^{(n)}$ the set of all ballot tableau pairs $Y \cup T$, with at most n rows, where Y is a Yamanouchi tableau. Let $\xi^{(n)}$ be an involution on $LR^{(n)}$ such that $\xi^{(n)}(Y_\mu \cup T) = Y_\nu \cup U$ with $Y_\mu \equiv U$ and $Y_\nu \equiv T$. Then, for all $Y \cup T \in \mathcal{LR}^{(n)}$,

$$\xi^{(n-1)}(Y \cup T)^- = \delta_n \xi^{(n)}(Y \cup T) \quad \text{iff} \quad \xi^{(n-1)}\delta_n(Y \cup T) = [\xi^{(n)}(Y \cup T)]^-.$$

Using the fact that ρ_1 is an involution.

- **Corollary**. $\bar{\rho}^{(n)}$ is an involution and by definition

$$\delta_n \bar{\rho}^{(n)}(Y \cup T) = \bar{\rho}^{(n-1)}(Y \cup T)^-.$$

  Then

$$\bar{\rho}^{(n-1)}\delta_n(Y \cup T) = [\bar{\rho}^{(n)}(Y \cup T)]^-.$$

- **Corollary**. $\rho^{(n)}$ is an involution and by definition

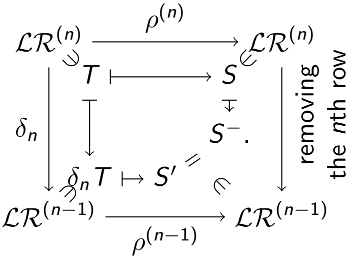
$$[\rho^{(n)}(Y_\mu \cup T)]^- = \rho^{(n-1)}\delta_n(Y_\mu \cup T).$$

  Then

$$\delta_n \rho^{(n)}(Y \cup T) = \rho^{(n-1)}(Y \cup T)^-.$$

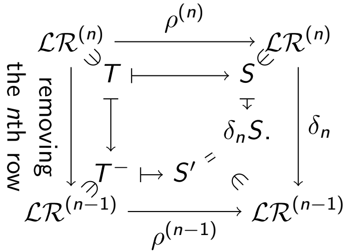
Without using the switching map ρ_1: the bijection $\rho^{(n)}$

- By definition of $\rho^{(n)}$

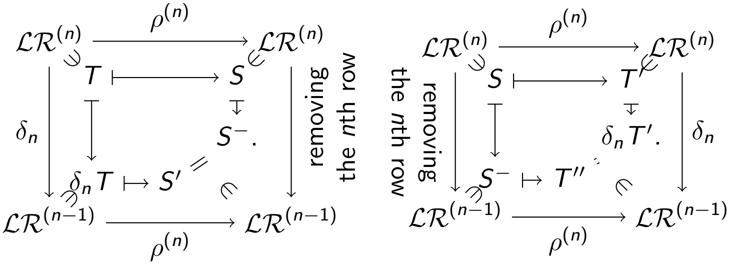


$$\rho^{(n-1)}\delta_n(Y \cup T) = [\rho^{(n)}(Y \cup T)]^-.$$

- Theorem (A. 2000); A., King, Terada (2016)



$$\rho^{(n-1)}(Y \cup T)^- = \delta_n\rho^{(n)}(Y \cup T).$$



$$\delta_n {\rho^{(n)}}^2 = {\rho^{(n)}}^2 \delta_n.$$

$\rho^{(n)}$ is an involution

Theorem
(A., King, Terada, 2016) ${\rho^{(n)}}^2 = id$.

Proof. By induction on n.

$$n = 1, \quad T = \begin{array}{|c|c|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & 1 & 1 & 1 & 1 \\ \hline \end{array} \underset{\rho^{(1)}}{\rightarrow} S = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & {\color{red}1} & {\color{red}1} \\ \hline \end{array} \underset{\rho^{(1)}}{\rightarrow} T = \begin{array}{|c|c|c|c|c|c|}\hline {\color{red}1} & {\color{red}1} & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

Let $n > 1$. By induction on n,

$$\begin{aligned} & {\rho^{(n)}}^2(\delta_n(Y \cup T)) = \delta_n(Y \cup T) \\ \Leftrightarrow \quad & \delta_n({\rho^{(n)}}^2(Y \cup T)) = {\rho^{(n)}}^2(\delta_n(Y \cup T)) = \delta_n(Y \cup T) \\ \Rightarrow \quad & {\rho^{(n)}}^2(Y \cup T) = Y \cup T. \end{aligned}$$