A Sundaram type bijection for $\mathrm{SO}(3)$

vacillating tableaux and pairs of SYTs and LR-tableaux

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September 12th, 2016

Background

$$V^{\otimes r} = \bigoplus_{\substack{\mu \text{ a partition} \\ l(\mu)\leq n \\ \mu'_1+\mu'_2\leq n}} V(\mu) \otimes \Big(\bigoplus_{\substack{\lambda\vdash r \\ l(\lambda)\leq n}} \mathrm{c}^{\mu}_{\lambda}(\mathfrak{o})\ S(\lambda)\Big)$$

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- $V = \mathbb{C}^n$ the vector representation of $\mathrm{SO}(n)$
- $V(\mu)$ ... an irreducible representation of $\mathrm{SO}(n)$
- $S(\lambda)$ ... an irreducible representation of $\mathfrak{S}_r$
- $\mathrm{c}^{\mu}_{\lambda}(\mathfrak{d})$ multiplicities counted by so called type $\mathfrak{d}$ Littlewood Richardson tableaux introduced by Kwon in 2015

Overview

Orthogonal Robinson Schensted

$$V^{\otimes r} = \bigoplus_{\mu} V(\mu) \otimes \Big(\bigoplus_{\lambda} \mathrm{c}_{\lambda}^{\mu}(\mathfrak{d})\ S(\lambda) \Big)$$

$$\{0, \pm 1\}^r \leftrightarrow \bigcup_{\mu} \Big(\text{orthogonal SSYT, vacillating tableaux} \Big)$$

we are interested in:

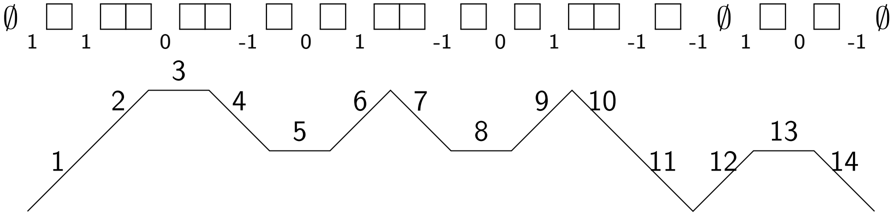
vacillating tableaux $\leftrightarrow \bigoplus_{\lambda} \mathrm{c}_{\lambda}^{\mu}(\mathfrak{d})\ S(\lambda) \leftrightarrow$ (LR-tableaux, SYT)

Vacillating Tableaux and Lattice Paths

Definition

a vacillating tableau $\emptyset = \lambda_0, \lambda_1, \dots, \lambda_r$
is a sequence of Young diagrams
with at most k rows: $n = 2k + 1 \rightarrow$ for us $k = 1$
λ_i and λ_{i+1} differ in at most one position
$\lambda_i = \lambda_{i+1}$ only occurs if the k^{th} row is nonempty

Example (with $\lambda_r = \emptyset$)



sufficient to find a bijection between

SYT all rows even

1	2	3	4	5	6

1	2	3	4
5	6		

1	2	3	5
4	6		

1	2	4	5
3	6		

1	3	4	5
2	6		

1	2	3	6
4	5		

1	2	4	6
3	5		

1	3	4	6
2	5		

1	2	5	6
3	4		

1	3	4	5
2	4		

1	2
3	4
5	6

1	3
2	4
5	6

1	2
3	5
4	6

1	3
2	5
4	6

1	4
2	5
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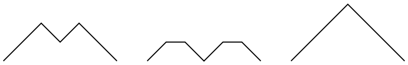
vacillating tableaux $\lambda_r = \emptyset$



Descents

SYT

d is a descent
if $d + 1$ is in a row below d

Example

1	3	7	8
2	5		
4	6		

has descents 1,3 and 5

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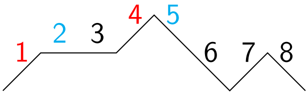
Vacillating Tableaux

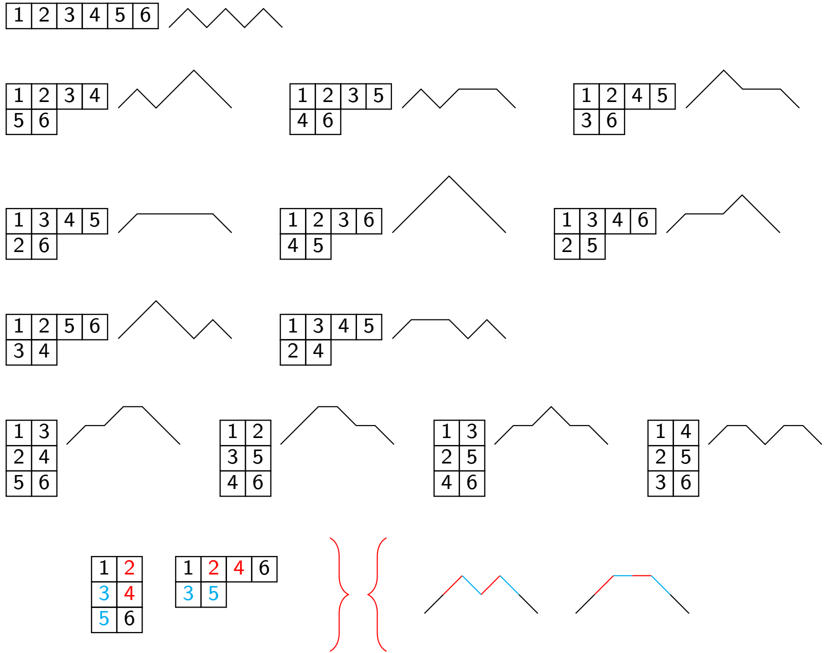
i is a descent
if the i^{th} position is:

- 1 followed by 0
- 0 followed by -1
- 1 followed by -1
  except if $\lambda_i = \square$

Example



has descents 1 and 4


1 2 3 4 5 6
1 2 3 4
5 6
1 2 3 5
4 6
1 2 4 5
3 6
1 3 4 5
2 6
1 2 3 6
4 5
1 3 4 6
2 5
1 2 5 6
3 4
1 3 4 5
2 4
1 3
2 4
5 6
1 2
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4 6
1 4
2 5
3 6
1 2
3 4
5 6
1 2 4 6
3 5

Bonus:

with a descent preserving bijection we also get the quasi symmetric expansion of the Frobenius character:

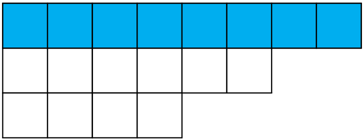
$$\mathrm{ch}\Big(\bigoplus_{\lambda\vdash r} \mathrm{c}^{\mu}_{\lambda}(\mathfrak{d})\ S(\lambda)\Big) = \sum_{\lambda} \mathrm{c}^{\mu}_{\lambda}(\mathfrak{d})\ s_{\lambda} = \sum_{w} \mathrm{F}_{\mathrm{Des}(w)}$$

- s_{λ} Schur functions
- $\mathrm{F}_D = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_r \\ j\in D: i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_r}$ fundamental quasi symmetric functions

Idea

Insert the SYT row by row into a path:

- insert first row



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Insert the SYT row by row into a path:

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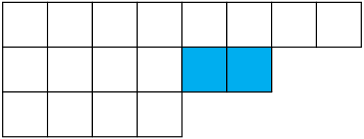

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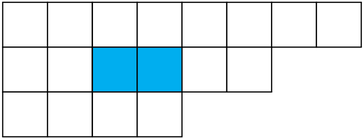

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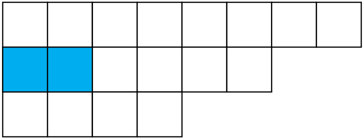

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Insert the SYT row by row into a path:

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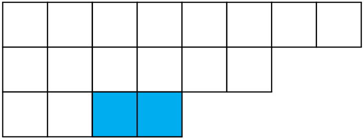
- insert third row in pairs $\boxed{a \mid b}$ starting with the right most



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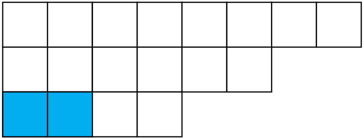
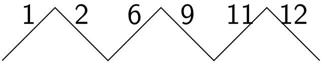


First Row

1	2	6	9	11	12
3	5	8	13		
4	7	10	14		

we create a path 1, -1, 1, -1, . . .
labeled with the first row entries



Second Row

1	2	6	9	11	12
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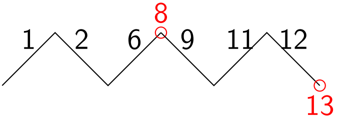
- insert b with -1
- insert a with
  - 0 if the step right of a is -1 but not b, change this step into 0
  - -1 otherwise, change the next -1 to the left into 1
- change pairs of $1, -1$ between a and b into $0, 0$



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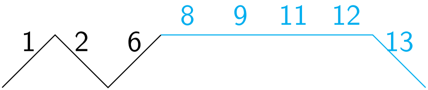

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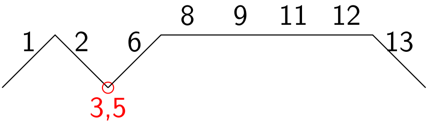

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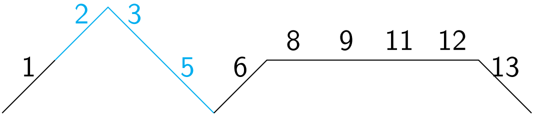

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- insert b with -1
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    or "unused" 0 into 1
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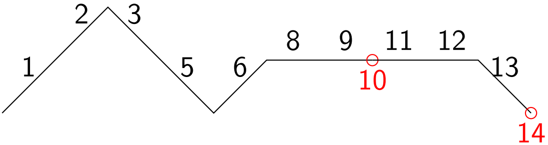

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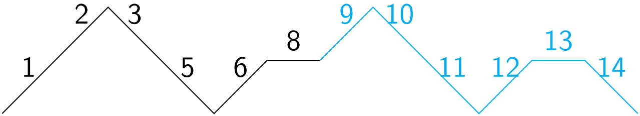

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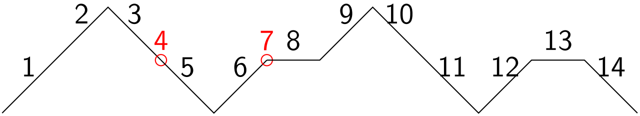

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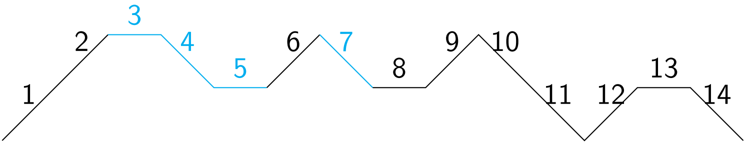

Result

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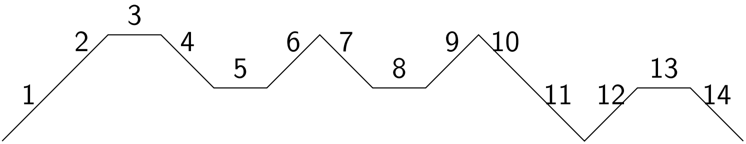

The Bijection

(LR,SYT)$\leftrightarrow$(SYT all even/odd rows,$\mathbb{N}_{\leq r}$) $\leftrightarrow$ vacillating tableaux

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$\mathrm{SO}(n)$, $n = 2k + 1$, $n > 3$: work in progress

- vacillating tableaux are k-tuples of paths with dependencies
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Other Groups

- $\mathrm{SO}(n)$, n even, vector rep. (descent set conj. by Rubey)
- G_2, vector rep. (descent set conj. by Rubey)
- $\mathrm{Sp}(n)$, vector rep., using Kwon's LR-tableaux