On Drinfel'd associators[1]

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INTRODUCTION

Zeta functions with several complex indices

Let $\mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\}$, for $r \in \mathbb{N}_+$, the following zeta function converges for $(s_1, \ldots, s_r) \in \mathcal{H}_r$

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}.$$

By a theorem by Abel, for $N \in \mathbb{N}, z \in \mathbb{C}, |z| < 1$, it can be obtained as

$$\zeta(s_1, \ldots, s_r) = \lim_{N \to +\infty} \mathrm{H}_{s_1, \ldots, s_r}(N) = \lim_{z \to 1} \mathrm{Li}_{s_1, \ldots, s_r}(z),$$

where the following functions are well defined for $(s_1, \ldots, s_r) \in \mathbb{C}^r$

$$\mathrm{Li}_{s_1, \ldots, s_r}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} \text{ and } \frac{\mathrm{Li}_{s_1, \ldots, s_r}(z)}{1 - z} = \sum_{N \geq 0} \mathrm{H}_{s_1, \ldots, s_r}(N) z^N.$$

They do appear in the *regularization* of solutions of the following differential equation with noncommutative indeterminates in $X = \{x_0, x_1\}$

$$(DE) \quad dG = MG, \text{ with } M = \omega_0 x_0 + \omega_1 x_1, \omega_0(z) = \frac{dz}{z}, \omega_1(z) = \frac{dz}{1 - z}.$$

Drinfel'd stated that[2] (DE) has a unique solution G_0 (resp. G_1), being group-like series, s.t. $G_0(z) \sim_0 e^{x_0 \log(z)}$ (resp. $G_1(z) \sim_1 e^{-x_1 \log(1-z)}$). There is then a unique group-like series $\Phi_{KZ} \in \mathbb{R}\langle\langle X \rangle\rangle$, so-called Drinfel'd associator, such that $G_0 = G_1 \Phi_{KZ}$.

[2] **V. Drinfel'd**, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

Knizhnik-Zamolodchikov differential equations

The linear representations of the braid group B_n, via the monodromies of the KZ equations over $\mathbb{C}_*^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$, yield

$$\begin{cases} (KZ_n) & dF(z_1, \ldots, z_n) = \Omega_n(z_1, \ldots, z_n) F(z_1, \ldots, z_n), \\ \text{where} & \Omega_n(z_1, \ldots, z_n) = \dfrac{1}{2\mathrm{i}\pi} \displaystyle\sum_{1 \leq i < j \leq n} t_{i,j} \frac{d(z_i - z_j)}{z_i - z_j}. \end{cases}$$

The system (KZ_n) is completely integrable iff[3] $d\Omega_n - \Omega_n \wedge \Omega_n = 0$.
This condition is equivalent to the fact that the elements of $\mathcal{T}_n = \{t_{i,j}\}_{1 \leq i,j \leq n}$ satisfy the following infinitesimal braid relations

$$\begin{cases} t_{i,j} = 0 & \text{for} & i = j, \\ t_{i,j} = t_{j,i} & \text{for} & i \neq j, \\ [t_{i,j}, t_{i,k} + t_{j,k}] = 0 & \text{for distinct} & i, j, k, \\ [t_{i,j}, t_{k,l}] = 0 & \text{for distinct} & i, j, k, l. \end{cases}$$

Example (KZ_3)

$\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}, [t_{1,3}, t_{1,2} + t_{2,3}] = [t_{2,3}, t_{1,2} + t_{1,3}] = 0$ and

$$\begin{cases} \Omega_3(z_1, z_2, z_3) = \dfrac{1}{2\mathrm{i}\pi}\left[t_{1,2}\dfrac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\dfrac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\dfrac{d(z_2 - z_3)}{z_2 - z_3}\right]. \\ F(z_1, z_2, z_3) = G((z_1 - z_2)/(z_1 - z_3))(z_1 - z_3)^{(t_{1,2} + t_{1,3} + t_{2,3})/2\mathrm{i}\pi}, \end{cases}$$

where G satisfies (DE) with $x_0 := t_{1,2}/2\mathrm{i}\pi, x_1 := -t_{2,3}/2\mathrm{i}\pi$.

[3] **P. Cartier**, *Développements récents sur les groupes de tresses. Applications à la topologie et à l'algèbre.* Sém BOURBAKI, 42ème 1989-1990, n°716.

Indexing by words

Let $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ and $1_\Omega : \Omega \to \mathbb{C}$ (mapping z to 1). The polylogarithms can be viewed as iterated integrals, w.r.t. ω_0, ω_1 and associated to words in X^* : $\mathrm{Li}_{s_1,\ldots,s_r}(z) = \alpha_0^z(x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1)$, where

$$\alpha_{z_0}^z(1_{X^*}) = 1_\Omega \quad \text{and} \quad \alpha_{z_0}^z(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^z \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k)$$

(here, $(z_0, z_1 \ldots, z_k, z)$ is a subdivision of the path $z_0 \rightsquigarrow z$).

They are single valued on Ω and alternatively they can be analytically continued and appear as multivalued functions over $B := \mathbb{C} - \{0, 1\}$ (more rigorously, we have analytic functions on the universal cover $\widetilde{B}$).

Now, introducing $Y = \{y_k\}_{k \geq 1}$ and using the correspondence

$$(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y^* \leftrightarrow x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1,$$

we will denote $\mathrm{H}_{y_{s_1}\ldots y_{s_r}} := \mathrm{H}_{s_1,\ldots,s_r}$ and $\mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} := \mathrm{Li}_{s_1,\ldots,s_r}$ and $\zeta(y_{s_1} \ldots y_{s_r}) := \zeta(s_1, \ldots, s_r) =: \zeta(x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1)$.

Let $Y_0 = Y \cup \{y_0\}$. For $(s_1, \ldots, s_r) \in \mathbb{N}^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y_0^*$, one defines

$$\begin{aligned}
\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}(z) &:= \mathrm{Li}_{-s_1,\ldots,-s_r}(z) = \sum_{n_1 > \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r}\, z^{n_1}, \\
\mathrm{H}^-_{y_{s_1}\ldots y_{s_r}}(N) &:= \mathrm{H}_{-s_1,\ldots,-s_r}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r}, \\
\zeta^-(y_{s_1} \ldots y_{s_r}) &:= \zeta(-s_1, \ldots, -s_r) \leftrightarrow \sum_{n_1 > \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r}.
\end{aligned}$$

Noncommutative, co-commutative bialgebras

$\mathbb{C}\langle X\rangle, \mathbb{C}\langle Y\rangle, \mathbb{C}\langle Y_0\rangle$ (resp. $\mathbb{C}\langle\langle X\rangle\rangle, \mathbb{C}\langle\langle Y\rangle\rangle, \mathbb{C}\langle\langle Y_0\rangle\rangle$) : sets of polynomials (resp. formal power series) over X, Y, Y_0.

- $(\mathbb{C}\langle X\rangle, ., \Delta_{\shuffle}, 1_{X^*}, \mathrm{e})$: for $x, y \in X$ and $u, v \in X^*$, $u \shuffle 1_{X^*} = u$ and $xu \shuffle yv = x(u \shuffle yv) + y(xu \shuffle v)$; $\Delta_{\shuffle}(x) = x \otimes 1_{X^*} + 1_{X^*} \otimes x$.
$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{S_l \otimes P_l} \quad \text{(MRS-factorization)},$$
where $\mathcal{L}ynX$ is the set of Lyndon words over X (with $x_1 > x_0$), $\{P_l\}_{l \in \mathcal{L}ynX}$ is a basis of Lie algebra of primitive elements and $\{S_l\}_{l \in \mathcal{L}ynX}$ is a pure transcendence basis of $(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*})$.
- $(\mathbb{C}\langle Y\rangle, ., \Delta_{\stuffle}, 1_{Y^*}, \mathrm{e})$: for $y_i, y_j \in Y$ and $u, v \in Y^*$, $u \stuffle 1_{Y^*} = u$ and $y_i u \stuffle y_j v = y_i(u \stuffle y_j v) + y_j(y_i u \stuffle v) + y_{i+j}(u \stuffle v)$; $\Delta_{\stuffle}(y_i) = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_k$.
$$\mathcal{D}_Y := \sum_{w \in Y^*} w \otimes w = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\Sigma_l \otimes \Pi_l} \quad (\stuffle-\text{extended MRS-factorization}),$$
where $\mathcal{L}ynY$ is the set of Lyndon words over Y (with $y_1 > \ldots$), $\{\Pi_l\}_{l \in \mathcal{L}ynY}$ is a basis of Lie algebra of primitive elements and $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ is a pure transcendence basis of $(\mathbb{C}\langle Y\rangle, \stuffle, 1_{Y^*})$.
- $\pi_Y : (\mathbb{C}\langle X\rangle, .) \to (\mathbb{C}\langle Y\rangle, .)$, $\quad x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \mapsto y_{s_1} \ldots y_{s_r}$.

First structures of polylogarithms and harmonic sums

1. Completing with $\mathrm{Li}_{x_0^k}(z) := \log^k(z)/k!$, the family $\{\mathrm{Li}_w\}_{w\in X^*}$ is $\mathbb{C}$-linearly independent. Hence, the following morphism of algebras is injective
$$\mathrm{Li}_\bullet : (\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) \to (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1), \quad u \mapsto \mathrm{Li}_u .$$
It follows that, $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\mathrm{Li}_{S_l}\}_{l\in\mathcal{L}ynX}$) is algebraically independent and, for any $u, v \in \mathcal{L}ynX - X$, $\zeta(u \sqcup\!\sqcup v) = \zeta(u)\zeta(v)$.

2. The following morphism of algebras is injective
$$\mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, \uplus, 1_{Y^*}) \to (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1), \quad u \mapsto \mathrm{H}_u.$$
Hence, $\{\mathrm{H}_w\}_{w\in Y^*}$ is linearly independent. It follows that, $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ (resp. $\{\mathrm{H}_{\Sigma_l}\}_{l\in\mathcal{L}ynY}$) is algebraically independent and, for any $u, v \in \mathcal{L}ynY - \{y_1\}$, $\zeta(u \uplus v) = \zeta(u)\zeta(v)$).

3. There exists, at least, an associative law of algebra $\top$, in $\mathbb{Q}\langle Y_0\rangle$, not dualizable such that the following morphism is a surjective
$$\mathrm{Li}^-_\bullet \quad : \quad (\mathbb{Q}\langle Y_0\rangle, \top) \quad \to \quad (\mathbb{Q}\{\mathrm{Li}^-_w\}_{w\in Y_0^*}, .), \quad w \mapsto \mathrm{Li}^-_w,$$
and $\ker \mathrm{Li}^-_\bullet = \mathbb{Q}\{w - w\top 1_{Y_0^*} | w \in Y_0^*\}$.

   Moreover, let $\top' : \mathbb{Q}\langle Y_0\rangle \times \mathbb{Q}\langle Y_0\rangle \to \mathbb{Q}\langle Y_0\rangle$ be a law such that $\mathrm{Li}^-_\bullet$ is a morphism for $\top'$ and $(1_{Y_0^*} \top' \mathbb{Q}\langle Y_0\rangle) \cap \ker(\mathrm{Li}^-_\bullet) = \{0\}$. Then $\top' = g \circ \top$, where $g \in GL(\mathbb{Q}\langle Y_0\rangle)$ such that $\mathrm{Li}^-_\bullet \circ g = \mathrm{Li}^-_\bullet$.

Bi-integro-differential algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$

Let $\mathcal{C} := \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$ be the differential ring (with $\partial_z := d/dz$).
Let $\theta_0 := z\partial_z$ and $\theta_1 := (1-z)\partial_z$ be differential operators.
Let ι_0 and ι_1 be sections of them[4], *i.e.* $\theta_0\iota_0 = \theta_1\iota_1 = \mathrm{Id}$. Then
$[\theta_0, \theta_1] = \theta_0 + \theta_1 = \partial_z$, $(\theta_0\iota_1)(\theta_1\iota_0) = (\theta_1\iota_0)(\theta_0\iota_1) = \mathrm{Id}$, $[\theta_0\iota_1, \theta_1\iota_0] = 0$.

Proposition

1. *If* $w = x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1$ *and* $u = y_{t_1} \ldots y_{t_r} \in Y_0^*$ *then* $\mathrm{Li}_w = (\iota_0^{s_1-1}\iota_1 \ldots \iota_0^{s_r-1}\iota_1)1_\Omega$ *and* $\mathrm{Li}_u^- = (\theta_0^{t_1+1}\iota_1 \ldots \theta_0^{t_r+1}\iota_1)1_\Omega$.
2. *For* $w \in Y_0^*$, $\mathrm{Li}_w^-(z) \in \mathbb{Z}[(1-z)^{-1}] \subsetneq \mathcal{C}$ *and* $\mathrm{H}_w^-(N) \in \mathbb{Q}[N]$ *of degree* $|w| + (w)$ *and of valuation* 1.
3. *The families* $\{\mathrm{Li}_{y_k}^-\}_{k\geq 0}$ *and* $\{\mathrm{H}_{y_k}^-\}_{k\geq 0}$ *are* $\mathbb{Q}$*-linearly independent.*
4. *The alg.* $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*} \cong \mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$ *is closed by* $\{\theta_i, \iota_i\}_{i=0,1}$.
5. *The bi-integro differential ring* $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ *is closed under the action of the group of transformations permuting* $\{0, 1, +\infty\}$,
$\mathcal{G} := \{z \mapsto z, z \mapsto 1-z, z \mapsto z^{-1}, z \mapsto (1-z)^{-1}, z \mapsto 1-z^{-1}, z \mapsto z(1-z)^{-1}\}$:
$$\forall h \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \quad \forall g \in \mathcal{G}, \quad h(g(z)) \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

[4] *i.e.* these operators take primitives for the corresponding differential operators w.r.t. to the function.

GLOBAL RENORMALIZATIONS
OF DIVERGENT ZETAS VALUES
INDEXED BY INTEGRAL MULTI-INDICES

Chen series and noncommutative generating series

$$S_{z_0\rightsquigarrow z} := (\alpha_{z_0}^z \otimes \mathrm{Id})\mathcal{D}_X \quad \mathrm{L} := (\mathrm{Li}_\bullet \otimes \mathrm{Id})\mathcal{D}_X \quad \mathrm{H} := (\mathrm{H}_\bullet \otimes \mathrm{Id})\mathcal{D}_Y$$

$$= \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\alpha_{z_0}^z(S_l)P_l}, \quad = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l} P_l}, \quad = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l}.$$

$$Z_{ш} := \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\zeta(S_l)P_l} \quad \text{and} \quad Z_{⧢} := \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\zeta(\Sigma_l)\Pi_l}.$$

$S_{z_0\rightsquigarrow z}, \mathrm{L}, Z_{ш}$ (resp. $\mathrm{H}, Z_{⧢}$) are goupe-like, for $\Delta_{ш}$ (resp. $\Delta_{⧢}$).

Theorem (Drinfel'd associator)

1. L *is a solution of* (DE) *satisfying* $\lim_{z\to 0}\mathrm{L}(z)e^{-x_0\log(z)} = 1_{X^*}$ *and* $\lim_{z\to 1} e^{x_1\log(1-z)}\mathrm{L}(z) = Z_{ш}$.
2. $\mathrm{L}(1-z) = \sigma(\mathrm{L}(z))Z_{ш}$, *where* $\sigma(x_0) = -x_1$ *and* $\sigma(x_1) = -x_0$. *Hence,* $\mathrm{L}(1-z)\widetilde{_{z\to 0}}e^{-x_1\log(z)}Z_{ш}$.
3. $S_{z_0\rightsquigarrow z}$ *is the unique solution of* (DE) *satisfying* $S_{z_0\rightsquigarrow z_0} = 1_{X^*}$, *and* $S_{z_0\rightsquigarrow z} = \mathrm{L}(z)(\mathrm{L}(z_0))^{-1}$.
4. L *is the unique solution of* (DE) *satisfying the asymptotic condition at 0. Hence,* $\Phi_{KZ} = Z_{ш}$ *is unique.*

Abel like theorems for noncommutative generating series

Theorem (HNM, 2005)

$$\lim_{z\to 1}\exp\left[-y_1\log\frac{1}{1-z}\right]\pi_Y \mathrm{L}(z) = \lim_{n\to\infty}\exp\left[\sum_{k\geq 1}\mathrm{H}_{y_k}(n)\frac{(-y_1)^k}{k}\right]\mathrm{H}(n) = \pi_Y Z_{\sqcup\!\sqcup}.$$

Theorem (Duchamp, HNM, Ngô, 2015)

$$\mathrm{L}^- := \sum_{w\in Y_0^*}\mathrm{Li}_w^-\, w, \mathrm{H}^- := \sum_{w\in Y_0^*}\mathrm{H}_w^- w, C^- := \sum_{w\in Y_0^*}\left(\prod_{w=uv, v\neq 1_{Y_0^*}}\frac{1}{(v)+\mid v\mid}\right)w.$$

$$\lim_{z\to 1}\Lambda^{\odot -1}((1-z)^{-1})\odot \mathrm{Li}^-(z) = \lim_{n\to+\infty}\Upsilon^{\odot -1}(n)\odot \mathrm{H}^-(n) = C^-,$$

where $\Lambda(t) = \sum_{w\in Y_0^*}((w)+\mid w\mid)! t^{(w)+|w|} w$ *and* $\Upsilon(t) = \sum_{w\in Y_0^*} t^{(w)+|w|} w$.
Moreover, H^- *and* C^- *are group-like, respectively, for* $\Delta_{\boxplus}$ *and* $\Delta_{\sqcup\!\sqcup}$.

Let $\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\zeta(s_1,\ldots,s_r)\}_{(s_1,\ldots,s_r)\in\mathcal{H}_r\cap\mathbb{N}^r}^{r\in\mathbb{N}}$. Then, for any $w\in X^*x_1$, there exists $a_i, b_{i,j}\in\mathcal{Z}$ and $\gamma_{\pi_Y w}, \alpha_i, \beta_{i,j}\in\mathcal{Z}[\gamma]$ such that

$$\mathrm{Li}_w(z) \underset{z\to 1}{\asymp} \sum_{i=1}^{(w)} a_i\log^i(1-z) + \langle Z_{\sqcup\!\sqcup} | \pi_X w\rangle + \sum_{i\in\mathbb{N}_+, j\in\mathbb{N}_-} b_{i,j}\frac{\log^i(1-z)}{(1-z)^j},$$

and $$\mathrm{H}_{\pi_Y w}(n) \underset{n\to+\infty}{\asymp} \sum_{i=1}^{|w|}\alpha_i\log^i(n) + \gamma_{\pi_Y w} + \sum_{i,j\in\mathbb{N}_+}\beta_{i,j}\frac{\log^i(n)}{n^j}.$$

Generalized Euler constants $\{\gamma_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{N}^r}$

The map $\gamma_{\bullet} : (\mathbb{C}\langle Y\rangle, \uplus, 1_{Y^*}) \to (\mathbb{C}, ., 1)$ is a character since its graph

$$Z_\gamma := \sum_{w\in Y^*} \gamma_w w \quad \text{satisfies} \quad \Delta_{\uplus}(Z_\gamma) = Z_\gamma \otimes Z_\gamma \text{ and } \langle Z_\gamma | 1_{Y^*}\rangle = 1.$$

The first Abel like theorem leads to

$$Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}, \quad \text{where} \quad B(y_1) = \exp\Big(\gamma y_1 - \sum_{k\geq 2} \zeta(k)\frac{(-y_1)^k}{k}\Big)$$

and then, by **cancellation** via the $\uplus$-extended MRS-factorization, one has $Z_{\uplus} = B'(y_1)\pi_Y Z_{\sqcup\!\sqcup}$, where $B'(y_1) = \exp(-\gamma y_1)B(y_1)$.

Let $B_{i,j}$'s are Bell polynomials. For $k \in \mathbb{N}_+, w \in Y^+$, one has

$$\gamma_{y_1^k} = \langle Z_\gamma | y_1^k \rangle = \sum_{\substack{s_1,\ldots,s_k>0 \\ s_1+\ldots+ks_k=k}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\Big(-\frac{\zeta(2)}{2}\Big)^{s_2}\ldots\Big(-\frac{\zeta(k)}{k}\Big)^{s_k},$$

$$\gamma_{y_1^k w} = \langle Z_\gamma | y_1^k w\rangle = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} \sqcup\!\sqcup \pi_X w])}{i!}\Big(\sum_{j=1}^{i} B_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\Big).$$

Example (generalized Euler constants)

$$\begin{aligned}
\gamma_{1,1} &= [\gamma^2 - \zeta(2)]/2, \\
\gamma_{1,1,1} &= [\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)]/6, \\
\gamma_{1,1,1,1} &= [80\zeta(3)\gamma - 60\zeta(2)\gamma^2 + 6\zeta(2)^2 + 10\gamma^4]/240, \\
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - 54\zeta(2)^4 175, \\
\gamma_{1,1,6} &= 4\zeta(2)^3\gamma^2/35 + [\zeta(2)\zeta(5) + 2\zeta(3)\zeta(2)^2 - 4\zeta(7)]\gamma/5 + \zeta(6,2) + 19\zeta(2)^4/35 + \zeta(2)\zeta(3)^2/2 - 4\zeta(3)\zeta(5), \\
\gamma_{1,1,1,5} &= 3\zeta(6,2)/4 - 14\zeta(3)\zeta(5)/3 + 3\zeta(2)\zeta(3)^2/4 + 809\zeta(2)^4/1400 \\
&- (2\zeta(7) - 3\zeta(2)\zeta(5)/2 + \zeta(3)\zeta(2)^2/10)\gamma + (\zeta(3)^2/4 - \zeta(2)^3/5)\gamma^2 + \zeta(5)\gamma^3/6.
\end{aligned}$$

POLYLOGARITHMS AND HARMONIC SUMS INDEXED BY NONCOMMUTATIVE RATIONAL SERIES

Rational series, $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$

$\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$: set of noncommutative rational series[5].

Theorem (Schützenberger, 1961)

$S \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ iff there is a linear representation, (ν, μ, η) of dimension $n > 0$, i.e. $\nu \in M_{1,n}(\mathbb{C})$, $\eta \in M_{n,1}(\mathbb{C})$ and $\mu : X^* \to M_{n,n}(\mathbb{C})$ such that

$$S = \nu\Big(\sum_{w \in X^*} \mu(w)\, w\Big)\eta = \nu((\mu \otimes \mathrm{Id})\mathcal{D}_X)\eta.$$

Theorem (HNM, 1995)

Let (ν, μ, η) be a linear representation of $S \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$.
Then the series $\sum_{w \in X^*} \langle S|w\rangle \alpha_{z_0}^z(w)$ is convergent.
Noting this extension by $\alpha_{z_0}^z(S)$, one has

$$\alpha_{z_0}^z(S) = \nu\Big(\prod_{l \in \mathcal{L}ynX} e^{\alpha_{z_0}^z(S_l)\, \mu(P_l)}\Big)\eta.$$

And, for any $T = R$ or $S \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, one has the convergent series $\alpha_{z_0}^z(T) = \sum_{w \in X^*} \langle T|w\rangle \alpha_{z_0}^z(w)$ and identity $\alpha_{z_0}^z(R \sqcup\!\sqcup S) = \alpha_{z_0}^z(R)\alpha_{z_0}^z(S)$.

[5] A series S is called *rational* iff it belongs to the closure, by $\{+, \mathtt{conc}, {}^*\}$, of $\mathbb{C}\langle X\rangle$ in $\mathbb{C}\langle\langle X\rangle\rangle$.

Exchangeable series, $\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$

The power series S belongs to $\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$, iff

$$(\forall u,v\in X^*)((\forall x\in X)(|u|_x=|v|_x)\quad\Rightarrow\quad\langle S|u\rangle=\langle S|v\rangle).$$

If $\quad S=\sum_{i_0,i_1\geq 0} s_{i_0,i_1}x_0^{i_0}\,⧢\,x_1^{i_1}\quad$ then $\quad\alpha_{z_0}^z(S)=\sum_{i_0,i_1\geq 0} s_{i_0,i_1}\dfrac{(\alpha_{z_0}^z(x_0))^{i_0}}{i_0!}\dfrac{(\alpha_{z_0}^z(x_1))^{i_1}}{i_1!}.$

Lemma (Duchamp, HNM, Ngô, 2016)

1. $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle:=\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle\bigcap\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle=\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\,⧢\,\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$.
2. *For any* $x\in X$, *one has* $\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle=\mathrm{span}_{\mathbb{C}}\{(ax)^*\,⧢\,\mathbb{C}\langle x\rangle|a\in\mathbb{C}\}$.
3. *The family* $\{x_0^*,x_1^*\}$ *is algebraically independent over* $(\mathbb{C}\langle X\rangle,⧢,1_{X^*})$ *within* $(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle,⧢,1_{X^*})$.
4. *The module* $(\mathbb{C}\langle X\rangle,⧢,1_{X^*})[x_0^*,x_1^*,(-x_0)^*]$ *is* $\mathbb{C}\langle X\rangle$*-free and* $\{(x_0^*)^{⧢ k}\,⧢\,(x_1^*)^{⧢ l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ *forms a* $\mathbb{C}\langle X\rangle$*-basis of it.*

   Hence, $\{w\,⧢\,(x_0^*)^{⧢ k}\,⧢\,(x_1^*)^{⧢ l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}_{w\in X^*}$ *is a* $\mathbb{C}$*-basis of it.*

Theorem (extension of $\mathrm{Li}_\bullet$, Duchamp, HNM, Ngô, 2016)

$$\begin{array}{rcl}\mathrm{Li}_\bullet:(\mathbb{C}[x_0^*,x_1^*,(-x_0)^*]\,⧢\,\mathbb{C}\langle X\rangle,⧢,1_{X^*}) & \longrightarrow & (\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*},.,1_\Omega),\\ R & \longmapsto & \mathrm{Li}_R\,.\end{array}$$

$\mathrm{Li}_\bullet$ *is surjective and* $\ker\mathrm{Li}_\bullet$ *is the shuffle ideal generated by* $x_0^*\,⧢\,x_1^*-x_1^*+1$.

Polylogarithms and harmonic sums by rational series

One has, $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \subset \mathrm{Dom}(\mathrm{Li}_\bullet) \cap \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, and, on it,

$$\begin{aligned}\mathrm{Li}_S(z) &= \sum_{n\geq 0} \frac{\langle S|x_0^n\rangle}{n!}\log^n(z) + \sum_{k\geq 1}\sum_{w\in x_0^* \sqcup\!\sqcup\, x_1^k} \langle S|w\rangle\,\mathrm{Li}_w(z)\\ &= \nu\Big(\prod_{l\in\mathcal{L}ynX-\{x_0\}} e^{\mathrm{Li}_{S_l}(z)\mu(P_l)}\Big) e^{\log(z)\mu(x_0)}\eta,\end{aligned}$$

where (ν,μ,η) is a linear representation of $S\in\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$.

Example (of polylogarithms indexed by rational series)

For any $a,b\in\mathbb{C}$, $\alpha_1^z((ax_0)^*) = z^a$ and $\alpha_0^z((bx_1)^*) = (1-z)^{-b}$. Then

1. $\mathrm{Li}_{x_0^*}(z) = z$, $\mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}$, $\mathrm{Li}_{(x_0+x_1)^*}(z) = z(1-z)^{-1}$.
2. Since $(x^*)^{\sqcup\!\sqcup\, n} = (nx)^*$, one has $\mathrm{Li}_{(x_0^*)^{\sqcup\!\sqcup\, n}}(z) = z^n$,
   $\mathrm{Li}_{(x_1^*)^{\sqcup\!\sqcup\, k}}(z) = (1-z)^{-k}$, $\mathrm{Li}_{(x_0^*)^{\sqcup\!\sqcup\, n}\,\sqcup\!\sqcup\,(x_1^*)^{\sqcup\!\sqcup\, k}}(z) = z^n(1-z)^{-k}$.
3. Since $(ax)^{*n} = (ax)^* \sqcup\!\sqcup\, (1-ax)^{n-1}$ then

$$\mathrm{Li}_{(ax_0)^{*n}}(z) = z^a\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(a\log z)^k}{k!},$$

$$\mathrm{Li}_{(ax_1)^{*n}}(z) = \frac{1}{(1-z)^a}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{k!}\Big(a\log\frac{1}{1-z}\Big)^k.$$

Indexing polylogarithms by rational series

$$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\ldots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\ldots\binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}({\theta_0}^{k_1}\lambda)\ldots({\theta_0}^{k_r}\lambda),$$

$$ {\theta_0}^{k_i}(\lambda(z)) = \frac{1}{1-z}\sum_{j=1}^{k_i} S_2(k_i,j)j!(\lambda(z))^j, \quad \text{for } k_i>0,$$

where $\lambda : z \mapsto z(1-z)^{-1}$ and S_2 are Stirling numbers of second kind.

Lemma (Encoding polylogarithms by rational series)

$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} = \mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}$, *where* $R_{y_{s_1}\ldots y_{s_r}} \in (\mathbb{C}[x_1^*], \shuffle, 1_{X^*})$ *given by*

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\ldots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\ldots\binom{\sum_{i=1}^r s_i-\sum_{i=1}^{r-1}k_i}{k_r}\rho_{k_1}\shuffle\ldots\shuffle\rho_{k_r},$$

$$\rho_{k_i} = \begin{cases} x_1^* - 1_{X^*}, & \textit{if} \quad k_i=0,\\ x_1^*\shuffle\sum_{j=1}^{k_i} S_2(k_i,j)j!(x_1^*-1_{X^*})^{\shuffle j}, & \textit{if} \quad k_i>0.\end{cases}$$

By linearity, $R_\bullet$ *is extended over* $\mathbb{Q}\langle Y_0\rangle$*. Hence, for any* $k,l\in\mathbb{N}$*, one has*

$$\mathrm{Li}_{R_{y_k}\shuffle R_{y_l}} = \mathrm{Li}_{R_{y_k}}\mathrm{Li}_{R_{y_l}} = \mathrm{Li}^-_{y_k}\mathrm{Li}^-_{y_l} = \mathrm{Li}^-_{y_k\top y_l} = \mathrm{Li}_{R_{y_k\top y_l}}.$$

Euler-Mac Laurin constants $\{\gamma_{-s_1,\ldots,-s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{N}^r, r\in\mathbb{N}}$

Definition (Double regularization)

$\zeta_{⧢} : (\mathbb{C}\langle X\rangle, ⧢, 1_{X^*}) \to (\mathbb{C}, ., 1)$ and $\gamma_{\bullet} : (\mathbb{C}\langle Y\rangle, \uplus, 1_{Y^*}) \to (\mathbb{C}, ., 1)$, s.t. $\zeta_{⧢}(x_1) = 0, \gamma_{y_1} = \gamma$ and $\forall l \in \mathcal{L}ynX - X, \zeta_{⧢}(l) = \gamma_{\pi_Y l} = \zeta(l)$.

For any $k \geq 1$, there exists unique $R_{y_k} \in (\mathbb{C}[x_1^*], ⧢, 1_{X^*})$ such that $\mathrm{Li}_{-k} = \mathrm{Li}^-_{y_k} = \mathrm{Li}_{R_{y_k}}$ and $\mathrm{H}_{-k} = \mathrm{H}^-_{y_k} = \mathrm{H}_{\pi_Y(R_{y_k})}$.

Theorem (Extended double regularization)

$$\zeta_{⧢}((tx_1)^*) = 1 \text{ and}^6\ \gamma_{\pi_Y(tx_1)^*} = \exp\left(\gamma t - \sum_{n\geq 2}\zeta(n)\frac{(-t)^n}{n}\right) = \frac{1}{\Gamma(1+t)}.$$

Example (of regularizations)

$$\begin{aligned}
\mathrm{Li}_{-1,-1} &= -\mathrm{Li}_{x_1^*} + 5\,\mathrm{Li}_{(2x_1)^*} - 7\,\mathrm{Li}_{(3x_1)^*} + 3\,\mathrm{Li}_{(4x_1)^*},\\
\mathrm{Li}_{-2,-1} &= \mathrm{Li}_{x_1^*} - 11\,\mathrm{Li}_{(2x_1)^*} + 31\,\mathrm{Li}_{(3x_1)^*} - 33\,\mathrm{Li}_{(4x_1)^*} + 12\,\mathrm{Li}_{(5x_1)^*},\\
\mathrm{Li}_{-1,-2} &= \mathrm{Li}_{x_1^*} - 9\,\mathrm{Li}_{(2x_1)^*} + 23\,\mathrm{Li}_{(3x_1)^*} - 23\,\mathrm{Li}_{(4x_1)^*} + 8\,\mathrm{Li}_{(5x_1)^*},\\
\mathrm{H}_{-1,-1} &= -\mathrm{H}_{\pi_Y(x_1^*)} + 5\mathrm{H}_{\pi_Y((2x_1)^*)} - 7\mathrm{H}_{\pi_Y((3x_1)^*)} + 3\mathrm{H}_{\pi_Y((4x_1)^*)},\\
\mathrm{H}_{-2,-1} &= \mathrm{H}_{\pi_Y(x_1^*)} - 11\mathrm{H}_{\pi_Y((2x_1)^*)} + 31\mathrm{H}_{\pi_Y((3x_1)^*)} - 33\mathrm{H}_{\pi_Y((4x_1)^*)} + 12\mathrm{H}_{\pi_Y((5x_1)^*)},\\
\mathrm{H}_{-1,-2} &= \mathrm{H}_{\pi_Y(x_1^*)} - 9\mathrm{H}_{\pi_Y((2x_1)^*)} + 23\mathrm{H}_{\pi_Y((3x_1)^*)} - 23\mathrm{H}_{\pi_Y((4x_1)^*)} + 8\mathrm{H}_{\pi_Y((5x_1)^*)}.
\end{aligned}$$

Therefore, $\zeta_{⧢}(-1,-1) = 0$, $\zeta_{⧢}(-2,-1) = -1$, $\zeta_{⧢}(-1,-2) = 0$, and

$$\begin{aligned}
\gamma_{-1,-1} &= -\Gamma^{-1}(2) + 5\Gamma^{-1}(3) - 7\Gamma^{-1}(4) + 3\Gamma^{-1}(5) = 11/24,\\
\gamma_{-2,-1} &= \Gamma^{-1}(2) - 11\Gamma^{-1}(3) + 31\Gamma^{-1}(4) - 33\Gamma^{-1}(5) + 12\Gamma^{-1}(6) = -73/120,\\
\gamma_{-1,-2} &= \Gamma^{-1}(2) - 9\Gamma^{-1}(3) + 23\Gamma^{-1}(4) - 23\Gamma^{-1}(5) + 8\Gamma^{-1}(6) = -67/120.
\end{aligned}$$

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[6] $\mathrm{H}_{\pi_Y(tx_1)^*} = \sum_{n\geq 0}\mathrm{H}_{y_1^n}t^n = \exp(-\sum_{n\geq 1}\mathrm{H}_{y_n}(-t)^n/n)$.

Candidates for associators with rational coefficients

$\Upsilon := ((\mathrm{H}_\bullet \circ \pi_Y \circ R_\bullet) \otimes \mathrm{Id})\mathcal{D}_Y$ and $\Lambda := ((\mathrm{Li}_\bullet \circ R_\bullet \circ \hat{\pi}_Y) \otimes \mathrm{Id})\mathcal{D}_X$,
$Z_\gamma^- := ((\gamma_\bullet \circ \pi_Y \circ R_\bullet) \otimes \mathrm{Id})\mathcal{D}_Y$ and $Z_{\shuffle}^- := ((\zeta_{\shuffle} \circ R_\bullet \circ \hat{\pi}_Y) \otimes \mathrm{Id})\mathcal{D}_X$,
where, the morphism of algebras $\hat{\pi}_Y$ is defined, over an algebraic basis, by

$$\forall l \in \mathcal{L}ynX - \{x_0\}, \hat{\pi}_Y S_l = \pi_Y S_l \quad \text{and} \quad \hat{\pi}_Y(x_0) = x_0$$

(such that $\mathrm{Li}_{R_{\hat{\pi}_Y x_0}}(z) = \log(z)$ and then $\zeta(R_{\hat{\pi}_Y x_0}) = 0$).
Hence, $Z_{\shuffle}^- \in \mathbb{Z}\langle\langle X \rangle\rangle$ and $Z_\gamma^- \in \mathbb{Q}\langle\langle Y \rangle\rangle$. One also has

Theorem (Associators with rational coefficients)

$$\Delta_{\stuffle}(\Upsilon) = \Upsilon \otimes \Upsilon \quad \textit{and} \quad \Delta_{\shuffle}(\Lambda) = \Lambda \otimes \Lambda,$$
$$\Delta_{\stuffle}(Z_\gamma^-) = Z_\gamma^- \otimes Z_\gamma^- \quad \textit{and} \quad \Delta_{\shuffle}(Z_{\shuffle}^-) = Z_{\shuffle}^- \otimes Z_{\shuffle}^-,$$

and all constant terms are 1*. It follows then*

$$\Upsilon = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\pi_Y R_{\Sigma_l}} \Pi_l} \quad \textit{and} \quad \Lambda = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{R_{\hat{\pi}_Y S_l}} P_l},$$
$$Z_\gamma^- = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\gamma_{\pi_Y R_{\Sigma_l}} \Pi_l} \quad \textit{and} \quad Z_{\shuffle}^- = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\zeta_{\shuffle}(R_{\hat{\pi}_Y S_l}) P_l}.$$

Moreover, for any $g \in \mathcal{G}$, there exists a morphism of linear substitution, σ_g, and a Lie series $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle$, such that $\Lambda(g) = \sigma_g(\Lambda)e^C$.

THANK YOU FOR YOUR ATTENTION