Shifted symmetric functions II:
expansions in multi-rectangular coordinates

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Universität Zürich [UZH]

Second lecture

Yesterday, we have seen

- Two nice bases of the shifted symmetric function ring: shifted Schur functions $s_\mu^\star$ and normalized characters Ch_μ.

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Today:

- several sets of coordinates of Young diagrams on which we can write shifted symmetric functions (writing in terms of the λ_i is not necessarily the best thing!);

- investigate expansions in one of these sets, multirectangular coordinates.

First part

Equivalent descriptions
of the shifted symmetric function ring

A generating function point of view (Ivanov, Kerov, Olshanski, 2003)

With a Young diagram λ, we associate the function

$$\Phi(\lambda; z) = \prod_{i \geq 1} \frac{z + i}{z - \lambda_i + i}.$$
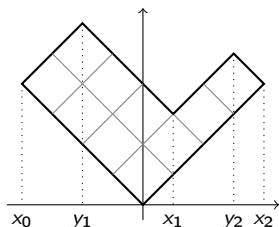
Around $z = \infty$,

$$\log\left(\Phi(\lambda; z)\right) = \sum_{k \geq 1} \tfrac{1}{k} p_k^\star(\lambda) \, z^{-k},$$

where $p_k^\star(\lambda) = \sum_{i \geq 1} \left[(\lambda_i - i)^N - (-i)^N\right]$.

Proposition

The shifted symmetric ring $\Lambda^\star$ is algebraically generated by the coefficients of the expansion of $\Phi(\lambda; z)$ at $z = \infty$.

Kerov's interlacing coordinates

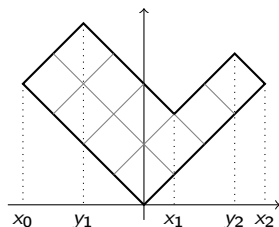Alternative description of Young diagrams

x-coordinates of lower corners

$$x_0 = -4, \; x_1 = 1, \; x_2 = 4$$

x-coordinates of higher corners

$$y_1 = -2, \; y_2 = 3$$

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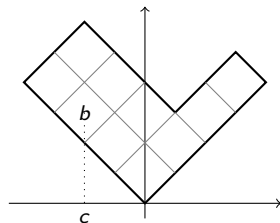
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Proposition (IKO '03)

$$\frac{\Phi(\lambda; z-1)}{\Phi(\lambda; z)} = \frac{z \prod_{i=1}^{m}(z - y_i)}{\prod_{i=0}^{m}(z - x_i)}.$$

As a consequence, $\Lambda^\star$ is algebraically generated by the coefficients of the expansion of the RHS at $z = \infty$ (or of $\log(RHS)$).
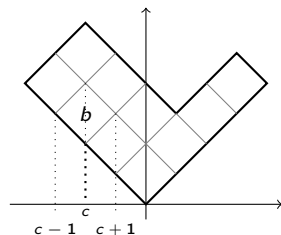
Proof: $\frac{\Phi(\lambda; z-1)}{\Phi(\lambda; z)} = \frac{z}{z-\lambda_1} \prod_{i \geq 1} \frac{z-\lambda_i+i}{z-\lambda_{i+1}+i}$. Only factors corresponding to corners do not cancel out.

Contents

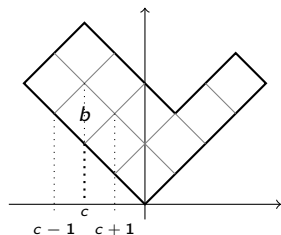Content c is the x-coordinate of the box center.

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Claim:
$$\prod_{\square \in \lambda} \frac{(z - c(\square))^2}{(z - c(\square) - 1)(z - c(\square) + 1)} = \frac{z \prod_{i=1}^{m}(z - y_i)}{\prod_{i=0}^{m}(z - x_i)} = \frac{\Phi(\lambda; z - 1)}{\Phi(\lambda; z)}.$$
Indeed, in LHS, only factors corresponding to corners of the diagram do not cancel out.

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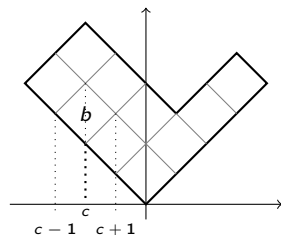Content c is the x-coordinate of the box center.

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Indeed, in LHS, only factors corresponding to corners of the diagram do not cancel out. We deduce from this (exercise!)

Proposition

$\Lambda^\star$ is the set of symmetric functions in $\mathcal{C}_\lambda = (c(\square))_{\square \in \lambda}$ with coefficients that are polynomials in $|\lambda|$.

Character on cycles

There is a efficient way to compute $Ch_{(k)}$ using the function Φ.
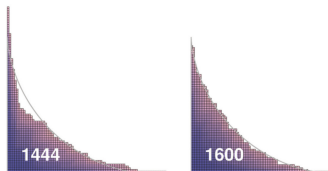
Proposition (Frobenius, '00)

$$Ch_{(k)}(\lambda) = -\frac{1}{k}\,[z^{-1}]\,(z \downharpoonright k)\frac{\Phi(\lambda; z)}{\Phi(\lambda; z - k)}.$$

- Generalized by Rattan and Śniady ('06) to several cycles.

A motivation: random Young diagram

Plancherel measure

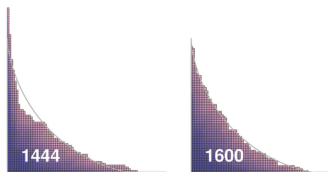$$\mathbb{P}(\lambda) = \frac{\dim(\rho^\lambda)^2}{n!}$$

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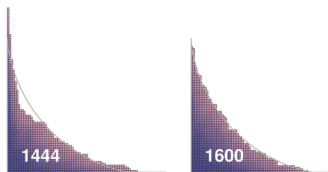$$\mathbb{P}(\lambda) = \frac{\dim(\rho^\lambda)^2}{n!}$$

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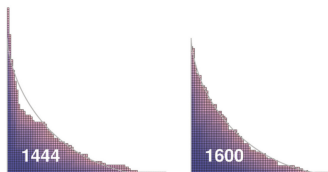

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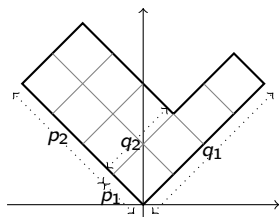
Doable with the formulas of this section and the combinatorial description of the multiplication table of Ch_μ (IKO, '03).

Transition

Multirectangular coordinates – expansion of normalized characters

Multirectangular coordinates (Stanley, 2003)

We see partitions as obtained by piling up rectangles of size $p_i \times q_i$.

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Connection to Kerov's interlacing coordinates

$$x_1 = q_1, \qquad y_1 = q_1 - p_1$$
$$x_2 = q_2 - p_1, \quad y_2 = q_2 - p_1 - p_2$$
$$x_3 = -p_1 - p_2$$

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- Contains expression in parts (by setting $p_i = 1$), but is more convenient when taking transpose or dilatation of Young diagrams.
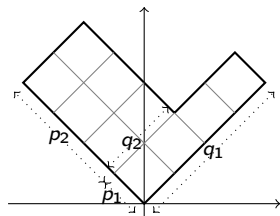- It turns out that these expressions have nice positivity/combinatorial properties!

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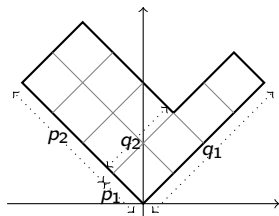
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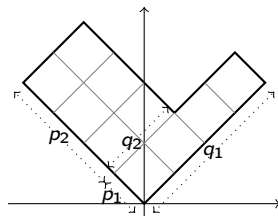$$x_2 = q_2 - p_1, \quad y_2 = q_2 - p_1 - p_2$$
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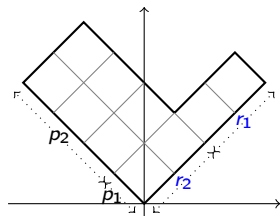$\rightarrow$ Shifted symmetric functions are polynomials in multirectangular coordinates.

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Embeddings of bipartite graphs
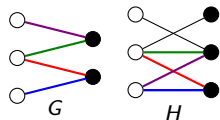
Definition: A bipartite graph embedding $G \to H$

maps edges of G to edges of H respecting incidence relations: i.e. edges sharing a black (resp. white) extremity in G are mapped to edges sharing a black (resp. white) extremity in H.

Notation

$$N_G(H) := \#\{\text{embeddings of } G \text{ in } H\}$$
$$\widetilde{N_G}(H) := \#\{\text{injective embeddings of } G \text{ in } H\}$$

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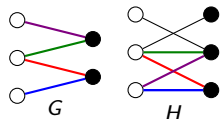
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Note: an embedding also maps black (resp. white) vertices of G to black (resp. white) vertices of H

Two families of bipartite graphs associated . . .

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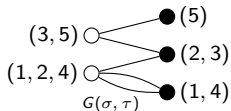
$\sigma = (1,2,4)(3,5), \ \tau = (1,4)(2,3)(5)$

Vertices share as many edges as the size of their intersection.

with partitions

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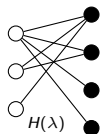
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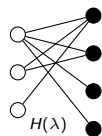
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Short notation: $N_{\sigma,\tau}(\lambda) := N_{G(\sigma,\tau)}\big(H(\lambda)\big); \quad \widetilde{N_{\sigma,\tau}}(\lambda) := \widetilde{N_{G(\sigma,\tau)}}\big(H(\lambda)\big).$

Lemma

$N_{\sigma,\tau}(\lambda)$ is a *polynomial with nonnegative coefficients in multirectangular coordinates*. It has degree $|C(\sigma)|$ in **p** and degree $|C(\tau)|$ in **q** (or **r**).

A formula for normalized characters

Theorem (Stanley, 2006, F., Śniady, 2007)

Let π be a permutation in S_k of type μ

$$\mathsf{Ch}_\mu = \sum_{\substack{\sigma,\tau \\ \sigma\,\tau=\pi}} (-1)^\tau N_{\sigma,\tau} = \sum_{\substack{\sigma,\tau \\ \sigma\,\tau=\pi}} (-1)^\tau \widetilde{N_{\sigma,\tau}}.$$

$(-1)^\tau = $ sign of $\tau = (-1)^k (-1)^{|C(\tau)|}$.

Note: second equality is relatively easy (show by sign-reversing involution that non-injective embeddings do not contribute to the total sum).

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Corollary

$(-1)^k \, \mathsf{Ch}_\mu$ *is a polynomial with nonnegative coefficients in $\boldsymbol{p}$ and $-\boldsymbol{q}$ (and thus in $\boldsymbol{p}$ and $-\boldsymbol{r}$).*

The formula is also suited for finding upper bounds for characters (F., Śniady, 2007).

A formula for normalized characters - sketch of proof

Use vanishing characterization: we define

$$F = \sum_{\substack{\sigma, \tau \\ \sigma\,\tau = \pi}} (-1)^\tau N_{\sigma,\tau} = \sum_{\substack{\sigma, \tau \\ \sigma\,\tau = \pi}} (-1)^\tau \widetilde{N_{\sigma,\tau}}$$

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3. The component of F of degree $|\mu|$ comes from the term $\sigma = \pi$ and $\tau = \mathrm{id}$ (in the first expression). But in this term is

$$N_{\pi,\mathrm{id}}(\lambda) = p_\mu(\lambda),$$

so that the top-component of F is p_μ as wanted.

Connection with maps combinatorics

- Reminder: if π has type μ, then $\mathrm{Ch}_\mu = \displaystyle\sum_{\substack{\sigma,\tau \\ \sigma\,\tau=\pi}} (-1)^\tau N_{\sigma,\tau}$.

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- But the set $\{\sigma, \tau;\ \sigma\, \tau = \pi\}$ is in bijection with maps (=graphs embedded in oriented surfaces) with prescribed face-degree. In other terms, Ch_μ is a signed weighted enumeration of maps.

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- But the set $\{\sigma, \tau;\ \sigma\,\tau = \pi\}$ is in bijection with maps (=graphs embedded in oriented surfaces) with prescribed face-degree. In other terms, Ch_μ is a signed weighted enumeration of maps.

- Maps with prescribed face-degree are better understood in the unicellular case, corresponding to $\mu = (k)$. For example, using a bijection of Chapuy ('11), we get this suprising relation:

$$\left(\sum_{\substack{i \geq 1 \\ h \geq 0}} \frac{p_i}{(2h+1)!} \frac{\partial^{2h+1}}{\partial p_i^{2h+1}} + \frac{r_i}{(2h+1)!} \frac{\partial^{2h+1}}{\partial r_i^{2h+1}} \right) \mathsf{Ch}_{(k)} = (k+1)\,\mathsf{Ch}_{(k)}\,.$$

Representation-theoretical consequence/interpretation? similar formulas for several parts?

Transition

Multirectangular coordinates –
expansion of shifted Schur functions

The falling factorial basis

Reminder: $s_\mu^\star(\lambda) = 0$ if $\lambda \not\supset \mu$.

$\to$ we cannot expect $s_\mu^\star$ to expand positively on multirectangular coordinates since it vanishes on a lot on Young diagrams (i.e. for a lot of positive specializations of multirectangular coordinates).

The falling factorial basis

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We introduce the falling factorial basis of multirectangular coordinates

$$(p_1 \downarrow a_1) \ldots (p_m \downarrow a_m)(r_1 \downarrow b_1) \ldots (r_m \downarrow b_m).$$

(Basis of polynomial ring in $\boldsymbol{p}$ and $\boldsymbol{r}$ when $a_1, \ldots, a_m, b_1, \ldots, b_m$ runs over lists of nonnegative integers.)

A nonnegative expansion for shifted Schur functions

Theorem (Alexandersson, F., 2017)

$s_\mu^\star$ expands positively on the falling factorial basis of multirectangular coordinates.

Corollary

$s_\mu^\star(x_1, \ldots, x_n)$ expands positively on the basis

$$\left((x_1 - x_2)_{b_1} \cdots (x_{\ell-1} - x_\ell)_{b_\ell - 1} (x_\ell)_{b_\ell} \right)_{b_1, \ldots, b_\ell \geq 0}.$$

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🙂 lifts to the polynomial $s_\mu^\star$ the nonnegativity of $s_\mu^\star(\lambda)$, for all partitions λ.

🙁 No combinatorial interpretation (although the coefficients of $|\mu|! \, s_\mu^\star$ are integers).

Main step of proofs

Using the previous formula for Ch_μ and the relation $s_\mu^\star = \sum_\nu \chi_\nu^\mu \, \mathrm{Ch}_\nu$, we get

$$s_\mu^\star = \frac{1}{k!} \sum_{\sigma, \tau \in \mathfrak{S}_k} \chi^\mu(\sigma\tau) \, (-1)^\tau \, N_{\sigma, \tau}.$$

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From there, we can extract the coefficient in FF-basis and reduce their nonnegativity to the one of

$$B_{S,T}^\mu = \sum_{\substack{\sigma \in \mathfrak{S}_S \\ \tau \in \mathfrak{S}_T}} \chi^\mu(\sigma\tau) \, (-1)^\tau,$$

for set-partitions S and T.

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We conclude with representation-theoretic arguments.

Transition

Quasi-symmetric functions
on Young diagrams

In which algebra lives N_G?

Expressing the shifted symmetric functions Ch_μ and $s_\mu^\star$ in terms of N_G gives nice expression.

But N_G is not shifted symmetric!

In the following slides, we study $Q\Lambda^\star := \mathrm{Span}(N_G)$ and connect it with quasi-symmetric functions.

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Definition

A polynomial F is   symmetric if
for any exponents $a_1, \ldots, a_r$, the coefficients of $x_{i_1}^{a_1} \ldots x_{i_r}^{a_r}$ is the same for all repetition-free sequences $i_1, \ldots, i_r$.

Example: $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_2^2 x_1 + x_3^2 x_1 + x_3^2 x_2 + \ldots$

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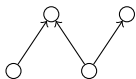
Definition

A polynomial F is quasi-symmetric if
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Detour: Quasi-symmetric function of an acyclic graph

Take an unlabelled acyclic directed graph $G_{ex} =$.

Definition

A function $f : V_G \to \mathbb{N}$ is order-preserving if

$$(i, j) \in E_G \Rightarrow f(i) \le f(j).$$

We consider the multivariate generating function in $x_1, x_2, \ldots$

$$\Gamma(G) = \sum_{\substack{f : V \to \mathbb{N} \\ f \text{ order-preserving}}} \prod_{v \in V} x_{f(v)} \quad \in \mathsf{QSym}.$$

Detour: Quasi-symmetric function of an acyclic graph

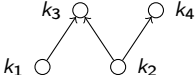Take an unlabelled acyclic directed graph $G_{ex} =$.

Definition

A function $f : V_G \to \mathbb{N}$ is order-preserving if

$$(i,j) \in E_G \ \Rightarrow \ f(i) \leq f(j).$$

We consider the multivariate generating function in $x_1, x_2, \ldots$

$$\Gamma(G) = \sum_{\substack{f : V \to \mathbb{N} \\ f \text{ order-preserving}}} \prod_{v \in V} x_{f(v)} \quad \in \mathsf{QSym}.$$

On the example above:

$$\Gamma(G_{ex}) = \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1 \leq k_3, \ k_2 \leq k_3, \ k_2 \leq k_4}} x_{k_1} x_{k_2} x_{k_3} x_{k_4}.$$

It is a quasisymmetric function (studied by Stanley, Gessel, . . .).

Quasisymmetric functions as functions on Young diagrams

Theorem (Aval, F., Novelli, Thibon, 2015)

There is an isomorphism QSym $\simeq Q\Lambda^\star = \mathrm{Span}(N_G)$ such that

- Sym $\subset$ QSym *is mapped to* $\Lambda^\star \subset Q\Lambda^\star$;
- *for bipartite graphs G, the function $\Gamma(G)$ is mapped to N_G.*

(Bipartite graphs are seen as acyclic graphs, by orienting edges from white to black.)

We have an "explicit" construction of the isomorphism using the virtual alphabet framework.

Quasisymmetric functions as functions on Young diagrams

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There is an isomorphism $QSym \simeq Q\Lambda^\star = Span(N_G)$ *such that*

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As a consequence,

$$\sum c_G N_G \quad \Leftrightarrow \quad \sum c_G \Gamma(G)$$
is shifted symmetric is symmetric

$\rightarrow$ more standard problem in symmetric function literature.

Conclusion

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- Nice combinatorics when using multirectangular coordinates;

- Tomorrow we will discuss Jack (and Macdonald) analogues of Ch_μ and $s_\mu^\star$; the extension of the positivity results shown today are only conjectured!