A convolution formula for Tutte polynomials of arithmetic matroids and other combinatorial structures

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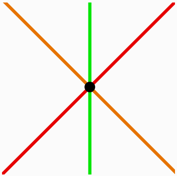

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Tutte polynomial and region counting

Tutte polynomial of the matroid (M, rk):

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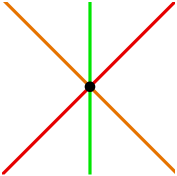

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Then $\mathcal{A}$ divides $\mathbb{R}^d$ into $\mathfrak{T}_M(2,0)$ regions.

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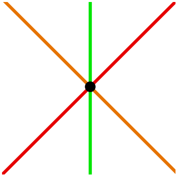

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Remark

- There is a proof using Hopf algebras (Duchamp–Hoang-Nghia–Krajewski–Tanasa: *Recipe theorem for the Tutte polynomial for matroids, renormalization group-like approach*)
- Presented at SLC 70 in 2013 in Ellwangen

Toric arrangements

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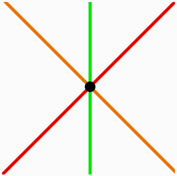

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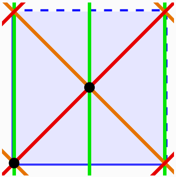

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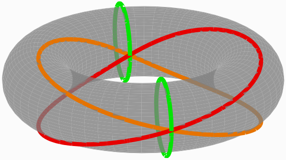

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Remark

Hyperplane arrangements are related to the problem of measuring volumes of polytopes, while toric arrangements are related to counting the number of lattice points.

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Theorem (Moci (2012), Lawrence (2011))

Let $\mathcal{A}$ be a toric arrangement in the real torus $(S^1)^d$ and let (M, rk, m) be the corresponding arithmetic matroid. Then $\mathcal{A}$ divides the torus into $\mathfrak{M}_M(1, 0)$ regions.

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Restriction and contraction for the multiplicity function:

- $m|_A(S) = m(S)$ for $S \subseteq M$
- $m_{/A}(S) = m(A \cup S)$ for $S \subseteq M \setminus A$

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(M, rk, m) ranked set with multiplicity. Let $\mathfrak{M}_M$ denote its arithmetic Tutte polynomial and $\mathfrak{T}_M$ its Tutte polynomial. Then

$$\begin{aligned} \mathfrak{M}_M(x, y) &= \sum_{A \subseteq M} \mathfrak{M}_{M|_A}(0, y)\mathfrak{T}_{M/A}(x, 0) \\ &= \sum_{A \subseteq M} \mathfrak{T}_{M|_A}(0, y)\mathfrak{M}_{M/A}(x, 0). \end{aligned}$$

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Remark

- (M, rk, m) *defined by* j*th boundary operator of a CW complex* $\rightsquigarrow$ $\mathfrak{M}_{(M,\mathrm{rk},m^2)}$ *known as modified* j*th Tutte–Krushkal–Renardy polynomial*
- *Bajo–Burdick–Chmutov asked if* (M, rk, m^2) *is an arithmetic matroid.*

Zonotopes

- For any polytope $P \subseteq \mathbb{R}^d$: $\left|P \cap \mathbb{Z}^d\right| = \sum_{F \preccurlyeq P} \left|\mathrm{relint}(F) \cap \mathbb{Z}^d\right|$.

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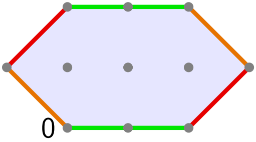
$$X = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$



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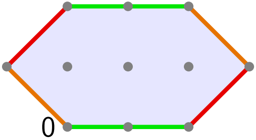

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$$\left|Z(X) \cap \mathbb{Z}^d\right| = \mathfrak{M}_X(2,1) = \sum_{A \subseteq X} \mathfrak{M}_{M|_A}(0,1)\mathfrak{T}_{M/A}(2,0)$$
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where the last sum is over all faces of $Z(X)$.

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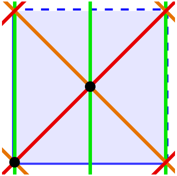

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- *Related to two decomposition formulas in the theory of splines and vector partition functions: Dahmen–Micchelli (1985), ML (2016)*

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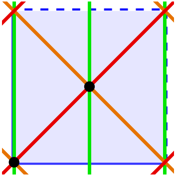

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Remark

This generalises flows and colourings on CW complexes (Beck–Breuer–Godkin–Martin, Beck–Kemper), which in turn generalise flows and colourings on graphs.

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Theorem (Brändén–Moci (2014))

X finite list of elements of $\mathbb{Z}^d$. There are infinite sets $\mathbb{Z}_M(X) \subseteq \mathbb{Z}_{>0}$ and $\mathbb{Z}_A(X) \subseteq \mathbb{Z}_{>0}$ s. t.

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- For $B \subseteq X$, G_B denotes the torsion subgroup of the quotient $\mathbb{Z}^d / \langle \{x : x \in B\} \rangle$.

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Thank you!

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Arithmetic matroids

Definition (D'Adderio–Moci (2013), Brändén–Moci (2014))

- An arithmetic matroid is a triple (M, rk, m)
  - (M, rk) is a matroid
  - $m : 2^M \to \mathbb{Z}_{\geq 1}$ is the *multiplicity function* that satisfies the axioms (A1), (A2) and (P) below.
- Let $R \subseteq S \subseteq M$. The set $[R, S] := \{A : R \subseteq A \subseteq S\}$ is called a *molecule* if S can be written as the disjoint union $S = R \cup F_{RS} \cup T_{RS}$ and for each $A \in [R, S]$, $\mathrm{rk}(A) = \mathrm{rk}(R) + |A \cap F_{RS}|$ holds.
- (A1) For all $A \subseteq E$ and $e \in E$: If $\mathrm{rk}(A \cup \{e\}) = \mathrm{rk}(A)$, then $m(A \cup \{e\}) | m(A)$. Otherwise $m(A) | m(A \cup \{e\})$.

  (A2) If $[R, S]$ is a molecule, then $m(R)m(S) = m(R \cup F)m(R \cup T)$.

  (P) for each molecule $[R, S]$, the following inequality holds

$$\rho(R, S) := (-1)^{|T_{RS}|} \sum_{A \in [R,S]} (-1)^{|S|-|A|} m(A) \geq 0.$$